Kumamoto J. Sci. (Math.)
Vol. 15, 7~10 March (1982)

A NOTE ON THE CLASSICAL SOLUTIONS OF SEMI-DISCRETE QUASI-STATIC PLASTICITY PROBLEMS

Tetsuhiko MIYOSHI

(Received Oct. 24, 1981)

One of the basic assumptions for solving the plasticity problems in engineering is that each point of the material chooses a definite state in the deformation process and that the elastic and plastic states continue for a while once they have been chosen by the point. Mathematically, this is nothing but to assume the existence of a classical solution. In [1] we discussed this assumption for dynamic finite element problems. The present paper deals with a quasi-static case. We consider a finite element problem to illustrate our basic idea of the proof, but the method and the results are valid to other various semi-discrete problems. Also, as is already shown in [1] and [2], our approach is immediately applicable to analyze both the fully discrete and fully continuous problems.

We consider so called a plane stress problem with "kinematic" hardening rule, but our method is valid to other kind of problems. Let Ω be a region obtained by a triangulation of the original region. We shall call the triangles in Ω the elements. Let $\{\varphi_p\}$ be the usual piecewise linear finite element basis defined on Ω. The approximate displacements of the material at time t is sought in the following form.

$$u_i(t) = \sum_{p\in P} u_i^p(t)\varphi_p \qquad (i=1, 2),$$

where P is the set of all nodes of Ω excepting those at which zero displacements are given. We employ the Prandtl-Reuss equation and the Ziegler's rule. Then the unknowns $\{u_i^p(t)\}$ are determined by solving the following system of equations.

$$(1) \qquad \sum_j (\sigma_{ij}, \varphi_{p,j})_{L^2(\Omega)} = (b_i, \varphi_p)_{L^2(\Omega)} \qquad p\in P,$$

$$(2_a) \qquad \dot\sigma = D\dot\varepsilon,\ \dot\alpha=0 \qquad \text{if } f(\sigma-\alpha)<\bar\sigma, \text{ or } f(\sigma-\alpha)=\bar\sigma \text{ and } \partial f^*\dot\sigma<0,$$

$$(2_b) \qquad \dot\sigma=(D-D')\dot\varepsilon,\ \dot\alpha=(\sigma-\alpha)\frac{\partial f^*\dot\sigma}{f(\sigma-\alpha)} \quad \text{if } f(\sigma-\alpha)=\bar\sigma \text{ and } \partial f^*\dot\sigma\geqslant 0.$$

In the above, σ, ε and α are used to denote the stresses, strains and parameters representing the center of the yield surface, and

$$f^2(\sigma)=\sigma_{11}^2+\sigma_{22}^2-\sigma_{11}\sigma_{22}+3\sigma_{12}^2$$

$$\varepsilon=(u_{1,1},\ u_{2,2},\ u_{1,2}+u_{2,1})$$

$$D'=\frac{D\partial f\partial f^* D}{\eta+\partial f^* D\partial f} \qquad (\eta\text{: positive constant, } \partial f=\frac{\partial f(\sigma-\alpha)}{\partial\sigma}).$$

For the details of this formulation, we refer to [3] and [4]. As regard b_i we assume that it is piecewise smooth and that, for any t in the time interval I on which the problem is considered, there exists an interval $I_t=[t,\ t+\delta]$ $(\delta>0)$ such that $b_i(t)$ is equal to an analytic function on I_t. Now the integration of (1)~(2) under the initial condition $(u,\ \sigma,\ \alpha)=(0,\ 0,\ 0)$ proceeds as follows. For any element the elastic stress-strain relation (2_a) is employed as far as $f(\sigma)<\bar{\sigma}$ is satisfied. Suppose that $f(\sigma)=\bar{\sigma}$ holds at $t=t_0$ for the elements $e_1,\ e_2,\ \ldots,\ e_m$ $(m\geqslant 1)$. Let E and E_1 be the set of all elements of Ω and the set consisting of the above elements which may "yield" at $t=t_0$, respectively. We say that an element is in elastic (resp. plastic) state if (2_a) (resp. (2_b)) holds at the moment considered. The main result is

THEOREM 1. *The state of each element of E is uniquely determined beyond $t=t_0$ and the system* (1)~(2) *has a unique analytic solution in a certain time interval* $[t_0,\ t_0+\delta]$ $(\delta>0)$.

We shall sketch below a proof of this theorem. The key of the proof is to construct a series of quadratic forms with respect to the derivatives of u. First, we note that the next state of the elements of E_1 is determined clearly by the sign of $\partial f^*\dot{\sigma}\,|_{t_0+0}$. To find this sign we hence consider the following system of linear equations for the velocity $\dot{u}$ at $t=t_0+0$.

$$(3) \qquad \sum_j (\dot{\sigma}_{ij},\varphi_{p,j})_{L^2(\Omega)}=(\dot{b}_i(t_0+0),\ \varphi_p)_{L^2(\Omega)} \qquad p\in P,$$

where $\dot{\sigma}=D\dot{\varepsilon}$ for the elements of $E-E_1$ and

$$\begin{cases}\dot{\sigma}=D\dot{\varepsilon} & \text{in } D_-^1=\{\dot{u}:\ \partial f^* D\dot{\varepsilon}<0\}\\ \dot{\sigma}=(D-D')\dot{\varepsilon} & \text{in } D_+^1=\{\dot{u}:\ \partial f^* D\dot{\varepsilon}\geqslant 0\}\end{cases}$$

for the elements of E_1.

LEMMA 1. *The system* (3) *has a unique solution* $\dot{u}=(\dot{u}_i^p)$.

This lemma is proved by minimizing a standard quadratic form with respect to $\dot{u}$. Now let $\dot{u}$ be the solution of (3) and $\dot{\varepsilon}$ be the strain velocity calculated from $\dot{u}$. Since the sign of $\partial f^*\dot{\sigma}$ is the same to that of $\partial f^* D\dot{\varepsilon}$, if $\partial f^* D\dot{\varepsilon}<0$ (resp. >0) then this elemet must be elastic (resp. plastic) after $t=t_0$, that is, (2_a) (resp. (2_b)) must be chosen. If there is no element of E_1 which satisfies $\partial f^* D\dot{\varepsilon}=0$, then the next state is completely determined for all elements. However, there might be such non-empty subset E_2 of E_1 that $\partial f^* D\dot{\varepsilon}=0$. In this case we have to examine the sign of $\frac{d}{dt}(\partial f^*\dot{\sigma})$ at $t=t_0+0$. We remark that the choice of the next state of E_2 gives no influence on the value of $\dot{u}(t_0+0)$ for any element of E.

To determine the sign of $\frac{d}{dt}(\partial f^*\dot{\sigma})=\frac{d}{dt}(\partial f^* D\dot{\varepsilon})\cdot(1-\zeta)$ $(0\leqslant\zeta<1)$ at t_0+0 for E_2, we consider the following linear system for $\ddot{u}$ at $t=t_0+0$.

$$(4)\qquad \sum_j (\ddot{\sigma}_{ij},\ \varphi_{p,j})_{L^2(\Omega)} = (\ddot{b}_i(t_0+0),\ \varphi_p)_{L^2(\Omega)} \qquad p\in P.$$

In this equation $\ddot{\sigma}$ is connected with $\ddot{\varepsilon}$ by the established stress-strain relation for the elements of $E-E_2$ and

$$\begin{cases} \ddot{\sigma}=D\ddot{\varepsilon} & \text{in } D_-^2=\{\ddot{u}:\ \frac{d}{dt}(\partial f^* D\dot{\varepsilon})<0\} \\ \ddot{\sigma}=\frac{d}{dt}[(D-D')\dot{\varepsilon}] & \text{in } D_+^2=\{\ddot{u}:\ \frac{d}{dt}(\partial f^* D\dot{\varepsilon})\geqslant 0\} \end{cases}$$

for those of E_2.

LEMMA 2. *The system* (4) *has a unique solution* $\ddot{u}=(\ddot{u}_i^p)$.

For the proof of this lemma, consider the quadratic form

$$F(\ddot{u}) = \sum_{e\in E}[\frac{1}{2}(\ddot{\sigma},\ \ddot{\varepsilon})_e - \frac{1}{2}(\theta,\ \ddot{\varepsilon})_e - (\ddot{b}(t_0+0),\ \ddot{u})_e]$$

where $(\ ,\ )_e$ denotes $L^2(e)$ inner product of vector functions. The functions θ and $\ddot{\varepsilon}$ are defined as follows.

$$\theta = (\frac{d}{dt}D')\dot{\varepsilon}|_{t_0+0}$$

$$\ddot{\varepsilon} = \begin{cases} 0 \text{ (resp. } \ddot{\varepsilon}) & \text{for the elastic (resp. plastic) elements of } E-E_2 \\ \left.\begin{array}{ll} \ddot{\varepsilon}_0 & \text{in } D^2_- \\ \ddot{\varepsilon} & \text{in } D^2_+ \end{array}\right\} & \text{for the elements of } E_2 \end{cases}$$

The vector $\ddot{\varepsilon}_0$ is an arbitrary fixed 3-dimensional vector which satisfies the equaution $\frac{d}{dt}(\partial f^* D\dot{\varepsilon})=0$. It is shown that the stationary condition of $F(\ddot{u})$ is the equation (4). This lemma hence implies that the next state of the elements of E_2 can be determined if there is no element which satisfies $\frac{d}{dt}(\partial f^* D\dot{\varepsilon})=0$.

To prove Theorem 1, continue this argument. Then if $(\partial f^* \dot{\sigma})^{(k)}<0$ (resp. >0) at $t=t_0+0$ for a certain $k<\infty$, this element must be elastic (resp. plastic) after $t=t_0+0$. If there is no finite k, then this element is neutral after t_0 and the stress-strain relation is arbitrary. This situation is just the same as in the dynamic case considered in [1] and we have the theorem. Our conclusion is

THEOREM 2. *There exists a unique function* $(u, \varepsilon, \sigma, \alpha)$ *which satisfies* (1) ~ (2) *except at most countable* $t \in I$.

It is evident that the proof of Theorem 1 is valid also to the case that there are some elements for which the unloading may occur. Therefore, since some energy inequalities are derived easily, we can continuate the solution through I and hence the theorem follows. The countability of the exceptional t follows from the fact that such point is always an end of a time interval of positive length. We note that absolute continuity is enough to set up the class of functions in which the solution is sought.

References

[1] Miyoshi, T.: On existence proof in plasticity theory. Kumamoto J. Sci. (Math.), 14, 18–33 (1980).
[2] Miyoshi, T.: Numerical stability in dynamic elastic-plastic problems. R. A. I. R. O. Analyse numérique, 14, 175–188 (1980).
[3] Yamada, T.: Plasticity, Visco-elasticity, Baifukan, Tokyo (1972).
[4] Ziegler, H.: A modification of Prager's hardening rule. Quart. Appl. Math., 17, 55–65 (1959).