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PICARD-VESSIOT GROUP OF APPELL'S SYSTEM OF HYPERGEOMETRIC DIFFERENTIAL EQUATIONS AND INFINITENESS OF MONODROMY GROUP

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§ 1. Introduction and results

In this paper we will compute Picard-Vessiot group of Appell's systems of hypergeometric differential equations (F_1), (F_2), (F_3) and (F_4). We prove also the infiniteness of monodromy groups of (F_2) and (F_3). To begin with, we will explain Appell's systems. These systems are defined on the projective space $\boldsymbol{P}^2$ and given by equations

$$(F_1)\begin{cases} \theta(\theta+\theta'+\gamma-1)\,z-x(\theta+\theta'+\alpha)(\theta+\beta)\,z=0 \\ \theta'(\theta+\theta'+\gamma-1)\,z-y(\theta+\theta'+\alpha)(\theta'+\beta')\,z=0, \end{cases}$$

$$(F_2)\begin{cases} \theta(\theta+\gamma-1)\,z-x(\theta+\theta'+\alpha)(\theta+\beta)\,z=0 \\ \theta'(\theta'+\gamma'-1)\,z-y(\theta+\theta'+\alpha)(\theta'+\beta')\,z=0, \end{cases}$$

$$(F_3)\begin{cases} \theta(\theta+\theta'+\gamma-1)\,z-x(\theta+\alpha)(\theta+\beta)\,z=0 \\ \theta'(\theta+\theta'+\gamma-1)\,z-y(\theta'+\alpha')(\theta'+\beta')\,z=0, \end{cases}$$

$$(F_4)\begin{cases} \theta(\theta+\gamma-1)\,z-x(\theta+\theta'+\alpha)(\theta+\theta'+\beta)\,z=0 \\ \theta'(\theta'+\gamma'-1)\,z-y(\theta+\theta'+\alpha)(\theta+\theta'+\beta)\,z=0, \end{cases}$$

where $\theta=x\dfrac{\partial}{\partial x}$, $\theta'=y\dfrac{\partial}{\partial y}$, $(x,y)\in\boldsymbol{P}^2$ being inhomogeneous coordinates, and Greek letters α, β, γ, $\cdots$ are complex parameters. Each system is of Fuchsian type; that is, every solution is regular singular at the singular set S: S consists of the line at infinity and $\{x=0\}\cup\{x=1\}\cup\{y=0\}\cup\{y=1\}\cup\{x=y\}$, $\{x=0\}\cup\{x=1\}\cup\{y=0\}\cup\{y=1\}\cup\{x+y=1\}$, $\{x=0\}\cup\{x=1\}\cup\{y=0\}\cup\{y=1\}\cup\{x+y-xy=0\}$ or $\{x=0\}\cup\{y=0\}\cup\{1-2(x+y)+(x-y)^2=0\}$ for (F_1), (F_2), (F_3) or (F_4) respectively. Also we consider the system called (F_D), which is a generalization of (F_1) given by G. Lauricella:

$$(F_D)\ \theta_i(\theta_1+\cdots+\theta_n+\gamma-1)\,z-x_i(\theta_1+\cdots+\theta_n+\alpha)(\theta_i+\beta_i)z=0\ (1\leqq i\leqq n),$$

where $\theta_i=x_i\dfrac{\partial}{\partial x_i}$, $(x_1,\cdots,x_n)\in\boldsymbol{P}^n$. This system is again of Fuchsian type and its singular set is the union of $\bigcup_i \{x_i=0\} \bigcup_i \{x_i=1\} \bigcup_{i,j} \{x_i=x_j\}$ and the line at infinity. Since (F_D) is (F_1) when $n=2$, we will not mention (F_1) in the following. So (F_i) denotes one of (F_2), (F_3) and (F_4).

By k we denote the order of each system, which is $n+1$ for (F_D) and 4 for others. The monodromy representation of the fundamental group $\pi_1(P^n-S)(n=2$ for $(F_i))$ has values in $GL(k, C)$. Its image, the monodromy group, gives an abelian aspect of the system.

The Picard-Vessiot group of the system is, by definition, the automorphism group of the solution space over the coefficients field which is, in our case, the rational function field of n variables. These are algebraic subgroups of $GL(k, C)$.

From now on we denote by (F) one of systems (F_i) and (F_D), by Γ(resp. Γ_i, Γ_D) the monodromy group of (F)(resp. (F_i), (F_D)) and by P(resp. P_i, P_D) the Picard-Vessiot group of (F)(resp. (F_i), (F_D)).

The fundamental result about the Picard-Vessiot group is the following: If the system of linear differential equations is of Fuchsian type, then the monodromy group is Zariski dense in the Picard-Vessiot group. Therefore, since every system we are considering is of Fuchsian type, the monodromy group Γ is Zariski dense in the Picard-Vessiot group P. So, to compute P, it is desirable to know the structure of the monodromy group Γ. Fortunately, we know generators of Γ explicitly. E.Picard and T. Terada gave generators of Γ_D ([4], [8]). Recently E. Nakagiri, K. Takano and J. Kaneko computed generators of Γ_i ([3],[6],[2]). Making use of these generators, we can prove the following

THEOREM 1. *When complex parameters take general values, the Picard-Vessiot group P is equal to the full general linear group $GL(k, C)$.*

Concerning the monodromy group itself we prove

THEORME 2. *If Γ_i, $i=2,3$, is irreducible, then it cannot be finite.*

We should remark here that Γ_D can be a finite irreducible group when parameters take special values and $n=1,2,3$ ([5]). But, so far, the author does not know whether Γ_4 can be finite or not.

We prove Theorem 1 in Part I and Theorem 2 in Part II.

Part I. Picard-Vessiot group

§2. The Picard-Vessiot theory of linear differential equations is developed by E.R. Kolchin extensively. But we know few examples whose Picard-Vessiot groups are computed. Professor K. Aomoto advised the author to calculate these groups for the systems (F) and conjectured Theorem 1. Really he raised a problem to characterize systems whose Picard-Vessiot groups are equal to the general linear group. At this stage, the author does not know why Theorem 1 holds, i.e. what kind of structures of the system assure this result.

The idea of the calculation of Picard-Vessiot groups is very simple which came to our notice in the conversation with Dr. S. Mukai. Namely, take one generator γ of the monodromy group Γ. Then, in general, the group $\{\gamma^n;\ n\in Z\}$ has a continuous group as its Zariski closure. Since we know γ explicitly, it is easy to get the infinitesimal generators of this group. And, obviously, these infinitesimal generators are contained in the Lie algebra of the Zariski closure of Γ. Then we see that the Lie algebra containing these generators is the Lie algebra of the general linear group.

There is one more remark. In the following we deal with, instead of Γ itself, a certain subgroup of Γ. Let z be a solution of the system (F). It is a function of (x, y) or $(x_1, \dots, x_n)$. If we consider z as a function of x or x_1 only, then z satisfies an ordinary differential equation of the same order with respect to x or x_1. If we denote by Γ' the monodromy group of this equation, then Γ' is a subgroup of Γ which we would like to consider. For the systems (F_i), $i=2, 3$, and (F_D) we will prove the following which is stronger than Theorem 1.

THEOREM 1'. *When complex parameters take general values, the Zariski closure of the group* Γ_2', Γ_3' or Γ_D' *is equal to the general linear group* $GL(k, C)$.

REMARK 1. Let K be the field over the rational number field extended by exponentials of $2\pi i$ times of complex parameters: $exp(2\pi i\alpha)$, $exp(2\pi i\beta)$, ... Then Γ is contained in $GL(k, K)$ and the Zariski closure over K equals to $GL(k, K)$.

REMARK 2. It is known that the equation obtained above from the system (F_D) restricting variables to x_1 is the equation of Jordan-Pochhammer type:

$$(F_{JP})\qquad \begin{aligned} &\phi(x)z^{(n+1)}-\mu\phi'(x)z^{(n)}+\mu(\mu+1)/2\phi''(x)z^{(n-1)}-\cdots\\ &\quad-\psi(x)z^{(n)}+(\mu+1)\psi'(x)z^{(n-1)}-\cdots=0.\end{aligned}$$

Here, $\phi(x)=(x-a_1)(x-a_2)\dots(x-a_{n+1})$, $\psi(x)/\phi(x)=\sum_{j=1}^{n+1}\alpha_j/(x-a_j)$, $a_1=0$, $a_{n+1}=1$, $a_i=$ the value of x_i which we fixed, $2\leqq i\leqq n$; $\alpha_1=\beta_1+\cdots+\beta_n-\gamma+1$, $\alpha_{n+1}=\gamma-\alpha$, $\alpha_i=-\beta_{i-1}+1$, $2\leqq i\leqq n$, and $\mu=\gamma-\alpha-1-n$.

§ 3. Generators of the monodromy group

Since the proof of the above Theorems is given by calculations, we list generators of Γ in order to do that. Γ_2 and Γ_3 are generated by 5 generators and Γ_4 is generated by 3 generators as follows. The expression $e(a)=exp(2\pi ia)$ is used.

(Γ_2) ([3])

$$T_1=\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & e(-\gamma) & 0 & e(-\beta)-1\\ e(-\beta)-1 & 0 & e(-\gamma) & 0\\ 0 & 0 & 0 & 1\end{bmatrix} \quad T_2=\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & e(-\gamma') & e(-\beta')-1 & 0\\ 0 & 0 & 1 & 0\\ e(-\beta')-1 & 0 & 0 & e(-\gamma')\end{bmatrix}$$

$$T_3=\begin{bmatrix}1 & e(-\alpha)-1 & 0 & 0\\ 0 & e(\delta) & 0 & 0\\ 0 & 1-e(\delta+\beta'-\gamma') & 1 & 0\\ 0 & 1-e(\delta+\beta-\gamma) & 0 & 1\end{bmatrix} \quad T_4=\begin{bmatrix}1 & 0 & 1-e(-\alpha) & 0\\ 0 & e(\delta+\beta') & e(\delta+\beta')(1-e(-\beta')) & 0\\ 0 & 0 & e(\delta+\beta'-\gamma') & 0\\ 0 & 1-e(\gamma'-\alpha) & -e(\gamma'-\alpha)(1-e(-\beta')) & 1\end{bmatrix}$$

$$T_5=\begin{bmatrix}1 & 0 & 0 & 1-e(-\alpha)\\ 0 & e(\delta+\beta) & 0 & e(\delta+\beta)(1-e(-\beta))\\ 0 & 1-e(\gamma-\alpha) & 1 & -e(\gamma-\alpha)(1-e(-\beta))\\ 0 & 0 & 0 & e(\delta+\beta-\gamma)\end{bmatrix}$$

where $\delta=-\alpha-\beta-\beta'-\gamma+\gamma'$.

(Γ_3) ([3])

$$S_1=\begin{bmatrix}e(-\beta) & 0 & 0 & 0\\ 0 & e(-\alpha) & 0 & e(-\alpha)(1-e(-\beta))\\ e(-\alpha)(1-e(-\beta)) & 0 & e(-\alpha) & 0\\ 0 & 0 & 0 & e(-\beta)\end{bmatrix}$$

$$S_2=\begin{bmatrix}e(-\beta') & 0 & 0 & 0\\ 0 & e(-\alpha') & e(-\alpha')(1-e(-\beta')) & 0\\ 0 & 0 & e(-\beta') & 0\\ e(-\alpha')(1-e(-\beta')) & 0 & 0 & e(-\alpha')\end{bmatrix}$$

$$S_3=\begin{bmatrix}1 & e(\delta+\alpha+\alpha') & 0 & 0\\ 0 & e(\delta) & 0 & 0\\ 0 & 1-e(\delta+\alpha') & 1 & 0\\ 0 & 1-e(\delta+\alpha) & 0 & 1\end{bmatrix} \quad S_4=\begin{bmatrix}1 & 0 & 1-e(\delta+\alpha+\alpha') & 0\\ 0 & e(\delta+\beta') & e(\delta+\beta')(1-e(-\beta')) & 0\\ 0 & 0 & e(\delta+\alpha') & 0\\ 0 & 1-e(\delta+\alpha+\beta') & -e(\delta+\alpha+\beta')(1-e(-\beta')) & 1\end{bmatrix}$$

$$S_5=\begin{bmatrix}1 & 0 & 0 & 1-e(\delta+\alpha+\alpha') \\ 0 & e(\delta+\beta) & 0 & e(\delta+\beta)(1-e(-\beta)) \\ 0 & 1-e(\delta+\alpha'+\beta) & 1 & -e(\delta+\alpha'+\beta)(1-e(-\beta)) \\ 0 & 0 & 0 & e(\delta+\alpha)\end{bmatrix}$$

where $\delta=\alpha-\alpha'-\beta-\beta'+\gamma$.

(Γ_4) ([6])

$$U_1=\begin{bmatrix}1 & 0 & -(1-e(-\alpha)) & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -e(\gamma+\gamma'-\alpha-\beta) & 0 \\ 0 & 0 & 1 & 1\end{bmatrix} \quad U_2=\begin{bmatrix}1 & 1-e(-\alpha) & 0 & 0 \\ 0 & e(-\gamma) & 0 & 0 \\ a & b & e(-\gamma) & c \\ 0 & 0 & 0 & 1\end{bmatrix}$$

$$U_3=\begin{bmatrix}1 & 0 & 0 & 1-e(-\alpha) \\ 0 & 1 & 0 & 0 \\ a' & c' & e(-\gamma') & b \\ 0 & 0 & 0 & e(-\gamma')\end{bmatrix}$$

where

$$a=-(1-e(-\beta))(1-e(\gamma))e(\gamma')(1-e(\gamma+\gamma'))^{-1},$$
$$a'=-(1-e(-\beta))(1-e(\gamma'))e(\gamma)(1-e(\gamma+\gamma'))^{-1},$$
$$b=(1-e(-\alpha))(1-e(-\beta))e(\gamma+\gamma')(1-e(\gamma+\gamma')^{-1},$$
$$c=-(1+e(\gamma')-e(\gamma'-\alpha)-e(\gamma'-\beta)-e(\gamma+\gamma')+e(\gamma+2\gamma'-\alpha-\beta))(1-e(\gamma+\gamma'))^{-1},$$
$$c'=-(1+e(\gamma)-e(\gamma-\alpha)-e(\gamma-\beta)-e(\gamma+\gamma')+e(2\gamma+\gamma'-\alpha-\beta))(1-e(\gamma+\gamma'))^{-1}.$$

(Γ'_D) ([7])

Γ_D has n(n+3)/2 generators. Here, instead of listing these generators, we write generators of Γ'_D, the monodromy group of (F_{JP}) in Remark 2. Γ'_D is generated by following n+1 elements.

$$g_j=\begin{bmatrix}1 & & & & & & & \\ & \ddots & & & & & & \\ & & 1 & & & & & \\ -(1-\varepsilon_1)\cdots & & -(1-\varepsilon_{j-1}) & \varepsilon_0\varepsilon_j & -\varepsilon_0(1-\varepsilon_{j+1})\cdots & & -\varepsilon_0(1-\varepsilon_{n+1}) \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1\end{bmatrix}$$

where $\varepsilon_0=e(\gamma-\alpha)$, $\varepsilon_1=e(\beta_1+\dots+\beta_n-\gamma)$ and $\varepsilon_j=e(-\beta_{j-1})$, $2\leqq j\leqq n+1$.

REMARK 3. To compute generators explicitly, it is necessary to fix a fundamental set of solutions. In that occasion, we assumed the parameters of the system satisfy the conditions below:

(1) γ, γ', $\gamma-\alpha$, $\gamma'-\alpha$, $\gamma+\gamma'-\alpha$ are not integers for (F_2),
(2) $\alpha-\beta$, $\alpha'-\beta'$, $\gamma-\alpha-\alpha'$, $\gamma-\alpha'-\beta$, $\gamma-\alpha-\beta'$ are not integers for (F_3),
(3) γ, γ', $\gamma+\gamma'$ are not integers and $\gamma\neq\gamma'$ for (F_4),
(4) ε_i is not equal to 1, $0\leqq i\leqq n+1$ for (F_D).

§ 4. Case (F_2) and (F_3).

The systems (F_2) and (F_3) have similar natures (see § 9) and, since the proof for (F_3) is the same as that for (F_2), we deal with (F_2) in this section. Let us denote by G_2 the Lie algebra of P_2.

First, note that Γ_2'(see § 3) is generated by T_1, T_2 and T_4.

By direct calculations, under the assumption (1), we see

$$T_1^n=\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & e(-\gamma)^n & 0 & a_n\\ a_n & 0 & e(-\gamma)^n & 0\\ 0 & 0 & 0 & 1\end{bmatrix},$$

where $a_n=(e(-\beta)-1)(1-e(-\gamma)^n)/(1-e(-\gamma))$. If we assume

(5) $e(\gamma)$ is not of finite order,

the Zariski closure of $\{T_1^n;\ n\in \boldsymbol{Z}\}$ is an one-parameter group generated by

$$\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & t & 0 & a(t)\\ a(t) & 0 & t & 0\\ 0 & 0 & 0 & 1\end{bmatrix},$$

where $a(t)=(e(-\beta)-1)(1-t)/(1-e(-\gamma))$. Hence its infinitesimal generator is contained in G_2 and it is equal to

$$X=\begin{bmatrix}0 & 0 & 0 & 0\\ 0 & 1 & 0 & a\\ a & 0 & 1 & 0\\ 0 & 0 & 0 & 0\end{bmatrix},\qquad a=(1-e(-\beta))/(1-e(-\gamma)).$$

By the same method for T_3 we get another element in G_2:

$$Y=\begin{bmatrix}0 & b & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & c & 0 & 0\\ 0 & c' & 0 & 0\end{bmatrix}, \quad \begin{aligned} b &=(1-e(-\alpha))/(1-e(\delta)),\\ c &=(e(\delta+\beta'-\gamma')-1)/(1-e(\delta)),\\ c' &=(e(\delta+\beta-\gamma)-1)/(1-e(\delta)),\end{aligned}$$

assuming

(6) $e(\delta)$ is not of finite order.

The closure of $\{T_4^n;\ n\in \boldsymbol{Z}\}$ is a two-parameter subgroup of P_2, whose elements are

$$\begin{bmatrix}1 & 0 & d_2 & 0\\ 0 & t & d_3 & 0\\ 0 & 0 & s & 0\\ 0 & d_1 & d_4 & 1\end{bmatrix}$$

where $d_1=(1-e(\gamma'-\alpha))(1-t)/(1-e(\delta+\beta'))$, $d_2=(1-e(-\alpha))(1-s)/(1-e(\delta+\beta'-\gamma'))$, $d_3=(1-e(-\beta'))(t-s)/(1-e(-\gamma'))$ and $d_4=(1-e(\gamma'-\alpha))(1-e(-\beta'))(e(\delta+\beta')-t)/(1-e(-\beta'))(1-e(\delta+\beta'))+(1-e(-\beta'))\{(1-e(-\alpha))s-(e(\delta+\beta'-\gamma')-e(\delta+\beta'-\alpha)+e(\gamma'-\alpha)-e(-\alpha))\}/(1-e(-\gamma'))(1-e(\delta+\beta'-\gamma'))$.

Here we assumed

(7) $e(\delta+\beta'-\gamma')$ and $e(\delta+\beta')$ are not of finite order.

Then we obtain two infinitesimal generators:

$$Z=\begin{bmatrix}0 & 0 & 0 & 0\\ 0 & 1 & a' & 0\\ 0 & 0 & 0 & 0\\ 0 & d & da' & 0\end{bmatrix}, \quad \begin{aligned} a' &=(1-e(-\beta'))/(1-e(-\gamma')),\\ d &=(e(\gamma'-\alpha)-1)/(1-e(\delta+\beta')),\end{aligned}$$

$$U=\begin{bmatrix}0 & 0 & b & 0\\ 0 & 0 & -a'c & 0\\ 0 & 0 & c & 0\\ 0 & 0 & -a'b & 0\end{bmatrix}.$$

Now we show that the Lie algebra G generated by these four elements X, Y, Z and U is the Lie algebra of $GL(4, C)$. In the following E_{ij} means the matrix whose entries are zero except (ij)-entry which is equal to 1. Putting $V_1=[X, Z]+Z$, we have $[X, V_1]=-2a(1+ad)(a'E_{21}+E_{24})$. So, if

$$(8)\quad a(1+ad)\equiv a(e(-\beta)-e(-\gamma))(1-e(\delta+\beta+\beta'))/(1-e(-\gamma))(1-e(\delta+\beta'))\neq 0,$$

the matrix $V_2:=a'E_{21}+E_{24}$ belongs to G. Then we see $V_3:=E_{22}+a'E_{23}=V_1+a[Z,V_2]$. Hence V_3 is an element of G. Since $Z=V_3+dV_4$, where $V_4=E_{42}+a'E_{43}$, V_4 is also an element of G, if

$$(9)\quad d\neq 0.$$

Using the identy $[X, Z]\equiv -ad(a'E_{41}+E_{44})$ (mod. V_2, V_3, V_4), $V_5:=a'E_{41}+E_{44}$ belongs to G by (8) and (9). Putting $V_6:=[Y, V_2]-V_2-dV_5$ we have $[Y, V_6]\equiv-(a'b+c')(bE_{12}-E_{22}+cE_{23})$ (mod. Y). This means $V_7:=bE_{12}-E_{22}+cE_{23}\in G$ if

$$(10)\quad a'b+c'\equiv(e(-\gamma')-e(-\beta'))(1-e(\gamma'-\alpha))/(1-e(\delta))(1-e(-\gamma'))\neq 0.$$

Since $[V_5, V_7]=a'bE_{42}$, $E_{42}\in G$ if

$$(11)\quad a'b\neq 0.$$

Then, considering Y and V_7, we see $E_{22}\in G$ and $bE_{12}+cE_{23}=:V_8\in G$. Hence, by (11), E_{23}, E_{43} and $E_{12}\in G$. The identity $[E_{23}, X]=aE_{24}$ means $E_{24}\in G$.

Now $V_9:=aE_{31}+E_{33}\equiv X$ (mod. E_{22}, E_{24}). Making a bracket of V_9 with V_8, we see $(ab+c)E_{32}\in G$ which yields $E_{32}\in G$, if

$$(12)\quad ab+c\equiv(e(-\gamma)-e(-\beta))(1-e(\gamma-\alpha))/(1-e(\delta))(1-e(-\gamma))\neq 0.$$

Define $V_{10}:=bE_{13}+cE_{33}$ which is equal to U (mod. E_{23}, E_{43}). From V_9, V_{10} and E_{32} we obtain $a(ab+c)E_{31}$. Therefore $E_{31}\in G$. Summing up above calculations we see E_{ij} belongs to G except E_{14} and E_{34}. But G_2 is a Lie algebra containing C, it contains E_{14} and E_{34} also and finally we see that G_2 is the Lie algebra of $GL(4, C)$. This proves Theorem 1′ for the system (F_2) under the generality conditions (1) and (5)–(12).

§ 5. Case (F_4)

Calculations necessary to prove Theorem 1 for the system (F_4) are carried out analogously as in the previous section. Let G_4 be the Lie algebra of P_4 Assume (3) and the following

(13) $-e(\gamma+\gamma'-\alpha-\beta)$, $e(-\gamma)$ and $e(-\gamma')$ are not of finite order.

Considering the closure of the group generated by U_1 (resp. U_2, U_3) we get an element X(resp. Y, Z) in G_4:

$$X=\begin{bmatrix}0 & 0 & kd & 0\\ 0 & 0 & -k & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & -k & 0\end{bmatrix}\quad Y=\begin{bmatrix}0 & dl & 0 & 0\\ 0 & 1 & 0 & 0\\ al & bl & 1 & cl\\ 0 & 0 & 0 & 0\end{bmatrix}$$

$$Z=\begin{bmatrix}0 & 0 & 0 & dm\\ 0 & 0 & 0 & 0\\ a'm & c'm & 1 & bm\\ 0 & 0 & 0 & 1\end{bmatrix},$$

where $d=1-e(-\alpha)$, $k=1/(1+e(\gamma+\gamma'-\alpha-\beta))$, $l=-1/(1-e(-\gamma))$ and $m=-1/(1-e(-\gamma'))$.

Putting $V_1=[X, Y]-X$, $W_1=[X, Z]-X$, we define new elements $V_2=\{[X, V_1]-X-V_1\}/2k+fl\,X$ and $W_2=\{[X, W_1]-X-W_1\}/2k+f'mX$, where $f=ad-b-c=1-e(\gamma'-\alpha-\beta)$, $f'=a'd-b-c'=1-e(\gamma-\alpha-\beta)$. Then $V_2=-d(l+1)E_{13}+fl\,E_{33}+E_{43}$ and $W_2=-d(m+1)E_{13}+E_{23}+f'm\,E_{33}$. Therefore the element $\{X+kV_2+kW_2\}/k(l+m+1)$ which is equal to $-dE_{13}+(1-e(\gamma+\gamma'-\alpha-\beta))\,E_{33}$ belongs to G_4. Let us denote this element by V and set $e=1-e(\gamma+\gamma'-\alpha-\beta)$. Making brackets of Y with V twice we get an identity $[[Y, V], V]=e[Y, V]+2d(e-ald)\,E_{13}$. Now assume

(14) $de(b+e)\equiv de(1-e(\gamma+\gamma'-\alpha))(1-e(\gamma+\gamma'-\beta))/(1-e(\gamma+\gamma'))\neq 0.$

Then, since $-ald=b$, E_{13} belongs to G_4. Hence, by this assumption, E_{33}, E_{23} and E_{43} belong to G_4.

Calculations remained are easy. By making the bracket of Y with E_{i3} we have $S_i:=aE_{i1}+bE_{i2}+cE_{i4}$, $1\leqq i\leqq 4$, and $S_5:=dlE_{12}+E_{22}$. Also, from Z, we have $T_i:=a'E_{i1}+c'E_{i2}+bE_{i4}$, $1\leqq i\leqq 4$, and $T_5:=dmE_{14}+E_{44}$. Since $[S_1, T_5]=(adm+c)E_{14}$ and $[T_1, S_5]=(a'd+c')E_{12}$, if

(15) $adm+c\equiv(1-e(\gamma'-\alpha))(1-e(\gamma'-\beta))/(e(\gamma')-1\neq 0,$
$a'dl+c'\equiv(1-e(\gamma-\alpha))(1-e(\gamma-\beta))/(e(\gamma)-1)\neq 0,$

we conclude that E_{14} and E_{12} belong to G_4. Now, from S_5 and T_5, E_{22} and E_{44} also belong to G_4. Then, considering S_i and T_i, we can see finally that all E_{ij} belong to G_4. This proves the theorem under the generality conditions (3) and (13)-(15).

REMARK 4. In the case (F_4), the Zariski closure of Γ'_4, the group generated by X and Y, is *not* $GL(4, C)$, but it is a group of dimension 5.

§ 6. Case (F_D)

The method is the one employed in previous sections. Denote by G the Lie algebra of the Zariski closure of Γ'_D. Letting $a_i = \varepsilon_i - 1$, we see

$$g_i^k = \begin{bmatrix} 1 & & & & & & & \\ & \ddots & & & & & & \\ & & 1 & & & & & \\ a_1 b_k & \cdots & a_{i-1} b_k & (\varepsilon_0 \varepsilon_j)^k & \varepsilon_0 a_{i+1} b_k & \cdots & \varepsilon_0 a_{n+1} b_k \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{bmatrix},$$

Here, $b_k = (1 - (\varepsilon_0 \varepsilon_j)^k)/(1 - \varepsilon_0 \varepsilon_j)$. Assuming

(16) $\varepsilon_0 \varepsilon_j$, $1 \leqq j \leqq n+1$, are not of finite order,

we have elements X_i in G:

$$X_i = c_i \sum_{k=1}^{i-1} a_k E_{ik} + E_{ii} + c_i \sum_{k=i+1}^{n+1} \varepsilon_0 a_k E_{ik},$$

where $c_i = 1/(\varepsilon_0 \varepsilon_i - 1)$.

We prepare an easy lemma. For the matrix $Y = \sum d^k E_{ik}$, let Y^j be the matrix $\sum d^k E_{jk}$.

LEMMA 1. *Let* $Y = \sum d^k E_{ik}$, $Z = \sum e^k E_{ik}$ *and assume* $i \neq j$. *Then* $[Y, Z] = d^j Z^i - e^i Y^j$.

Using this lemma we have

(17) $[X_i, X_{i+1}] = \varepsilon_0 a_{i+1} c_i X_{i+1}^i - a_i c_{i+1} X_i^{i+1}$,

and we see

$$c_iX^i_{i+1}=c_{i+1}X_i+c_{i+1}(a_ic_i-1)E_{ii}+c_i(1-\varepsilon_0a_{i+1}c_{i+1})E_{ii+1},$$
$$c_{i+1}X^{i+1}_i=c_iX_{i+1}+c_{i+1}(1-a_ic_i)E_{i+1i}+c_i(\varepsilon_0a_{i+1}c_{i+1}-1)E_{i+1i+1}.$$

Putting $X_{ii+1}:=[X_i,X_{i+1}]-\varepsilon_0a_{i+1}c_{i+1}X_i+a_ic_iX_{i+1}$, we see

(18) $[X_i,X_{ii+1}]=2\varepsilon_0a_{i+1}c_iX^i_{i+1}(mod.\ X_i,\ X_{ii+1})$.

LEMMA 2. *Let us assume*

(19) $\varepsilon_0c_ia_k\neq 0,\ 1\leqq i,\ k\leqq n+1$.

Then X^i_k, $1\leqq i,\ k\leqq n+1$, *belongs to G.*

PROOF. For $k=i\pm 1$, X^i_{i+1} and X^{i+1}_i belong to G by (17) and (18). Lemma 1 shows that the bracket $[X_i,\ X^k_{i+1}]$ is the linear combination of X^i_{i+1} and X^k_i with non-zero coefficients by the assumption (19). Using this fact we can prove this lemma by the induction on $|k-i|$.

LEMMA 3. *Determinants of principal minors of the matrix* $({}^tX_1,\dots,{}^tX_{n+1})$ *are not zero for general values of* ε_0 *and* ε_j, $1\leqq j\leqq n+1$.

PROOF. Generalities mentioned are stated in the following. We prove this lemma for $det({}^tX_1,\dots,{}^tX_{n+1})$. It is seen that this determinant is equal to the determinant $p_{n+1}(b_1,\dots,b_{n+1},\varepsilon_0)$ of the matrix

$$\begin{bmatrix} 1 & \varepsilon_0b_2 & \varepsilon_0b_3 & \cdots\cdots & \varepsilon_0b_{n+1} \\ b_1 & 1 & \varepsilon_0b_3 & \cdots\cdots & \varepsilon_0b_{n+1} \\ b_1 & b_2 & 1 & & \\ & & & \ddots & \\ b_1 & b_2 & b_3 & \cdots\cdots & 1 \end{bmatrix},$$

where $b_i=a_ic_i=(\varepsilon_i-1)/(\varepsilon_0\varepsilon_i-1)$. If we denote by $p_n(b_2,\dots,b_{n+1},\varepsilon_0)$ the determinant of the minor $(2,\dots,n+1)$ of this matrix, we have

$$p_{n+1}(b_1,\dots,b_{n+1},\varepsilon_0)=p_n(b_2,\dots,b_{n+1},\varepsilon_0)+$$
$$b_1\{(1-\varepsilon_0b_2)\dots(1-\varepsilon_0b_{n+1})-p_n(b_2,\dots,b_{n+1},\varepsilon_0)\}.$$

This means that $p_{n+1}(b_1,\dots,b_{n+1},\varepsilon_0)$ cannot vanish when ε_1 takes a general value for fixed values of ε_0 and ε_j, $2\leqq j\leqq n+1$. Then, by induction, we see that $det({}^tX_1,\dots,{}^tX_{n+1})$ is not zero generally. For minors the proof is carried out in the same way.

Now we can prove Theorem $1'$ for (F_D). In fact, we can conclude by Lemma 2 and Lemma 3 that every E_{ij}, $1\leqq i, j\leqq n+1$, belongs to G, i. e. G is the Lie algebra of $GL(n+1, C)$. The generality conditions are (4), (16), (19) and the condition that all determinants of principal minors of the matrix $({}^tX_1, \ldots, {}^tX_{n+1})$ are not zero.

Part II Infiniteness of monodromy group

§ **7.** In this part we will prove Theorem 2. The proof is due to the simple fact that every finite group in the general linear group has a non-degenerate hermitian invariant form. In fact, we prove

PROPOSITION 1. *Γ_3 has generally no non-degenerate invariant form.*

Theorem 2 for (F_3) follows immediately from this proposition. The case (F_2) is proved making use of the case (F_3).

§ 8. **Case** (F_3)

For the sake of simplicity we set $d=1-e(-\beta-\beta'+\gamma)$. By the form of each generator we see easily

LEMMA 4. *Γ_3 is reducible if $d=0$.*

LEMMA 5. *If Γ_3 is finite, then parameters α, α', β, β' and γ are real and rational.*

Now let us look for a non-degenerate invariant form A under Γ_3, assuming that parameters are real. A is a 4×4 nonsingular hermitian matrix (a_{ij}) such that $SA^t\,\bar{S}=A$ for all elements S in Γ_3. Let S_i be generators of Γ_3 as listed in § 3. We denote by $(ij)_k$ the equation: (ij)-component of $S_kA^t\bar{S}_k-A=0$, $1\leqq i, j\leqq 4$, $1\leqq k\leqq 5$, which we do not write here, because it can be computed by the table in § 3.

Hereafter we assume $d\neq 0$. Equations $(ij)_3$, $1\leqq i, j\leqq 4$, can be solved as follows

$$(20)\quad \begin{aligned} &a_{22}=(1-e(\delta))a_{12}/d, && a_{23}=(e(\delta)-e(-\alpha'))a_{12}/d,\\ &a_{24}=(e(\delta)-e(-\alpha))a_{12}/d, && a_{21}=e(\beta+\beta'-\gamma)a_{12}.\end{aligned}$$

Since *det* $A\neq 0$, we have

$$(21)\quad a_{12}\neq 0.$$

By equations $(14)_5$ and $(12)_1$,

(22) $da_{44} = -(1-e(\delta+\alpha))\ a_{14}$,

(23) $(1-e(\beta))a_{14} = (e(\beta-\alpha)-1)\ a_{12}$.

Similarly by equations $(13)_4$ and $(12)_2$,

(24) $da_{33} = -(1-e(\delta+\alpha'))a_{13}$

(25) $(1-e(\beta'))a_{13} = (e(\beta'-\alpha')-1)a_{12}$.

LEMMA 6. $(1-e(\delta)((1-e(\delta+\alpha))=0$, (26)

$(1-e(\delta))(1-e(\delta+\alpha'))=0$. (27)

PROOF. The equation $(12)_5$ is

$$e(-\delta-\beta)\{a_{12}+da_{42}+(1-e(\beta))(a_{14}+da_{44})\} = a_{14}.$$

Substituting (20), (22), (23) in $(12)_5$, we have

$$(1-e(-\delta))(1-e(\delta+\alpha))\ a_{12}=0.$$

By (21) we have (26). Similarly by $(12)_4$, using (24) and (25), we get (27).

LEMMA 7. $e(\delta)=1$.

PROOF. Assume $e(\delta)\neq 1$. Then, by (26) and (27), $e(\delta+\alpha)=e(\delta+\alpha')=1$. Hence, by (20), (22) and (24), $a_{23}=a_{24}=a_{33}=a_{44}=0$. Substituting these values in $(13)_5$ and using (23), we have $a_{34}=a_{12}$. Also, by $(14)_4$, $a_{34}=e(\beta+\beta'-\gamma)a_{12}$. Since $a_{12}\neq 0$, we have $e(\beta+\beta'-\gamma)=1$. This means $d=0$, contradicting the assumption.

PROOF OF PROPOSITION 1. Obviously this follows from Lemma 7.

Now we can prove Theorem 2 for (F_3). More precisely we have

THEOREM 2-1. *Γ_3 is not finite if $d\neq 0$.*

PPOOF. Lemma 7 means that

$$S_3 = \begin{bmatrix} 1 & -d & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1-e(\delta+\alpha') & 1 & 0 \\ 0 & 1-e(\delta+\alpha) & 0 & 1 \end{bmatrix}.$$

If Γ_3 is finite, S_3 must be of finite order. But it is not of finite order if $d\neq 0$.

REMARK 5. Continuing above calculations, it is seen that Γ_3 has the following invariant form when $d(1-e(\beta))(1-e(\beta')\neq 0$ and $e(\delta)=1$:

$$\begin{bmatrix} ie(-\alpha-\alpha')kk'd & id & ie(-\alpha')k'd & ie(-\alpha)kd \\ -id & 0 & i(1-e(-\alpha')) & i(1-e(-\alpha)) \\ -ik'd & -i(1-e(\alpha')) & i(1-e(-\alpha'))k' & ie(-\alpha)d \\ -ikd & -i(1-e(\alpha)) & ie(-\alpha')) & i(1-e(-\alpha))k \end{bmatrix}$$

where $i=\sqrt{-1}$, $k=(e(\alpha-\beta)-1)/(1-e(-\beta))$, $k'=(e(\alpha'-\beta')-1)/(1-e(-\beta'))$. The determinant of this matrix is $(1-e(-\alpha-\alpha'))(1-e(\alpha'))(1-e(\alpha))(1-e(-\beta-\beta'))/(1-e(-\beta))(1-e(-\beta'))$ which is generally non-zero.

§ 9. Case (F_2)

First we note that any solution of (F_3) can be represented by solutions of (F_2). Precisely we have

PROPOSITION 2. ([1]) *Every solution of* (F_3) *with parameters* $(\underline{\alpha}, \underline{\alpha}', \underline{\beta}, \underline{\beta}', \underline{\gamma})$ *and variables* (x, y) *is written as a linear combination of solutions of* (F_2) *with parameters* $(\alpha, \beta, \beta', \gamma, \gamma')$ *and variables* $(1/x, 1/y)$, *whose coefficients are in* $\mathbf{C}[x^{-\underline{\alpha}}, x^{-\underline{\beta}}, y^{-\underline{\alpha}'}, y^{-\underline{\beta}'}]$. $(\alpha, \beta, \beta', \gamma, \gamma')$ *are given by eqations*

(28) $\alpha=\underline{\beta}+\underline{\beta}'-\underline{\gamma}+1,\ \beta=\underline{\beta},\ \beta'=\underline{\beta}',\ \gamma=\underline{\beta}-\underline{\alpha}+1,\ \gamma'=\underline{\beta}'-\underline{\alpha}'+1.$

By the same way as in § 8, we see

LEMMA 8. *Γ_2 is reducible if* $e(\alpha)\neq 1$.

LEMMA 9. *If Γ_2 is finite, parameters* α, β, β', γ *or* γ' *are real and rational.*

With these lemmas Theorem 2 for (F_2) follows from Proposition 2. We will prove

THEOREM 2-2. *Γ_2 is not finite if* $e(\alpha)\neq 1$.

PROOF. Assume Γ_2 is finite. Then all solutions of (F_2) are algebraic, because (F_2) is of Fuchsian type. Hence, by Proposition 2 and Lemma 6, every solution of (F_3) is also algebraic, i. e. Γ_3 is finite. Then, by Theorem 2-1, $e(\underline{\beta}+\underline{\beta}'-\underline{\gamma})=1$. This is equivalent to $e(\alpha)=1$ by (28), contradicting the assumption.

REMARK 6. Proposition 2 menas that (F_2) and (F_3) are the same in a function theoretic aspect. But, as differential equations, they are different. Compare Proposition 1 with the next

PROPOSITION 3. *Γ_2 has generally a non-degenrate invariant form, if parameters are real. Moreover if $e(\beta)$, $e(\beta')$ and $e(\delta)$ are not equal to 1, the following matrix is the invariant form*

$$\begin{bmatrix} pp'q & q & -p'q & -pq \\ e(\alpha+\delta)q & 1 & -e(\gamma'-\beta')r & -e(\gamma-\beta)r' \\ -e(\gamma-\beta)p'q & -r & p'r & e(\gamma-\beta)q \\ -e(\gamma'-\beta')pq & -r' & e(\gamma'-\beta')q & pr' \end{bmatrix}$$

where $p=(1-e(\gamma))/(1-e(\beta))$, $p'=(1-e(\gamma'))/(1-e(\beta'))$, $q=(1-e(-\alpha))/(1-e(\delta))$, $r=(1-e(-\alpha-\beta+\gamma))/(1-e(\delta))$ and $r'=(1-e(-\alpha-\beta'+\gamma'))$. The determinant of this matrix is equal to $\sin\pi\alpha\, \sin\pi(\alpha-\gamma-\gamma')\, \sin\pi(\alpha-\gamma)\, \sin\pi(\alpha-\gamma')\, \{\sin\pi(\beta-\gamma)\, \sin\pi(\beta'-\gamma')/\sin\pi\beta\, \sin\pi\beta'\, \sin\pi\gamma\, \sin\pi\gamma'\}^2$.

§ 10. Reducible Case

In the above two sections, we proved the monodromy group Γ_2 and Γ_3 are not finite if it is irreducible; note Lemmas 4 and 8. Thus they are finite only if reducible. But, doing careful calculations, we have

PROPOSITION 4. *Γ_2 and Γ_3 are not finite under the conditions* (1) *or* (2) *in* § 3.

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