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ON AN ESSENTIALLY COMPLETE CLASS OF THE ESTIMATORS UNDER CERTAIN RESTRICTIONS ON PARAMETERS

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Summary

There are some results dealing with statistical problem in the presence of the restrictions on parameters. However, it seems that there are no papers which discuss the complete class in the statistical estimation problem under some restrictions by inequalities so long as I know. Our aim is to show that the class of estimators satisfying the same inequalities as that of parameters is essentially complete.

1. Introduction

Suppose that $\boldsymbol{X}$ is the sample space of the observed random variables. $\boldsymbol{A}$ is a σ-field of subsets of $\boldsymbol{X}$ and $\{P_\theta;\ \theta \in \Theta\}$ is a family of probability measures on $\boldsymbol{A}$. Here it is assumed that Θ is an open and convex subset of a p-dimensional Euclidean space and $\boldsymbol{C}$ is the Borel σ-field on Θ. Let $p(x;\theta)$ designate the probability density function of P_θ with respect to a σ-finete measure μ on $\boldsymbol{A}$, for which it is assumed to be $\boldsymbol{C}$-measurable for each $x \in \boldsymbol{X}$.

As we are interested in the estimation of a parameter θ, we may suppose that the action space $\boldsymbol{Y}$ is identical to the parameter space Θ.

Moreover, assume that the loss function is $L(\theta, y)=(y-\theta)' M(y-\theta)$ where $(y-\theta)'=(y_1-\theta_1, \cdots, y_p-\theta_p)$ and M is a known $p\times p$ real symmetric positive-definite matrix. In the sequel, we consider the estimation problem in which there is the restricted condition on parameters so that θ belongs to the set

(1) $$F=\{\theta \in \Theta\colon f_i(\theta)\leqq c_i,\ i=1, 2,\cdots,k\}$$

where k is a given positive integer, $f_i{}'s$ are convex and continuous real-valued functions of θ and $c_i{}'s$ are constants.

Since the loss function $L(\theta, y)$ is convex in y for each θ, we may only consider the class D of non-randomized decision functions, i. e., $D=\{\delta;\ \delta: \boldsymbol{X}\rightarrow\boldsymbol{Y}$, measurable$\}$ (see Blackwell-Girshick, p. 294). The risk function of $\delta\in D$ is defined by

$$(2)\qquad r(\theta,\ \delta)=\int_X L(\theta,\ \delta(x))\ p(x;\ \theta)\ \mu(dx).$$

For a given $\delta\in D$, $r(\theta,\ \delta)$ is considered a function on Θ, which is C-measurable.

Let Ξ be the set of prior probability measures on $(\Theta,\ C)$. Then the prior risk of δ with respect to $\xi\in\Xi$ is defined by

$$(3)\qquad r(\xi,\ \delta)=\int_\Theta r(\theta,\ \delta)\ \xi(d\theta).$$

We donote the class of Bayes decision function by B.

Let $\Xi_0=\{\xi\in\Xi;\ r(\xi,\ \delta)<\infty$ *for some* $\delta\in D\}$ and $B_0=\{\delta\in D;\ \delta$ *is Bayes for some* $\xi\in\Xi_0\}$. A sequence $\{\delta_n\}$ of randomized decision functions is said to *converge regularly* to a randomized decision function δ if for every μ-integrable function $f(x)$ and every bounded continuous $g(y)$

$$\lim_{n\to\infty}\int_X\int_Y f(x)g(y)\ \delta_n(dy\mid x)\ \mu(dx)$$
$$=\int_X\int_Y f(x)\ g(y)\ \delta(dy\mid x)\ \mu(dx).$$

From LeCam (1955) the closure of B in the topology of regular convergence is an essentially complete class. If $\delta_n\in D$ converges regularly to $\delta\in D$, we have that $\delta_n(x)$ converges a. e. to $\delta(x)$. Accordingly, the closure $\bar{B}$ of B in the topology of convegence a. e. is an essentially complete class.

2. Complete class theorems

In order to achieve our aim, one theorem and some lemmas are prepared.

THEOREM 1. $\bar{B}_0$ is essentially complete.

PROOF. Let B_1 be the part of $\bar{B}$ which excludes those δ's with infinite risk. Then B_1 is essentially complete because $\bar{B}$ is. Let $\delta\in B_1$, i. e., $\delta\in\bar{B}$ and $r(\theta,\ \delta)<\infty$. Then there is a sequence $\{\delta_n\}$ of B such that δ_n converges regularly to δ and $r(\theta,\ \delta_n)<\infty$ for all n. In fact, for the regular convergence of $\{\delta_n\}$ to δ, we have that $\lim_{n\to\infty} r(\theta,\ \delta_n)=r(\theta,\ \delta)<\infty$ for every θ (see Farrell). Hence there is a $\xi_n\in\Xi$ such that δ_n is Bayes for ξ_n and $r(\xi_n,\ \delta_n)<\infty$; that is, $\xi_n\in\Xi_0$. This shows that there exists a sequence $\{\delta_n\}$ of B_0 such that δ_n converges regularly to δ; that is, $\delta \in B_0$. Hence $B_1 \subset \bar{B}_0$. This is the desirable result because B_1 is essentially complete.

We denote by D_F the class of all estimators δ which satisfy the inequalities $f_i(\delta) \leqq c_i$, $i=1, 2, \cdots, k$.

LEMMA 1. $B_0 \subset D_F$.

PROOF. Due to Girshick and Savage, $\delta \in B_0$ is given by

$$\delta(x) = \int_F \theta \xi(d\theta \mid x) \ a.\ e. \tag{4}$$

for some $\xi \in \Xi_0$.

From Jensen's inequality, we have

$$\begin{aligned} f_i(\delta(x)) &= f_i(\int_F \theta \xi(d\theta \mid x)) \\ &\leqq \int_F f_i(\theta)\ \xi(d\theta \mid x) \\ &\leqq c_i \end{aligned} \tag{5}$$

for $i=1, 2 \cdots, k$. This shows *Lemma* 1.

LEMMA 2. $\bar{D}_F = D_F$.

PROOF. Let $\delta(x)$ be any element of $\bar{D}_F$. Then there exists a sequence $\{\delta_n(x)\}$ in D_F such that $\delta_n(x)$ converges a. e. to $\delta(x)$. Since $f_i(\delta_n(x)) \leqq c_i$ for all n and f_i is continuous for $i=1, 2, \cdots, k$, it is clearly implied that $f_i(\delta(x)) \leqq c_i$ for $i=1, 2, \cdots, k$.

The following theorem is our aim.

THEOREM 2. D_F is an essentially complete class.

PROOF. From *Lemma* 1, $B_0 \subset D_F$. Hence $\bar{B}_0 \subset \bar{D}_F$. Hence, from *Lemma* 2, $\bar{B}_0 \subset D_F$. This shows *Theorem* 2.

3. Examples

EXAMPLE 1. Let $X_1, X_2 \cdots X_n$, be independent and identically distributed random variables with an unknown mean θ. When it is known previously that $a \leqq \theta \leqq b$ ($a = -\infty$ or $b = +\infty$ may be permitted), we consider the estimation problem of θ. Since the restriction $a \leqq \theta \leqq b$ can be written as follows: Let $f_1(\theta) = -\theta$ and $f_2(\theta)=\theta$, and let $c_1 = -a$ and $c_2 = b$. Then $F=[a, b]=\{\theta;\ f_1(\theta) \leqq c_1$ and $f_2(\theta) \leqq c_2\}$.

Since both f_1 and f_2 are convex and continuous in $\theta \in [a, b]$ we can apply *Theorom* 2 to this problem.

We can directly have from

$$\hat{\theta}_\xi(x) = \frac{\int_a^b \theta p(x;\ \theta)\ \xi(d\theta)}{\int_a^b p(x;\ \theta)\ \xi(d\theta)} \text{ and } a \leqq \theta \leqq b$$

that $a \leqq \hat{\theta}_\xi(x) \leqq b$.

EXAMPLE 2. Let X_1 and X_2 be independent random variables with means θ_1 and θ_2. Under the restriction $\theta_1^2 + \theta_2^2 \leqq a^2$ (a is a positive constant), we considər the estimation of θ_1 and θ_2. Such a restriction can be written as follows.

$$\begin{aligned} F &= \{\theta=(\theta_1,\ \theta_2):\ \theta_1^2+\theta_2^2 \leqq a^2\} \\ &= \{\theta;\ f_1\ (\theta_1,\ \theta_2) = \theta_2 - \sqrt{a^2-\theta_1^2} \leqq 0 \\ &\qquad \text{and } f_2(\theta_1,\ \theta_2) = -\theta_2 - \sqrt{a^2-\theta_1^2} \leqq 0\} \end{aligned}$$

We can easily check the conditions which f_1 and f_2 must satisfy. Indeed,

It is directly shown that *Theorem* 2 can be applied to this problem.

$$\hat{\theta}_1 = \frac{\int_F \theta_1\ p(x;\ \theta)\ \xi(d\theta)}{\int_F p(x;\ \theta)\ \xi(d\theta)} \text{ and } \hat{\theta}_2 = \frac{\int_F \theta_2\ p(x;\ \theta)\ \xi(d\theta)}{\int_F p(x;\ \theta)\xi(d\theta)}$$

where $x=(x_1,\ x_2)$.

From Schwarz's inequality, we have

$$\begin{aligned} \hat{\theta}_1^2 + \hat{\theta}_2^2 &= \frac{(\int_F \theta_1\ p(x;\ \theta)\ \xi(d\theta))^2 + (\int_F \theta_2\ p(x;\ \theta)\ \xi(d\theta))^2}{(\int_F p(x;\ \theta)\ \xi(d\theta))^2} \\ &\leqq \frac{\int_F \theta_1^2\ p(x;\ \theta)\xi(d\theta) + \int_F \theta_2^2\ p(x;\ \theta)\ \xi(d\theta)}{\int_F p(x;\ \theta)\ \xi(d\theta)} \\ &= \frac{\int_F (\theta_1^2 + \theta_2^2)\ p(x;\ \theta)\ \xi(d\theta)}{\int_F p(x;\ \theta)\ \xi(d\theta)} \\ &\leqq a^2. \end{aligned}$$

EXAMPLE 3. Let X_1 and X_2 be for as *Example* 2. We suppose that there is an order restriction $a \leqq \theta_1 \leqq \theta_2 \leqq b$ ($a=-\infty$ or $b=+\infty$ may be permitted). This restriction can be expressed using functions $f_1(\theta_1, \theta_2)=-\theta_1$, $f_2(\theta_1\ \theta_2)=\theta_1-\theta_2$ and $f_3(\theta_1, \theta_2)=\theta_2$. That is,

$$F=\{\theta=(\theta_1, \theta_2);\ a \leqq \theta_1 \leqq \theta_2 \leqq b\}$$
$$=\{\theta;\ f_1(\theta) \leqq -a,\ f_2(\theta) \leqq 0 \text{ and } f_3(\theta) \leqq b\}.$$

It is easily seen that f_1, f_2 and f_3 satisfy the conditions.

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