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EFFECT OF TWO TIME DELAYS ON PARTIALLY FUNCTIONAL DIFFERENTIAL EQUATIONS

Kiyoshi YOSHIDA and Kazuo KISHIMOTO

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In [8] we treated Hutchinson's equation with diffusion

(1) $$\dot{U}=d\Delta U+a(1-U(t-1)/K)U \quad \text{in } (0, \infty)\times\Omega$$

subject to zero Neumann boundary condition, where $\cdot=\partial/\partial t$ and Ω is a bounded domain in $\boldsymbol{R}^N$ with smooth boundary $\partial\Omega$, and considered the Hopf bifurcation and its stability in order to obtain a stable spatially inhomogeneous temporally periodic orbit from ecological and mathematical view points. But in [8] we could obtain only a spatially homogeneous temporally periodic orbit for (1) with zero Neumann boundary condition as the primary bifurcation when a varies in a neighborhood of $\pi/2$ as a bifurcation parameter. We did not there consider the effect of the magnitude of the diffusion constant d and so fixed d. Recently, Y. Morita [5] proved that this spatially homogeneous themporally periodic orbit becomes unstable when we make the diffusion constant d small after the Hopf bifurcation occured for a certain fixed a. This implies that an orbit of another type can appear near this orbit as the secondary bifurcation, which was suggested by J. Lin and P. B. Kahn [4].

The purpose of this note is to give an example that a stable spatially inhomogeneous temporally periodic orbit occurs as the primary bifurcation by the cause of two time delays. As is stated above, such a phenomenon does not occur for equations with only one time delay, while equations with multiple time delays seem to have various interesting aspects. Thus we tried here to study Hutchinson's equation with diffusion and two time delays r_1, r_2 $(0<r_1<r_2)$

(2) $$\dot{U}=d\Delta U+(c-aU(t-r_1)-bU(t-r_2))U.$$

Equations with two time delays are studied by J. K. Hale [2], R. D. Nussbaum [6], J. Ruiz-Claeyssen [7] and so on. They, however, treat ordinary functional differential equations and therefore they do not consider that an effect of two time delays have influence on space variables. As another model which exhibits an appearance of a stable spatially inhomogeous temporally periodic orbit as the primary bifurcation, the authors know only a system of competing reaction-diffusion equations of four species:

$$\dot{u}_i = d_i(u_i)_{xx} + \sum_{j=1}^{4}(c_i - a_{ij}u_j)u_i, \quad i=1, 2, 3, 4$$

with zero Neumann boundary condition given by K. Kishimoto, M. Mimura and K. Yoshida [3]. Thus the example given in this note is not only interesting as a phenomenon but also gives a direction in which we study equations with multiple time delays in future

1. Preliminaries

For the later purpose let us recall the discussion on the Hopf bifurcation theory developed in [8], which holds for (2) with zero Neumann boundary condition. Throughout this note we assume $c=a+b$ without loss of the generalty. Since the equation (2) with zero Neumann boundary condition arises in ecology, we are interesting in only the existence of the unique global non-negative solution for this equation, and this is proved by the same method as in [8, Proposition 1.1]. In order to investigate the Hopf bifurcation from the positive steady state $U=c/(a+b)(=1)$, we change the unknown function by $U=1+u$. Then our considering equation results in

$$(3) \qquad \begin{cases} \dot{u} = d\Delta u - (au(t-r_1) + bu(t-r_2))(1+u) \\ \left.\dfrac{\partial u}{\partial n}\right|_{\partial\Omega} = 0, \end{cases}$$

where $\partial/\partial n$ is the outer normal derivative on $\partial\Omega$. Let $W^{2,p}(\Omega)$, $1<p<\infty$, be the Sobolev space of real valued L^p functions whose derivatives of order up to 2 belong to $L^p(\Omega)$ and $W^{2,p}_N(\Omega)=\{u \in W^{2,p}(\Omega)\colon \partial u/\partial n=0 \text{ on } \partial\Omega\}$. As in [8] let $C=C([-r_2, 0]\colon L^p(\Omega))$ and $C_1=C([-r_2, 0]\colon W^{2,p}_N(\Omega))$. Throughout this note we assume $N<p<\infty$ (N being space dimension), which means that $W^{2,p}(\Omega)$ forms algebra (cf. [8]). As usual we write $u_t(\theta)=u(t+\theta)$ for $\theta \in [-r_2, 0]$. Let $T(t)$ be the solution map for

$$\begin{cases} \dot{u}=d\Delta u-au(t-r_1)-bu(t-r_2) & \text{in } (0,\infty)\times\Omega, \\ \left.\dfrac{\partial u}{\partial n}\right|_{\partial\Omega}=0, \\ u_0(\theta)=\phi(\theta)\in C, & -r_2\leqq\theta\leqq 0, \end{cases}$$

from C to itself which is defined by

$$T(t)\phi=u_t(\cdot).$$

Here we note that the solution is understood in the sense of "mild solution" (cf. [8. p. 324]). Then $\{T(t)\}_{t\geq 0}$ forms a strongly continuous semigroup on C and $T(t)$ is compact for each $t>r_2$. We also note that $T(t)$ is uniquely extended to a bounded operator from the space of piecewise continuous functions on $[-r_2, 0]$ with values in $L^p(\Omega)$ to C. Using $T(t)$ we can rewrite (3) as

$$u_t=T(t)u_0+\int_0^t T(t-s)X_0F(u_s)ds, \tag{4}$$

where

$$F(u_s)=-(au_s(-r_1)+bu_s(-r_2))u_s(0)$$

and $X_0=X_0(\theta,x)$ is such that $X_0=0$ for $-r_2\leqq\theta<0$ and $X_0=1$ for $\theta=0$. Let B be the infinitesimal generator of $T(t)$ and $D(B)$ the domain of B. Then,

$$D(B)=\{\phi\in C\colon \dot{\phi}\in C,\ \phi(0)\in W_N^{2,p}(\Omega),\ \dot{\phi}(0)=d\Delta\phi(0)-a\phi(-r_1)-b\phi(-r_2)\}$$
$$B\phi=\dot{\phi} \qquad \text{for } \phi\in D(B),$$

and the sectrum of B is composed of the roots of equations

$$\lambda+ae^{-r_1\lambda}+be^{-r_2\lambda}+d\xi_n=0, \qquad n=0,1,2,\ldots, \tag{5$_n$}$$

where $\{\xi_n\}$, $0=\xi_0<\xi_1\leqq\xi_2\leqq\cdots\to\infty$, is the set of eigenvalues for $-\Delta$ with zero Neumann boundary condition. We call these equations $(5)_n$ the characteristic equations.

2. The Hopf bifurcation theorem

Putting $C^*=C([0,r_2]\colon L^q(\Omega))$, $1/p+1/q=1$, we define a bilinear form $(\!(\cdot, \cdot)\!)$ by

$$(\!(\psi, \phi)\!) = <\psi(0), \phi(0)> - a\int_{-r_1}^{0} <\psi(\xi + r_1), \phi(\xi)> d\xi$$
$$- b\int_{-r_2}^{0} <\psi(\xi + r_2), \phi(\xi)> d\xi$$

for $\psi \in C^*$, $\phi \in C$, where $< \cdot, \cdot >$ is the duality between L^p and L^q. Let $B(\alpha)$ denote the infinitesimal generater B if it is necessary to specify α when the constants (a, b, d, r_1, r_2) varies along a certain line $(a(\alpha), b(\alpha), d(\alpha), r_1(\alpha), r_2(\alpha))$, $-\alpha_0 < \alpha < \alpha_0$. In the sequel we assume the following

HYPOTHESIS A. *Some characteristic equation* $(5)_l$ *has a pair of simple complex conjugate roots* $\{\lambda(\alpha), \overline{\lambda(\alpha)}\}$, $\lambda(\alpha) = \mu(\alpha) + i\nu(\alpha)$, *such that*

$$\mu(0) = 0 \quad \text{and} \quad \nu(0) = \nu_0 > 0, \tag{6}$$

$$\mu'(0) \neq 0, \tag{7}$$

and the other roots of $(5)_n$, $n = 0, 1, \cdots, l, \cdots$, *have negative real parts for* $\alpha \in (-\alpha_0, \alpha_0)$.

It is to be noticed that in Hypothesis A we implicitly assume that ξ_l is the simple eigenvalue. As in [8, Section 3] we can prove the followings:

if we put, with the eigenfunction $h(x)$ of ξ_l,

$$\Phi_\alpha = (\phi_{1,\alpha}, \phi_{2,\alpha})$$

$$\phi_{1,\alpha}(\theta) = h(x)e^{\mu(\alpha)}\cos\nu(\alpha)\theta, \quad \phi_{2,\alpha}(\theta) = h(x)e^{\mu(\alpha)}\sin\nu(\alpha)\theta, \quad -r_2 \leqq \theta \leqq 0,$$

$$\Psi_\alpha^* = \text{col}\,(\psi_{1,\alpha}^*, \psi_{2,\alpha}^*)$$

$$\psi_{1,\alpha}^*(s) = h(x)e^{\mu(\alpha)}\cos\nu(\alpha)s, \quad \psi_{2,\alpha}^*(s) = h(x)e^{\mu(\alpha)}\sin\nu(\alpha)s, \quad 0 \leqq s \leqq r_2,$$

$$(\!(\Psi_\alpha^*, \Phi_\alpha)\!) = \begin{bmatrix} (\!(\psi_{1,\alpha}^*, \phi_{1,\alpha})\!) & (\!(\psi_{1,\alpha}^*, \phi_{2,\alpha})\!) \\ (\!(\psi_{2,\alpha}^*, \phi_{1,\alpha})\!) & (\!(\psi_{2,\alpha}^*, \phi_{2,\alpha})\!) \end{bmatrix}$$

then the 2×2 matrix $(\!(\Psi_\alpha^*, \Phi_\alpha)\!)$ is nonsingular. if $\Psi_\alpha = (\!(\Psi_\alpha^*, \Phi_\alpha)\!)^{-1}\Psi_\alpha^*$, then $(\!(\Psi_\alpha, \Phi\ )\!) = I$ and C is decomposed as

$$C = P_\alpha \oplus Q_\alpha,$$

$$P_\alpha=\{\phi\in C:\phi=\Phi_\alpha \boldsymbol{a},\ \boldsymbol{a}\in \boldsymbol{R}^2\},$$

$$Q_\alpha=\{\phi\in C:\ (\!(\Psi_\alpha,\ \phi)\!)=0\}:$$

by making use of this decomposition, (4) is decomposed as

$$u_t^{P\alpha}=T_{P\alpha}(t)u_0^{P\alpha}+\int_0^t T_{P\alpha}(t-s)X_0^{P\alpha}F(u_s)ds, \tag{8}$$

$$u_t^{Q\alpha}=T_{Q\alpha}(t)u_0^{Q\alpha}+\int_0^t T_Q(t-s)X_0^{Q\alpha}F(u_s)\ ds, \tag{9}$$

where

$$u_t^{P\alpha}=\Phi_\alpha(\!(\Psi_\alpha,\ u_t)\!),\ u_t^{Q\alpha}=u_t-u_t^{P\alpha},$$

$$X_0^{P\alpha}F(u_s)=\Phi_\alpha<\Psi_\alpha(0),\ F(u_s)>,\ X_0^{Q\alpha}F(u_s)=X_0F(u_s)-X_0^{P\alpha}F(u_s),$$

and $T_{P\alpha}(t)$ (resp. $T_{Q\alpha}(t)$) is the restriction of $T(t)$ to P_α (resp. Q_α).

Applying Ψ_α to the both sides of (8) and then differentiating them, we have

$$\dot{x}_\alpha(t)=M_\alpha x_\alpha(t)+X(x_\alpha(t),\ u_t^{Q\alpha},\ \alpha), \tag{10}$$

where

$$x_\alpha(t)=(\!(\Psi_\alpha,\ u_t)\!),$$

$$M_\alpha=\begin{bmatrix}\mu(\alpha) & \nu(\alpha)\\ -\nu(\alpha) & \mu(\alpha)\end{bmatrix}$$

is the matrix representation for the restriction of B to P_α
and

$$X(x_\alpha(t),\ u_t^{Q\alpha},\ \alpha)=<\Psi_\alpha(0),\ F(\Phi_\alpha x_\alpha(t)+u_t^{Q\alpha})>.$$

Then we have

THEOREM 1 ([8, Theorem 4.1]). *Let k be an arbitrary fixed positive integer. Then there exist a zero neighborhood $\mathscr{U}\times I$ in $\boldsymbol{R}^2\times(-\alpha_0,\ \alpha_0)$ and a k times continuously differential function G on $\mathscr{U}\times I$ with values in $\tilde{Q}_\alpha=Q_\alpha\cap C_1$ satisfying the following conditons*:

i) $$G(0, \alpha)=0, \ D_x G(0, 0)=0.$$

ii) *For any* $\alpha \in I$ *let*

$$\mathscr{M}(\alpha)=\{\phi \in C_1 : \ \phi=\Phi_\alpha x+G(x, \alpha), \ x \in \mathscr{U}\}.$$

Then $\mathscr{M}(\alpha)$ *is locally invariant in the sence that if, for any* $\phi \in \mathscr{M}(\alpha)$ *such that* $(\!(\Psi_\alpha, \phi)\!) \in \mathscr{U}$, *the solution* $x(t)$ *of*

$$\dot{x}(t)=M_\alpha x(t)+X(x(t), \ G(x(t), \alpha), \alpha)$$

with

$$x(0)=(\!(\Psi^\alpha, \phi)\!)$$

stays in $\mathscr{U}$, *then*

$$u(t)=u_t(0)=\Phi_\alpha(0)x(t)+G(x(t), \alpha)|_{\theta=0}$$

is he unique so ution of (3) *with* $u_0=\phi$.

$\mathscr{M}(\alpha)$ *is locally attractive, i. e., if the solution* $u(t)$ *of* (3) *satisfies* $\|u_t\|_{C_1} \leqq \varepsilon$ *for some small* $\varepsilon>0$, *then there exist positive constants* K, γ *independent of* t *which satisfy*

$$\|u_t-\Phi_\alpha x(t), \ \alpha)\|_{C_1} \leqq K e^{-\gamma t}\|u_0\|_{C_1}$$

where

$$x(t)=(\!(\Psi_\alpha, \ u_t)\!).$$

By vertue of Theorem 1 we have the following Hopf bifurcation theorem.

THEOREM 2. *There exist a positive constant* ε_0, *real valued continuously differentiable functions* $\alpha(\varepsilon)$, $\omega(\varepsilon)$ *on* $(-\varepsilon_0, \ \varepsilon_0)$ *such that* $\alpha(0)=0$, $\omega(0)=2\pi/\nu_0$, *and an* $\omega(\varepsilon)$*-periodic solution* $x_{\alpha(\varepsilon)}(t)$ *of the equation*

(11) $$\dot{x}=M_{\alpha(\varepsilon)}x+X(x, \ G(x, \ \alpha(\varepsilon)), \ \alpha(\varepsilon))$$

such that $|x_{\alpha(\varepsilon)}(t)|=o(\varepsilon)$. *Thus*

$$u(t)=\Phi_{\alpha(\varepsilon)}(0)x_{\alpha(\varepsilon)}(t)+G(x_{\alpha(\varepsilon)}(t),\ \alpha(\varepsilon))|_{\theta=0}$$

is the l-th mode $\omega(\varepsilon)$*-periodic solution of* (3).

3. Stability of bifurcation orbits

In order to know whether the Hopf bifurcation is realized, one must investigate its stability. In this section we describe briefly S.-N. Chow and J. Mallet-Paret's theory that the stabily constant "K" ensures the Hopf bifurcation stable.

Let us consider, in general, an infinite evolution equation

$$\dot{z}=f(z,\ \alpha)=A(\alpha)z+F(z,\ \alpha) \qquad (12)$$
$$F(z,\ \alpha)=o(|z|^2)$$

in a certain Banach space X, where $A(\alpha)$ is an unbounded closed operator from X into itself with domain $Y\subset X$, Y being a Banach space continuously and densely defined in X. Assume that

$$F:\ Y\times(-\alpha_0,\ \alpha_0)\rightarrow X$$

is sufficiently smooth and further the spectrum of $A(\alpha)$ is composed of only the point spectrum with same property as in HYPOTHESIS A. Let

$$X=P_\alpha\oplus Q_\alpha$$

be the spectral decomposition, where P_α is two dimensional eigenspace of $A(\alpha)$ corresponding to $\{\lambda(\alpha),\ \overline{\lambda(\alpha)}\}$. By using this decomposition, (12) is rewritten as

$$\begin{cases}\dot{x}=A_P(\alpha)x+F_P(x,\ y,\ \alpha)\\ \dot{y}=A_Q(\alpha)y+F_Q(x,\ y,\ \alpha)\end{cases} \qquad (13)$$

where $z=x+y\in P_\alpha\oplus Q_\alpha$ and $A_P(\alpha)$ (resp. $A_Q(\alpha)$) and $F_P(x,\ y,\ \alpha)$ (resp. $F_Q(x,\ y,\ \alpha)$) are restrictions of $A(\alpha)$ and $F(z,\ \alpha)$ to P_α (resp. Q_α). Let us denote the matrix representation of $A_P(\alpha)$ by

$$\begin{bmatrix}\mu(\alpha) & \nu(\alpha)\\ -\nu(\alpha) & \mu(\alpha)\end{bmatrix}.$$

Expanding (13) in the Taylor series we have

$$\dot{x}_1 = \mu(\alpha)x_1 + \nu(\alpha)x_2 + \sum_{j=2}^{\infty} B_j^1(\boldsymbol{x}, \boldsymbol{y}, \alpha)$$

$$\dot{x}_2 = -\nu(\alpha)x_1 + \mu(\alpha)x_2 + \sum_{j=2}^{\infty} B_j^2(\boldsymbol{x}, \boldsymbol{y}, \alpha)$$

$$\begin{aligned}\dot{\boldsymbol{y}} &= A_Q(\alpha)\boldsymbol{y} + \sum_{j=2}^{\infty} B_Q^j(\boldsymbol{x}, \boldsymbol{y}, \alpha)\\ &= A_Q(\alpha)\boldsymbol{y} + J(\alpha)\boldsymbol{x}^2 + N(\alpha)\boldsymbol{xy} + E(\alpha)\boldsymbol{y}^2 + \Gamma_3(\boldsymbol{x}, \alpha)\boldsymbol{y}^3 + \cdots,\end{aligned}$$

which, in polar coordinates $\boldsymbol{x} = (r\cos\zeta, r\sin\zeta)$, become

$$(14)\qquad \begin{cases} \dot{r} = F_1(\zeta, \alpha)\boldsymbol{y}^2 + r\{\mu(\alpha) + G_2(\zeta, \boldsymbol{y}, \alpha)\boldsymbol{y}\} + r^2C_3(\zeta, \boldsymbol{y}, \alpha) \\ \qquad\qquad + r^3C_4(\zeta, \boldsymbol{y}, \alpha) + \cdots \\ \dot{\zeta} = -\nu(\alpha) + rD_3(\zeta, \boldsymbol{y}, \alpha) + r^2D_4(\zeta, \boldsymbol{y}, \alpha) + \cdots \\ \dot{\boldsymbol{y}} = \text{as above but with } \boldsymbol{x} = (r\cos\zeta, r\sin\zeta). \end{cases}$$

Here we used the notations in [1] as possible as we can.
Scaling (14) by

$$r \longrightarrow \varepsilon r, \quad \boldsymbol{y} \longrightarrow \varepsilon \boldsymbol{y}, \quad \alpha \longrightarrow \varepsilon\alpha,$$

we have

$$\begin{aligned}\dot{r} &= \varepsilon\{\alpha\mu'(0)r + r^2C_3(\zeta, \varepsilon\boldsymbol{y}, \varepsilon\alpha) + F_1(\zeta, \varepsilon\alpha)\boldsymbol{y}^2 + rG_2(\zeta, \varepsilon\boldsymbol{y}, \varepsilon\alpha)\boldsymbol{y}\}\\ &\qquad + \varepsilon^2 r^3 C_4(\zeta, \varepsilon\boldsymbol{y}, \varepsilon\alpha) + O(\varepsilon^3) + O(\varepsilon^2|\alpha|),\\ \dot{\zeta} &= -\nu_0 + \varepsilon\{\alpha\nu'(0) + rD_3(\zeta, \varepsilon\boldsymbol{y}, \varepsilon\alpha)\} + O(\varepsilon^2),\\ \dot{\boldsymbol{y}} &= A_Q\boldsymbol{y} + \varepsilon\{J(\varepsilon\alpha)\boldsymbol{x}^2 + N(\varepsilon\alpha)\boldsymbol{xy} + E(\varepsilon\alpha)\boldsymbol{y}^2\} + O(\varepsilon^2).\end{aligned}$$

Let us now define the stability constant K by

$$K = K^* + K^{**},$$

$$K^* = \frac{1}{2\pi}\int_0^{2\pi}\{C_4(\zeta, 0, 0) + \frac{1}{\nu_0}C_3(\zeta, 0, 0)\}d\zeta,$$

$$K^{**} = \frac{1}{2\pi}\int_0^{2\pi} w^*(\zeta)J(0)(\cos\zeta, \sin\zeta)^2 d\zeta,$$

where $w^*(\zeta)$ is the unique 2π-periodic solution of

(15) $$\nu_0\dot{w}^*(\zeta)-w^*(\zeta)A_Q(0)=G_2(\zeta,\ 0,\ 0).$$

We recall that for each $\alpha\in(-\alpha_0,\ \alpha_0)$, $J(\alpha)$ is a bilinear form in the $\boldsymbol{x}$-space $\boldsymbol{R}^2$, taking values in the $\boldsymbol{y}$-space; in the above definition $J(0)$ acts on the point $(\cos\zeta,\ \sin\zeta)\in\boldsymbol{R}^2$. Since $G_2(\zeta,\ 0,\ 0)$ arises as a coefficient of $\boldsymbol{y}$ in the differential equation involving $\dot{r}$, $G_2(\zeta,\ 0,\ 0)$ for each ζ is a linear functional on $\boldsymbol{y}$. Also note that the property $K\neq0$ depends on the differential equation at $\alpha=0$.

THEOREM (S.-N.Chow and J. Mallet-Paret [1]). *Suppose that there exists a center manifold taking value in* $Q_\alpha\cap Y$. *If* $\mu'(0)K<0$, *then the Hopf bifurcation is stable.*

4. An example.

In this section, as an application of Theorems 1, 2 and S-.N. Chow and J. Mallet-Paret's theorem, we give an example that a stable spatially inhomogeneous temporally periodic orbit occurs as the primary bifurcation. In what follows we treat the equation (2)(or (3)) in the case of one space dimension and so put $\Omega=(0,\ \pi)$. Then $\{\cos nx : n=0,\ 1,\ 2,\ \cdots\}$ is the set of eigenfunction for $-(d^2/dx^2)$ with zero Neumann boundary condition and n^2 is the eigenvalue for $\cos nx$. Thus $(5)_n$ become

$(16)_n$ $$\lambda+ae^{-r_1\lambda}+be^{-r_2\lambda}+dn^2=0,\ n=0,\ 1,\ \cdots.$$

Let us now determine the constants $(a,\ b,\ d,\ r_1,\ r_2)$ parametrized by α. Put

$$r_1=1,\quad r_2=3.05,$$

$$a=a_0+\alpha,\ a_0=2\pi/\{3(\sin\frac{2}{3}\pi+\frac{4}{11}\sin\frac{2}{3}\pi r_2)\},$$

$$b=\frac{4}{11}a_0,$$

$$d=-a_0\cos\frac{2}{3}\pi-\frac{4}{11}a_0\cos\frac{2}{3}\pi r_2>0.$$

Then these constants satisfy HYPOTHESIS A with $l=1$. In fact, if we write $\lambda=\mu+i\nu$, the equations $(16)_n$ are equivalent to the systems of eqations to find $(\mu,\ \nu)$ in $\boldsymbol{R}^2$:

$$(17)_n \qquad \mu + ae^{-r_1\mu}\cos r_1\nu + be^{-r_2\mu}\cos r_2\nu + dn^2 = 0,$$

$$(18) \qquad \nu - ae^{-r_1\mu}\sin r_1\nu - be^{-r_2\mu}\sin r_2\nu = 0.$$

It is easy to verify that $\left(0, \frac{2}{3}\pi\right)$ and $\left(0, -\frac{2}{3}\pi\right)$ are solutions of $(17)_1$ and (18). Next we must prove that the other solutions of $(17)_n$, $n=0, 1, \cdots$, and (18) are not located in the half plane $\mu \geqq 0$. Since

$$a_0 \sim 2.316717113$$

$$b \sim 0.842442587$$

$$d \sim 0.320530953,$$

we see that $(17)_n$ has no roots ν for any $\mu \geqq 0$ if $n \geqq 4$. Hence we have only to consider the cases $0 \leqq n \leqq 3$. But by the argument principle it is easily shown that each equation $(16)_n$ has only roots with negative real part. For example consider

$$(16)_0 \qquad \lambda + a_0 e^{-r_1\lambda} + be^{-r_2\lambda} = 0$$

and apply the argument principle along the semicircle:

$$\{(0, \nu):\ -R \leqq \nu \leqq -R\} \cup \{(\mu, \nu):\ \mu^2 + \nu^2 = R^2,\ \mu \geqq 0\},$$

where R is large enough. Then we see that the winding number is zero, which means that our assertion holds. Thus HYPOTHESIS A holds except the property (7). Since the root $\lambda(\alpha) = \mu(\alpha) + i\nu(a)$ satisfies

$$\mu(0) = 0 \ \text{ and } \ \nu(0) = \nu_0 \left(= \frac{3}{2}\pi\right),$$

it follows from $(16)_1$ that

$$\lambda'(0) + e^{-r_1\lambda(0)} - a_0 r_1 \lambda'(0) e^{-r_1\lambda(0)} - br_2\lambda'(0)e^{-r_2\lambda(0)} = 0,$$

which leads us to

$$(19) \qquad \mu'(0)(1 - a_0 r_1 \cos r_1\nu_0 - br_2\cos r_2\nu_0) = \nu'(0)(a_0 r_1 \sin r_1\nu_0 + br_2 \sin r_2\nu_0) - \cos r_1\nu_0,$$

(20) $$\nu'(0)(1-a_0r_1\cos r_1\nu_0-br_2\cos r_2\nu_0)$$
$$=\sin r_1\nu_0-\mu'(0)(a_0r_1\sin r_1\nu_0+br_2\sin r_2\nu_0).$$

From (19) and (20) we obtain

(21) $$\mu'(0)=A_1/B_1,$$

where

(22) $$A_1=\sin r_1\nu_0(a_0r_1\sin r_1\nu_0+br_2\sin r_2\nu_0)$$
$$-\cos\nu_0(1-a_0r_1\cos r_1\nu_0-br_2\cos r_2\nu_0),$$

(23) $$B_1=(a_0r_1\sin r_0\nu_0+br_2\sin r_2\nu_0)^2$$
$$+(1-a_0r_1\cos r_1\nu_0-br_2\cos r_2\nu_0)^2\ (\neq 0).$$

From (21), (22) and (23) we have, numerically,

$$A_1\sim 1.771627691$$
$$B_1\sim 5.332866583,$$

which lead us to

$$\mu'(0)\sim 0.332209266.$$

Consequently, there exists a positive constant α_0 such that HYPOTHESIS A holds. Therefore, by vertue of Theorem 2 we have the first mode $\omega(\varepsilon)$-periodic solution of (3)

$$u(t,\ x)=e^{\mu_{\alpha(\varepsilon)}}x_{1,\ \alpha(\varepsilon)}(t)h(x)+O(\varepsilon^2),$$

where

$$h(x)=\left(\frac{\pi}{2}\right)^{-\frac{1}{2}}\cos x$$

and $x_{1,\alpha(\varepsilon)}(t)$ is the first component of the $\omega(\varepsilon)$-periodic solution $\boldsymbol{x}_{\alpha(\varepsilon)}(t)$ of (11).

Let us now proceed to the proof of the stability of this periodic orbit following Chow and Mallet-Paret's theorem. To do so we first determine $\Psi_0(\xi)$. Since

$$\Phi_0(\theta)=(\phi_{1,0}(\theta),\ \phi_{2,0}(\theta)),$$

$$\phi_{1,0}(\theta)=h(x)\cos\nu_0\theta,$$

$$\phi_{2,0}(\theta)=h(x)\sin\nu_0\theta,$$

$$h(x)=\text{as above},$$

it follows from $(\!(\Psi_0,\ \Phi_0)\!)=I$ that

$$\Psi_0(\xi)=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} \cos\nu_0\xi \\ \sin\nu_0\xi \end{bmatrix} h(x),$$

where

(24) $$a_{11}=a_{22}=A_2/B_1,$$

(25) $$a_{12}=-a_{21}=A_3/B_1,$$

(26) $$A_2=-2(a_0r_1\cos\nu_0r_1+br_2\cos\nu_0r_2-1)$$

(27) $$A_3=-2(a_0r_1\sin\nu_0r_1+br_2\sin\nu_0r_2),$$

and B_1 is equal to (23). Next, in order to evaluate the stability constant K, we must determine $C_3(\zeta, 0, 0)$, $C_4(\zeta, 0, 0)$, $D_3(\zeta, 0, 0)$, $G_2(\zeta, 0, 0)$ and $J(0)(\cos\zeta, \sin\zeta)^2$. But, these are, by easy calculation, obtained as

(28) $$\begin{aligned} C_3(\zeta, 0, 0)=&a_{11}\{-a_0\cos\nu_0r_1\ \cos^3\zeta+a_0\sin\nu_0r_1\ \sin\zeta\ \cos^2\zeta \\ &-b\cos\nu_0r_2\ \cos^3\zeta+b\sin\nu_0r_2\ \sin\zeta\ \cos^2\zeta\}\int_0^\pi h^3(x)dx \\ &+a_{21}\{-a_0\cos\nu_0r_1\ \cos^2\zeta\ \sin\zeta+a_0\sin\nu_0r_1\ \sin^2\zeta\ \cos\zeta \\ &-b\cos\nu_0r_2\ \cos^2\zeta\ \sin\zeta+b\sin\nu_0r_2\ \sin^2\zeta\ \cos\zeta\}\int_0^\pi h^3(x)dx, \end{aligned}$$

(29) $$C_4(\zeta, 0, 0)=0,$$

(30) $$\begin{aligned} D_3(\zeta, 0, 0)=&a_{11}\{a_0\cos\nu_0r_1\ \sin\zeta\ \cos^2\zeta-a_0\sin\nu_0r_1\ \sin^2\zeta\ \cos\zeta \\ &+b\cos\nu_0r_2\ \sin\zeta\ \cos^2\zeta-b\sin\nu_0r_2\ \sin^2\zeta\ \cos\zeta\}\int_0^\pi h^3(x)dx \\ &+a_{21}\{a_0\sin\nu_0r_1\ \sin\zeta\ \cos^2\zeta-a_0\cos\nu_0r_1\ \cos^3\zeta \\ &+b\sin\nu_0r_2\ \sin\zeta\ \cos^2\zeta-b\cos\nu_0r_2\ \cos^3\zeta\}\int_0^\pi h^3(x)dx, \end{aligned}$$

(31) $$G_2(\zeta, 0, 0)=-a_{11}\langle a_0(\cos\nu_0r_1\ \cos^2\zeta-\sin\nu_0r_1\ \sin\zeta\ \cos\zeta)\phi(0)$$

$$+a_0\cos^2\zeta\phi(-r_1)+b(\cos\nu_0 r_2\cos^2\zeta-\sin\nu_0 r_2\sin\zeta\cos\zeta)\phi(0)$$

$$+b\cos^2\zeta\phi(-r_2),\ h^2>-a_{21}<a_0(\cos\nu_0 r_1\cos\zeta\sin\zeta-\sin\nu_0 r_1\sin^2\zeta)\phi(0)$$

$$+a_0\sin\zeta\cos\zeta\phi(-r_1)+b(\cos\nu_0 r_2\cos\zeta\sin\zeta-\sin\nu_0 r_2\sin^2\zeta)\phi(0)$$

$$+b\sin\zeta\cos\zeta\phi(-r_2),h^2>,\ \text{for } \phi\in\tilde{Q}$$

and

$$J(0)(\cos\zeta,\ \sin\zeta)^2=X_0^Q\{(a_0\sin\nu_0 r_1\sin\zeta\cos\zeta-a_0\cos\nu_0 r_1\cos^2\zeta+b\sin\nu_0 r_2\sin\zeta\cos\zeta-b\cos\nu_0 r_2\cos^2\zeta)h^2\}$$
$$=(a_0\sin\nu_0 r_1\sin\zeta\cos\zeta-a_0\cos\nu_0 r_1\cos^2\zeta+b\sin\nu_0 r_2\sin\zeta\cos\zeta-b\cos\nu_0 r_2\cos^2\zeta)X_0 h^2 \tag{32}$$

Since

$$\int_0^\pi h^3(x)dx=\left(\frac{\pi}{2}\right)^{-\frac{3}{2}}\int_0^\pi\cos^3 x dx=0,$$

it follows that

$$C_3(\zeta,\ 0,\ 0)=0 \text{ and } D_3(\zeta,\ 0,\ 0)=0,$$

which together with (30) lead us to

$$K^*=\frac{1}{2\pi}\int_0^{2\pi}\{C_4(\zeta,\ 0,\ 0)+\frac{1}{\nu_0}C_3(\zeta,\ 0,\ 0)\ D_3(\zeta,\ 0,\ 0)\}\ d\zeta=0.$$

In order to solve (15), write $G_2(\zeta,\ 0,\ 0)$ as a Foutrier series

$$G_2(\zeta,\ 0,\ 0)=\sum_{n=-\infty}^{\infty}g_n e^{in\zeta},\quad g_n\in\tilde{Q}^*,$$

where $\tilde{Q}^*$ is the dual space of $\tilde{Q}$, and next by expanding $w^*(\zeta)$ as a Fourier series

$$w^*(\zeta)=\sum_{n=-\infty}^{\infty}x_n e^{in\zeta},$$

inserting this into (15) and equating coefficients, we obtain

$$w_n=g_n(in\nu_0-B_Q)^{-1}.$$

In what follows, justifying this formal calculas, we apply this to find $w^*(\zeta)$. Since

$$\cos\zeta = (e^{i\zeta}+e^{-i\zeta})/2,\quad \sin\zeta=(e^{i\zeta}-e^{-i\zeta})/2i,$$

it follows from (31) that

$$G_2(\zeta,\ 0,\ 0)\phi = \sum_{n=-\infty}^{\infty} g_n(\phi)e^{in\zeta},$$

$$g_0(\phi) = -\frac{a_{11}}{2} < a_0\cos\nu_0 r_1\phi(0) + a_0\phi(-r_1) + b\cos\nu_0 r_2\phi(0) + b\phi(-r_2),\ h^2 >$$

$$+\frac{a_{21}}{2} < a\sin\nu_0 r_1\phi(0) + b\sin\nu_0 r_2\phi(0),\ h^2 >,$$

$$g_1(\phi) = \overline{g_{-1}(\phi)} = 0,$$

$$g_2(\phi) = \overline{g_{-2}(\phi)}$$

$$= -\frac{a_{11}}{4} < a_0(\cos\nu_0 r_1 + i\sin\nu_0 r_1)\phi(0) + a_0\phi(-r_1)$$

$$+ b(\cos\nu_0 r_2 + i\sin\nu_0 r_2)\phi(0) + b\phi(-r_2),\ h^2 >$$

$$-\frac{a_{21}}{4} < a_0(\sin\nu_0 r_1 - i\cos\nu_0 r_1)\phi(0) - ia_0\phi(-r_1)$$

$$+ b(\sin\nu_0 r_2 - i\cos\nu_0 r_2)\phi(0) - ib\phi(-r_2),\ h^2 >,$$

$$g_n(\phi) = \overline{g_{-n}(\phi)} = 0,\qquad n=3,\ 4,\ \cdots.$$

On the othere hand, since

$$a_0\cos\nu_0 r_1 + b\cos\nu_0 r_2 + d = 0,$$

$$\nu_0 - a_0\sin\nu_0 r_1 - b\sin\nu_0 r_2 = 0,$$

we obtain

$$(33)\qquad g_0(\phi) = -\frac{a_{11}}{2} < a_0\phi(-r_1) + b\phi(-r_2) - d\phi(0),\ h^2 > + \frac{a_{21}}{2} < \nu_0\phi(0),\ h^2 >,$$

$$(34)\qquad g_2(\phi) = -\frac{1}{4}(a_{11} - ia_{21}) < a_0\phi(-r_1) + b\phi(-r_2) - d\phi(0) + i\nu_0\phi(0),\ h^2 >.$$

There (33) and (34) together with (32) yield

$$w^*(\zeta)J(0)(\cos\zeta,\ \sin\zeta)^2 = B_2(\zeta)\{g_0(B_Q^{-1}(X_0h^2))$$

$$-g_2((2\nu_0 i-B_Q)^{-1}(X^0h))e^{2\zeta i}+g_{-2}((2\nu_0 i+B_Q)^{-1}(X_0h^2))e^{-2\zeta i}\},$$

where

$$B_2(\zeta)=a_0\cos\nu_0 r_1\cos^2\zeta-a_0\sin\nu_0 r_1\sin\zeta\cos\zeta$$
$$+b\cos\nu_0 r_2\cos^2\zeta-b\sin\nu_0 r_2\sin\zeta\cos\zeta.$$

Consequently, we have

$$\int_0^{2\pi} w^*(\zeta)J(0)(\cos\zeta,\ \sin\zeta)^2\,d\zeta=-d\pi g_0(B_Q^{-1}(X_0h^2))$$
$$-(\pi/2)g_2((2\nu_0 i-B_Q)^{-1}(X_0h^2))\{a_0(\cos\nu_0 r_1-i\sin\nu_0 r_1)$$
$$+b(\cos\nu_0 r_1-i\sin\nu_0 r_2)\}$$
$$+(\pi/2)g_{-2}((2\nu_0 i+B_Q)^{-1}(X_0h^2))\{a_0(\cos\nu_0 r_1+i\sin\nu_0 r_1)$$
$$+b(\cos\nu_0 r_2+i\sin\nu_0 r_2)\}.$$

Since $g_{-2}=\overline{g_2}$, it follows that

(35) $$K^{**}=\frac{1}{2\pi}\int_0^{2\pi} w^*(\zeta)J(0)(\cos\zeta,\ \sin\zeta)^2 d\zeta$$
$$=(d/2)g_0(-B_Q^{-1}(X_0h^2))+(d/2)\mathrm{Re}\ g_2((2\nu_0 i-B_Q)^{-1}(X_0h^2))$$
$$-(\nu_0/2)\mathrm{Im}\ g_2((2\nu_0 i-B_Q)^{-1}(X_0h^2)).$$

Let us determine $-B_Q^{-1}(X_0h^2)$ and $(2\nu_0 i-B_Q)^{-1}(X_0h^2)$. To do so calculate, more generally,

(36) $$\phi=(n\nu_0 i-B_Q)^{-1}\psi$$
$$=(n\nu_0 i-B)^{-1}\psi \qquad \text{for given } \psi\in\tilde{Q}_0.$$

If we remember the representation of B (cf. p. 93), then the problem (36) to find ϕ is reduced to the equation

(37) $$\dot{\phi}(\theta)=n\nu_0 i\phi(\theta)-\psi(\theta)$$

subject to the boundary conditions

(38) $$\dot{\phi}(0)=d\phi_{xx}(0)-a_0\phi(-r_1)-b\phi(-r_2),$$

(39) $$\phi_x(0, 0)=\phi_x(0, \pi)=0.$$

From (37) we have

$$\phi(\theta)=e^{n\nu_0 i\theta}\phi(0)-\int_0^\theta e^{n\nu_0 i(\theta-s)}\psi(s)ds,$$

which leads us to

(40) $$\phi(-r_1)=e^{-n\nu_0 r_1 i}\phi(0)-\int_0^{-r_1} e^{-n\nu_0 i(r_1+s)}\psi(s)ds,$$

(41) $$\phi(-r_2)=e^{-n\nu_0 r_2 i}\phi(0)-\int_0^{-r_2} e^{-n\nu_0 i(r_2+s)}\psi(s)ds.$$

On the other hand since, form (37) and (38),

$$n\nu_0 i\phi(0)-\psi(0)=d\phi_{xx}(0)-a_0\phi(-r_1)-b\phi(-r_2),$$

it follows from (40) and (41) that

$$-d\phi_{xx}(0)+n\nu_0 i\phi(0)+a_0\, e^{-n\nu_0 r_1 i}\phi(0)+be^{-n\nu_0 r_2 i}\phi(0)$$
$$=\psi(0)+a_0\int_0^{-r_1} e^{-n\nu_0 i(r_1+s)}\psi(s)ds+b\int_0^{-r_2} e^{-n\nu_0 i(r_2+s)}\,\psi(s)ds.$$

For $\psi=X_0h^2$ we then have

(42) $$-d\phi_{xx}(0)+n\nu_0 i\phi(0)+a_0 e^{-n\nu_0 r_1 i}\phi(0)+be^{-n\nu_0 r_2 i}\phi(0)=h^2.$$

First, consider the case $n=0$. Since $h^2=(\pi/2)^{-1}\cos^2 x$, the equation (42) results in

$$-d\phi_{xx}(0)+(a_0+b)\phi(0)=(1/\pi)(1+\cos^2 x),$$

which together with (39) admits the solution

$$\phi(0)=\frac{1}{\pi}\left\{\frac{1}{c_0}+\frac{1}{4d+c_0}\cos 2x\right\},$$

where $c_0=a_0+b$. Since

$$\phi(-r_1)=\phi(0) \quad\text{and}\quad \phi(-r_2)=\phi(0),$$

it follows from (33) that

(43) $$g_0(-B_Q^{-1}(X_0h^2)) = \frac{1}{2\pi}\left\{\frac{1}{c_0}+\frac{1}{2(c_0+4d)}\right\}\{a_{21}\nu_0-a_{11}(c_0-d)\}.$$

We proceed to the case $n=2$. Then, the equation (41) becomes

$$-d\phi_{xx}(0)+(2\nu_0 i+a_0e^{-2\nu_0 r_1 i}+be^{-2\nu_0 r_2 i})\phi(0)=(1/\pi)(1+\cos 2x),$$

which together with (39) admits the solution

$$\phi(0)=(1/\pi)(A_4+B_4\cos 2x),$$

where

$$A_4=(2\nu_0 i+a_0e^{-2\nu_0 r_1 i}+be^{-\nu_0 2 r_2 i})^{-1},$$

$$B_4=(4d+2\nu_0 i+a_0e^{-2\nu_0 r_1 i}+be^{-2\nu_0 r_2 i})^{-1}.$$

Since

$$\phi(-r_1)=e^{-2\nu_0 r_1 i}\phi(0) \quad\text{and}\quad \phi(-r_2)=e^{-2\nu_0 r_2 i}\phi(0),$$

it follows from (34) that

$$g_2((2\nu_0 i-B_Q)^{-1}(X_0h^2))=-(1/4\pi)\{A_4+(1/2)B_4\}\{a_0e^{-2\nu_0 r_1 i} + be^{-2\nu_0 r_2 i}-d+i\nu_0\}\{a_{11}-ia_{21}\}.$$

Put

(44) $$D=a_0\cos 2\nu_0 r_1+b\cos 2\nu_0 r_2,$$

(45) $$E=2\nu_0-a_0\sin 2\nu_0 r_1-b\sin 2\nu_0 r_2,$$

(46) $$F=D-d,$$

(47) $$G=E-\nu_0,$$

(48) $$H=D+4d.$$

Then, it follows that

(49) $$\begin{aligned}&\mathrm{Re}\ g_2((2\nu_0 i-B_Q)^{-1}(X_0h^2))\\ &=-(a_{11}/8\pi)\{2(D^2+E^2)^{-1}(DF+EG)+(H^2+E^2)^{-1}(FH+EG)\}\\ &+(a_{21}/8\pi)\{2(D^2+E^2)^{-1}(EF-DF)+(H^2+E^2)^{-1}(EF+GF)\},\end{aligned}$$

$$(50)\qquad \operatorname{Im} g_2((2\nu_0 i - B_Q)^{-1}(X_0 h^2))$$
$$= -(a_{11}/8\pi)\{2(D^2+E^2)^{-1}(DG-EF)+(H^2+E^2)^{-1}(GH-EF)\}$$
$$+(a_{21}/8\pi)\{2(D^2+E^2)^{-1}(DF+EG)+(H^2+E^2)^{-1}(FH+EG)\}.$$

Consequently it follows from (35), (43), (49) and (50) that

$$K^{**}=(d/4\pi)\left\{\frac{1}{c_0}+\frac{1}{2(c_0+4d)}\right\}\{a_{21}\nu_0-a_{11}(c_0-d)\}$$
$$-\frac{a_{11}d}{16\pi}\{2(D^2+E^2)^{-1}(DF+EG)+(H^2+E^2)^{-1}(FH+EG)\}$$
$$+\frac{a_{21}d}{16\pi}\{2(D^2+E^2)^{-1}(EF-DG)+(H^2+E^2)^{-1}(EF-GH)\}$$
$$+\frac{a_{11}\nu_0}{16\pi}\{2(D^2+E^2)^{-1}(DG-EF)(H^2+E^2)^{-1}(GH-EF)\}$$
$$-\frac{a_{21}\nu_0}{16\pi}\{2(D^2+E^2)^{-1}(DF+EG)+(H^2+E^2)^{-1}(FH+EG)\}.$$

This K^{**} is numerically evaluated as

$$K^{**}\sim -0.05130788.$$

Since

$$K=K^{*}+K^{**},$$

$$K^{*}=0,$$

$$\mu'(0)>0,$$

it follows from Chow and Mallet-Paret's theorem that the solution obtained in this section

$$u(t,\ x)=e^{\mu_{\alpha(\varepsilon)}}x_{1,\alpha(\varepsilon)}(t)h(x)+0(\varepsilon^2)$$

is stable for small ε.

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Department of Mathematics
Faculty of Science
Kumamoto University
and
Faculty of Engineering
Hiroshima University