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EXISTENCE, STABILITY AND CONVERGENCE IN A HEAT CONTROL PROBLEM

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Introduction

Let Ω and Γ be the open interval (0, 1) and its boundary, and T a fixed positive number. We use Q and Σ to denote the sets $(0, T)\times\Omega$ and $(0, T)\times\Gamma$, respectively. The heat control problem considered in this paper is as follows. Seek a function $u(t, x)$ satisfying

(0.1) $$u' - u_{xx} = 0 \quad \text{in } Q \quad \left(u' = \frac{\partial u}{\partial t}\right),$$

with the initial condition

(0.2) $$u(0, x) = a(x) \quad \text{in } \Omega,$$

and the boundary conditions

(0.3) $$u' \geqq 0, \quad \frac{\partial u}{\partial n} \geqq 0 \quad \text{on } \Sigma,$$

(0.4) $$\begin{cases} \partial u/\partial n = 0 & \text{if } u' > 0 \\ \partial u/\partial n \geqq 0 & \text{if } u' = 0 \end{cases} \quad \text{on } \Sigma,$$

where n is the outward normal to Γ.

The difficulty in proving the existence of a solution to this problem or approximating it numerically lies in the treating of the boundary conditions.

In [2] Duvaut-Lions presented an idea to formulate this kind of problems by using variational inequality.

In the present paper we employ the idea proposed in [4] to analyze plastic vibration. We therefore introduce first a finite element scheme, which is a system of ordinary differential equations. We then prove that an initial value problem for this system is well set up. As a result, some a priori estimates of the solution of this system are obtained, and the existence and uniqueness of the original problem are proved by a standard compactness argument.

Furthermore, we show the stability and convergence of an explicit finite difference scheme for approximating the problem. Our criterion for the stability is $k/h^2 \leqq 1/4$, being h and k the spatial and time increments, respectively.

1. A finite element system

We devide [0, 1] at the points $\{x_j\}$ $(j=0, 1, \cdots, J)$, where $x_0=0$, $x_J=1$. We assume $x_{j+1}-x_j=h$. We use P_h to denote the set $\{0, 1, \cdots, J\}$ which depends on h. Also, we use P_h^o to denote the set obtained by excluding both the first and last numbers from P_h. We shall use the following finite element basis.

(1) $\{\phi_j\}$ $(j \in P_h)$ is the system of continuous functions satisfying $\phi_j(x_j)=1$, $\phi_j(x)=0$ for $x \notin (x_{j-1}, x_{j+1}) \cap \bar{\Omega}$ and linear in each $[x_j, x_{j+1}]$, where $x_{-1}=-h$, $x_{J+1}=1+h$.

(2) $\{\chi_j\}$ $(j \in P_h)$ is the set of the characteristic functions of the interval $[x_j-h/2, x_j+h/2]$.

Now the solution u is approximated by $\hat{u}$ and $\bar{u}$ of the following forms.

$$\hat{u}(t, x) = \sum_{j \in P_h} u_j(t)\phi_j(x) \qquad \bar{u}(t, x) = \sum_{j \in P_h} u_j(t)\chi_j(x).$$

The unknown $\{u_j(t)\}$ is determined by solving the following system of ordinary differential equations, which correspond to (0. 1)—(0. 4).

$$(1.1) \qquad (\bar{u}', \chi_j) + \left(\frac{\partial \hat{u}}{\partial x}, \frac{d\phi_j}{dx}\right) = 0 \quad \text{for all } j \in P_h^o,$$

$$(1.2) \qquad u_j(0) = a(jh) = a_j \quad \text{for all } j \in P_h,$$

$$(1.3) \qquad \hat{u}' \geqq 0, \ \frac{\partial \hat{u}}{\partial n} \geqq 0 \ \text{and} \ \hat{u}' \cdot \frac{\partial \hat{u}}{\partial n} = 0 \quad \text{on } \Sigma.$$

Here, (,) and ‖ ‖ denote the inner product and norm of $L^2(\Omega)$, respectively. As well seen, (1.1) is equal to

$$(1.1)' \qquad u_j' - \frac{1}{h^2}(u_{j+1} - 2u_j + u_{j-1}) = 0 \quad \text{for all } j \in P_h^o.$$

We first show that this problem can be well set up as an initial value problem for the unknown $\{u_j(t)\}$. We start from $t=0$. The semi-discrete system (1.1)—

(1.3) has an unique analytic solution in a neighborhood of $t=0$ for given initial value, provided u_0 and u_J are given by analytic functions near $t=0$ in advance. Considering the continuity of $\{u_j\}$ and the boundary condition (1.3), we assume the next condition concerning the initial value.

ASSUMPTION 1. *The initial value* $a(x)$ *belongs to* $C^3[0, 1]$*-class and decreases at* $x=0$ *and increases at* $x=1$.

Under this assumption the value of u_0 and u_J near $t=0$ are determined as follows.

As regards u_0; By Assumption 1, if h is sufficiently small, we have $a_0 \geqq a_1$. Therefore we consider the following four cases for $\{a_j\}$,

(1) $a_0 > a_1$,

(2) $a_j = a_{j+1} (j=0, \cdots, J_0)$ and $a_{J_0+1} > a_{J_0+2}$,

(3) $a_j = a_{j+1} (j=0, \cdots, J_0)$ and $a_{J_0+1} < a_{J_0+2}$,

(4) $a_j = a_{j+1} (j=0, \cdots, J-1)$.

We use $u^{(n)}$ to denote the n-th derivative of u with respect to t.

In the case (1), $u_1(t) < a_0$ must hold in some time interval $[0, \delta]$ by the continuity of u_1. Therefore we determine $u_0(t) = a_0$ in this case.

To discuss about the case (2), we first prove

LEMMA 1. *In the case* (2) *we have*

$$u_1'(0) = \cdots = u_1^{(J_0)}(0) = 0 \quad \text{and} \quad u_1^{(J_0+1)}(0) < 0.$$

PROOF. By (1.1)$'$ we have

$$u_1'(0) = u_2'(0) = \cdots = u_{J_0}'(0) = 0 \quad \text{and} \quad u_{J_0+1}'(0) < 0.$$

We determine the value of u_0 by the following rule. If $u_1(t)$ is decreasing after $t=0$, then we determine $u_0(t) = a_0$. In other cases we determine $u_0(t) = u_1(t)$. In both cases we have $u_0'(0) = 0$ since $u_1'(0) = 0$. By an induction we can easily prove that

$$u_0^{(i)}(0) = u_1^{(i)}(0) = \cdots = u_{J_0+1-i}^{(i)}(0) = 0 \quad \text{and} \quad u_{J_0+2-i}^{(i)}(0) < 0 \quad \text{for } i=1, \cdots, J_0.$$

From this we have

$$u_1^{(J_0+1)}(0)<0.$$

This lemma shows that $u_1(t)$ is decreasing after $t=0$. Therefore we determine $u_0(t)=a_0$ in this case, too.

In the case (3), $u_1(t)$ is increasing after $t=0$ by the same reason as in the case (2). Therefore we determine $u_0(t)=u_1(t)$ in this case.

In the case (4), we have a trivial solution. In what follows, we exclude this case from our consideration.

Summarizing the above results, we have

$(a)_0$ if $a_0>a_1$ or $a_j=a_{j+1}(j=0, \cdots, J_0)$ and $a_{J_0+1}<a_{J_0+2}$, then $u_0(t)=a_0$ in some time interval, and

$(b)_0$ if $a_j=a_{j+1}(j=0, \cdots, J_0)$ and $a_{J_0+1}<a_{J_0+2}$, then $u_0(t)=u_1(t)$ in some time interval.

Since the value of u_J is determined analogously, we can integrate the system (1.1) and get an analytic solution satisfying the initial and boundary conditions (1.2)—(1.3) at least for a certain time interval. The solution exists until the time $t=t_1$ at which $\{u_j\}$ satisfies $u_0(t)-u_1(t)=u_0'(t-0)=0$ or $u_J(t)-u_{J-1}(t)=u_J'(t-0)$. After $t=t_1$, $\{u_j\}$ may destroy the boundary condition (1.3). We have to determine again the value of u_0 and u_J after $t=t_1$.

As regards u_0; If $\{u_0, u_1\}$ satisfies $u_0(t_1)-u_1(t_1)=u_0'(t_1-0)$, then the next value of u_0 can be determined as follows. We have to discuss about the next two cases.

(1) $$u_1'(t_1-0)>0,$$

(2) $$u_1'(t_1-0)=\cdots=u_1^{(n)}(t_1-0)=0 \text{ and } u_1^{(n+1)}(t_1-0)\neq 0.$$

In the case (1), $u_1'(t)>0$ in some time interval $[t_1, t_1+\delta]$ due to the continuity of u_1 at $t=t_1$. Therefore we determine $u_0(t)=u_1(t)$ in this case. We note that $u_0(t_1)=u_1(t_1)$ and $u_1(t_1)<u_2(t_1)$ by $(1.1)'$.

To discuss about the case (2), we first prove

LEMMA 2. *If* $u_0(t_1)-u_1(t_1)=u_0'(t_1-0)=0$ and $u_1'(t_1-0)=\cdots=u_1^{(n)}(t_1-0)=0$ $(n+1<J)$, *then*

① $u_0(t_1)=u_1(t_1)=\cdots=u_{n+1}(t_1)$,

② $u_0^{(i)}(t_1\pm 0)=u_1^{(i)}(t_1\pm 0)=\cdots=u_{n+1-i}^{(i)}(t_1\pm 0)=0$ $(i=1, \cdots, n)$ *for any* *choice of the next value of* u_0,

$$③ \quad u_1^{(n+1)}(t_1 \pm 0) = h^{-2(n+1)}\{u_{n+2}(t_1) - u_{n+1}(t_1)\}.$$

PROOF. We shall use an induction on n. The lemma is obviously correct for $n=1$. Assume the lemma is correct until n. By $u_1^{(n+1)}(t_1-0)=0$ and ③ we have ① for $n+1$. By $u_1^{(n+1)}(t_1-0)=0$ and ② we have ② for $n+1$. Finally, by ② for $n+1$ we have

$$\begin{aligned} u_1^{(n+2)}(t_1 \pm 0) &= h^{-2}\{u_2^{(n+1)}(t_1 \pm 0) - 2u_1^{(n+1)}(t_1 \pm 0) + u_0^{(n+1)}(t_1 \pm 0)\} \\ &= h^{-2} u_2^{(n+1)}(t_1 \pm 0) \\ &= h^{-2(n+2)}\{u_{n+3}(t_1) - u_{n+2}(t_1)\}, \end{aligned}$$

which proves ③ for $n+1$.

Now assume that $u_1^{(n+1)}(t_1-0)<0$ (i. e. $u_{n+1}(t_1)>u_{n+2}(t_1)$) and n is odd. By the above lemma, $u_1(t)$ is decreasing after $t=t_1$. Therefore we determine $u_0(t)=u_0(t_1)$ after $t=t_1$. If $u_1^{(n+1)}(t_1-0)>0$ (i. e. $u_{n+1}(t_1)<u_{n+2}(t_1)$) and n is even, then $u_1(t)$ is increasing after $t=t_1$. Therefore we determine $u_0(t)=u_1(t)$ after $t=t_1$. We remark that the other cases don't occur by the relation between u_0 and u_1 for $t<t_1$.

Summarizing the above results

(a)$_1$ if $u_j(t_1)=u_{j+1}(t_1)(j=0, \cdots, n)$ and $u_{n+1}(t_1)>u_{n+2}(t_1)$ (n is odd), then we determine $u_0(t)=u_0(t_1)$ after $t=t_1$, and

(b)$_1$ if $u_j(t_1)=u_{j+1}(t_1)$ $(j=0, \cdots, n)$ and $u_{n+1}(t_1)<u_{n+2}(t_1)$ (n is even), then we determine $u_0(t)=u_1(t)$ after $t=t_1$.

In both cases u_0 and u_1 satisfy the boundary condition (1.3) after $t=t_1$. In the case that u_J and u_{J-1} satisfy $u_J(t_1)-u_{J-1}(t_1)=u_J'(t_1-0)=0$, the value of u_J after $t=t_1$ is determined analogously. Hence we can set up the initial value problem at $t=t_1$ again and continuate the solution beyond $t=t_1$. We can continuate the solution successively by this procedure. By the energy inequalities, which are proved in §2, there is no bound beyond which this continuation is impossible. Therefore, the initial value problem for this semi-discrete system is posed over the whole time interval [0, T]. We note that $\{u_j\}$ $(j\in P_h)$ and $\{u_j'\}$ $(j\in P_h^0)$ are absolutely continuous and $\{u_j\}$ $(j\in P_h\setminus\{0, 1, J-1, J\})$ belongs to C^2-class.

2. Energy inequalities

We shall derive some energy inequalities for the semi-discrete solution $\hat{u}(t,x)$ obtained in the preceding section.

THEOREM 1. *Under Assumption* 1, *it holds that*

$$\|\hat{u}\|+\|\frac{\partial\hat{u}}{\partial x}\|\leqq C, \tag{2.1}$$

where C is a positive constant indpendent of h and t.

PROOF. According to (1,1), we have

$$(\bar{u}', \sum_{j\varepsilon P_h^o} u_j'\chi_j)+(\frac{\partial\hat{u}}{\partial x}, \sum_{j\varepsilon P_h^o} u_j'\frac{d\phi_j}{dx})=0.$$

We rewrite this equality as follows

$$\|\bar{u}'\|^2-(\bar{u}', u_0'\chi_0+u_J'\chi_J)+(\frac{\partial\hat{u}}{\partial x}, \frac{\partial\hat{u}'}{\partial x})-(\frac{\partial\hat{u}}{\partial x}, u_0'\frac{d\phi_0}{dx}+u_J'\frac{d\phi_J}{dx})=0.$$

By using the relation $(D_xu_0)u_0'=(D_xu_{J-1})u_J'=0$ and $(u_0')^2\leqq(u_1')^2$, $(u_J')^2\leqq(u_{J-1}')^2$, we have

$$\|\bar{u}'\|^2+\frac{d}{dt}\|\frac{\partial\hat{u}}{\partial x}\|^2\leqq 0$$

where D_x is the forward difference operator. By integrating with respect to t, we have

$$\int_0^t\|\bar{u}'\|^2dt+\|\frac{\partial\hat{u}}{\partial x}\|^2\leqq\|\frac{\partial\hat{u}}{\partial x}(0)\|^2=h\sum_{j=0}^{J-1}(D_xa_j)^2. \tag{2.2}$$

By the smoothness of $a(x)$, this inequality implies

$$\int_0^t\|\bar{u}'\|^2dt\leqq C, \tag{2.3}$$

and $\|\frac{\partial\hat{u}}{\partial x}\|\leqq C$. Furthermore, it follows that

$$\|\bar{u}\|^2\leqq 2\|\bar{u}(0)\|^2+2T\int_0^t\|\bar{u}'\|^2dt\leqq 2h\sum_{j\varepsilon P_h}(a_j)^2+2T\int_0^t\|\bar{u}'\|^2dt. \tag{2.4}$$

Hence, by (2.3) and Assumption 1, we have $\|\bar{u}\| \leqq C$ and $\|\hat{u}\| \leqq C$, which complete the proof of Theorem 1.

For estimating the higher derivatives, we introduce $\overset{o}{w}(t, x) = \sum_{j \in P_h^o} u_j(t) \chi_j(x)$ and assume the next condition.

ASSUMPTION 2. $\dfrac{d^2 a}{dx^2} = 0$ *at* $x=0$ and $x=1$.

REMARK. By $(1.1)'$, $u_1'(0) = d^2a/dx^2(0) + O(h)$. If h is sufficiently small then a_0, a_1 and a_2 satisfy $a_0 = a_1 = a_2$ or $a_0 > a_1 > a_2$. In both cases we have $u_0'(0) = 0$. Therefore if we assume $\lim_{h \to 0} u_1'(0) = u_0'(0)$ then $d^2a/dx^2(0) = 0$. Analogously we have $d^2a/dx^2(1) = 0$. This observation suggests that this assumption is not unnatural.

THEOREM 2. *Under Assumption* 1 *and* 2, *it holds that*

$$\|\hat{u}'\| + \| \frac{\partial \hat{u}'}{\partial x} \| + \| \tilde{D}_x (\frac{\partial \hat{u}}{\partial x}) \| \leqq C, \tag{2.5}$$

$$\int_0^t \| \overset{o}{w}'' \|^2 \, dt \leqq C, \tag{2.6}$$

where $\tilde{D}_x u(x) = \{u(x+h) - u(x)\}/h$. *Here, we define* $\tilde{D}_x u(x) = 0$ *for* $1 - h < x \leqq 1$.

PROOF. We first have

$$(\overset{o}{w}', \chi_j) + (\frac{\partial \hat{u}}{\partial x}, \frac{d\phi_j}{dx}) = 0 \quad \text{for all } j \in P_h^0. \tag{2.7}$$

Note that $\overset{o}{w}'$ is absolutely continuous with respect to t. Let $T_m = (t_{m-1}, t_m)$ be a time interval in which no change of the relation between u_0 and u_1, and u_{J-1} and u_J, which we considered in §1, occurs. In this interval we can differentiate (2.7), and have the following equality.

$$(\overset{o}{w}'', \sum_{j \in P_h^o} u_j'' \chi_j) + (\frac{\partial \hat{u}'}{\partial x}, \sum_{j \in P_h^o} u_j'' \frac{d\phi_j}{dx}) = 0.$$

By using $(D_x u_0')\, u_0'' = (D_x u_{J-1}')\, u_J'' = 0$ in T_m we have

$$\| \overset{o}{w}'' \|^2 + \frac{1}{2} \frac{d}{dt} \| \frac{\partial \hat{u}'}{\partial x} \|^2 = 0.$$

Hence, integrating in $[s, t] \subset T_m$, we have

$$\int_s^t \|\overset{o}{w}''\|^2 dt + \frac{1}{2} G(t) = \frac{1}{2} G(s), \tag{2.8}$$

where $G(t) = \| \frac{\partial \hat{u}'}{\partial x} (t)\|^2$, and hence

$$\int_{t_{m-1}}^{t_m} \|\overset{o}{w}''\|^2 dt + \frac{1}{2} G(t_m - 0) = \frac{1}{2} G(t_{m-1} + 0). \tag{2.9}$$

We want to show that $G(t)$ is non-increasing at $t = t_m$. Assume $u_0(t_m) - u_1(t_m) = u_0'(t_m - 0) = 0$. If u_0 is constant on T_m and $u_1'(t_m - 0) > 0$, then we have $(D_x u_0')^2 (t_m - 0) > 0$ and $(D_x u_0')(t_m + 0) = 0$. The analogous relation holds for u_J and u_{J-1}. In other cases $D_x u_0'$ and $D_x u_{J-1}'$ are equal to zero or continuous at $t = t_m \pm 0$. $\{D_x u_j'\}$ $(j = 1, 2, \cdots, J-2)$ are obviously continuous at $t = t_m$. Hence, in any situations $G(t)$ is non-increasing at $t = t_m$.

Therefore we have successively

$$\begin{aligned} G(t_{m-1}+0) &\leqq G(t_{m-1}-0) \\ &\leqq G(t_{m-2}+0) \\ &\quad \vdots \\ &\leqq G(0). \end{aligned}$$

We note that

$$G(0) = h \sum_{j=0}^{J-3} (D_{xxx} a_j)^2 + \frac{1}{h} \{u_1'(0) - u_0'(0)\}^2 + \frac{1}{h} \{u_J'(0) - u_{J-1}'(0)\}^2,$$

where D_{xxx} is the third forward difference operator.
By Assumption 1 the first term of $G(0)$ is bounded by a constant and by Assumption 2 the last two terms are also bounded by a constant. Therefore for any t it holds that

$$G(t) = \| \frac{\partial \hat{u}'}{\partial x} \|^2 \leqq G(0) \leqq C. \tag{2.10}$$

Therefore we have

(2.11) $$\int^{t} \|\overset{o}{w}''\|^2 dt \leqq C.$$

To prove (2.5) we first note that

$$\|\overset{o}{w}'\|^2 \leqq 2\|\overset{o}{w}'(0)\|^2 + 2T\int_0^t \|\overset{o}{w}''\|^2 dt = 2h \sum_{j=0}^{J-2} (D_{xx}a_j)^2 + 2T\int_0^t \|\overset{o}{w}''\|^2 dt.$$

Since $a(x)$ is sufficiently smooth, $\|\overset{o}{w}'(0)\|$ is bounded by a positive constant. Therefore we have $\|\overset{o}{w}'\| \leqq C$. Since $(u_0')^2 \leqq (u_1')^2$ and $(u_J')^2 \leqq (u_{J-1}')^2$, we have $\|\hat{u}'\| \leqq C$, and hence by (1.1) we have $\|\tilde{D}_x\left(\frac{\partial \hat{u}}{\partial x}\right)\| \leqq C$. The proof of Theorem 2 is complete.

3. Existence of a solution

THEOREM 3. *Under Assumption* 1 *and* 2, *there exists a function u such that* $u \in L^\infty(0, T; H^2(\Omega))$, $u' \in L^\infty(0, T; H^1(\Omega))$ and $u'' \in L^2(Q)$ *satisfying a. a. t*

(3.1) $$u' - u_{xx} = 0 \quad \text{in } \Omega,$$

(3.2) $$u(0, x) = a(x) \quad \text{in } \Omega,$$

(3.3) $$u' \geqq 0, \ \frac{\partial u}{\partial n} \geqq 0 \ \text{and} \ u' \cdot \frac{\partial u}{\partial n} = 0 \ \text{on } \Gamma.$$

PROOF. By the energy inequalities derived in the preceding section, we can extract a subsequence such that

(3.4) $$\left.\begin{array}{l} \hat{u}, \ \dfrac{\partial \hat{u}}{\partial x} \ \tilde{D}_x\left(\dfrac{\partial \hat{u}}{\partial x}\right) \longrightarrow u, \ \dfrac{\partial u}{\partial x}, \ \dfrac{\partial^2 u}{\partial x^2} \\ \hat{u}', \dfrac{\partial \hat{u}'}{\partial x} \longrightarrow u', \dfrac{\partial u'}{\partial x} \end{array}\right\} \text{weakly* in } L^\infty(0, T; L^2(\Omega)),$$

$$\overset{o}{w}'' \longrightarrow u'' \quad \text{weakly in } L^2(Q).$$

We first show that u satisfies the equation (3. 1). Let ψ be an arbitrary function of $C^\infty(\Omega)$ whose support is in Ω,and take an arbitrary $s \in (0, T)$. For positive integer k we define ψ_k, $\hat{\psi}_k$ and $\bar{\psi}_k$ as follows.

$$\psi_k = \begin{cases} \psi & \text{if } t \in \Theta_k = (s-1/k,\ s+1/k) \\ 0 & \text{if } t \notin \Theta_k, \end{cases}$$

$$\hat{\psi}_k = \sum_{j \in P_h} \psi_k(t,\ jh)\phi_j(x), \qquad \bar{\psi}_k = \sum_{j \in P_h} \psi_k(t,\ jh)\chi_j(x).$$

Then (1.1) implies

$$\int_0^T (\bar{u}',\ \bar{\psi}_k)dt + \int_0^T \left(\frac{\partial \hat{u}}{\partial x},\ \frac{\partial \hat{\psi}_k}{\partial x}\right) dt = 0.$$

As well known, $\bar{\psi}_k \longrightarrow \psi_k$, $\partial\hat{\psi}_k/\partial x \longrightarrow \partial\psi_k/\partial x$ strongly in $L^2(Q)$ when h tends to 0. Therefore we have

$$\int_0^T (u',\ \psi_k)dt + \int_0^T \left(\frac{\partial u}{\partial x},\ \frac{\partial \psi_k}{\partial x}\right) dt = 0,$$

that is,

$$\left(\int_{\Theta_k} u' dt,\ \psi\right) + \left(\int_{\Theta_k} \frac{\partial u}{\partial x}\, dt,\ \frac{d\psi}{dx}\right) = 0.$$

From which we have (3.1).

It is easily to show that u satisfies the initial condition (3.2).

To prove that u satisfies the boundary condition (3.3) we introduce the following functions.

$$\tilde{u}(t,\ x) = \sum_{j \in P_h^0} u_j(t)\phi_j(x) + u_1(t)\phi_0(x) + u_{J-1}(t)\phi_J(x),$$

$$\tilde{v}(t,\ x) = \sum_{j \in P_h} (D_x u_j)\phi_j,$$

where $D_x u_J = D_x u_{J-1}$.

The energy inequalities obtained in §2 are valid also for these functions. Therefore we can extract a subsequence such that

$$\text{(3.5)} \qquad \left.\begin{aligned} \tilde{u}' &\longrightarrow u' \\ \tilde{v} &\longrightarrow \frac{\partial u}{\partial x} \end{aligned}\right\} \text{weakly in } H^1(Q).$$

As well known (see for example Aubin [1] or Lions-Magens [3]) the canonical injection from $H^1(Q)$ to $H^{3/4}(Q)$ is compact and the trace mapping is continuous linear from $H^{3/4}(Q)$ to $L^2(\Sigma)$. Therefore we have

$$\left.\begin{aligned}\tilde{u}'|_\Sigma &\longrightarrow u'|_\Sigma \\ \tilde{v}|_\Sigma &\longrightarrow \frac{\partial u}{\partial x}\Big|_\Sigma\end{aligned}\right\} \text{ strongly in } L^2(\Sigma). \tag{3.6}$$

Also the following estimates hold

$$\tilde{u}'|_\Sigma \geqq -C\sqrt{h}, \tag{3.7}$$

$$|(\tilde{u}', \tilde{v})_{L^2(\Sigma)}| \leqq C\sqrt{h}, \tag{3.8}$$

where C is a positive constant independent of h and t.

To prove (3.7) we first have

$$h\{(u_1'-u_0')/h\}^2 \leqq C, \tag{3.9}$$

by (2.5). On the other hand, if $u_0=u_1$ then $u_1'\geqq 0$ and if $u_0\neq u_1$ then $u_0'=0$. In the later case we have $(u_1')^2\leqq Ch$ by (3.9). Therefore for any cases we have $u_1'\geqq -C\sqrt{h}$. Samely $u_{J-1}'\geqq -C\sqrt{h}$. Hence we have (3.7).

To prove (3.8), we have on the boundary $x=0$

$$\begin{aligned}\left|\int_{u_0\neq u_1} u_1'(D_x u_0)\, dt\right| &\leqq \left(\int_{u_0\neq u_1} (u_1')^2 dt\right)^{1/2}\left(\int_o^T (D_x u_0)^2 dt\right)^{1/2} \\ &\leqq \left(\int_{u_0\neq u_1} Ch\, dt\right)^{1/2} C\|\tilde{v}\|_{H_1(Q)} \leqq C\sqrt{h}.\end{aligned}$$

Samely we have an estimate on the boundary $x=1$. Hence we have (3.8).

By using (3.6) and (3.7) we have $u'|_\Sigma\geqq 0$, and samely $\partial u/\partial n|_\Sigma \geqq 0$. Also by (3.6) and (3.8) we have $(u', \partial u/\partial n)_{L^2(\Sigma)}=0$, which completes the proof.

4. Uniqueness of the solution

THEOREM 4. *The solution of* (3.1)—(3.3) *is unique.*

PROOF. Let u and u^* be two solutions and put $w=u-u^*$. Then we have

$$\|w'\|^2 + \left(\frac{\partial w}{\partial x}, \frac{\partial w'}{\partial x}\right) - \int_\Gamma \frac{\partial w}{\partial n} w' d\Gamma = 0.$$

By using the boundary condition, we have

$$\|w'\|^2+\frac{1}{2}\frac{d}{dt}\|\frac{\partial w}{\partial x}\|^2\leqq 0.$$

Since $w(0)=0$, it follows that

$$2\int_0^t\|w'\|^2dt+\|\frac{\partial w}{\partial x}\|^2\leqq 0,$$

which proves the uniqueness.

Therefore, the convergence holds not only for a subsequence, but also for the original sequence.

5. Error estimation

In this section we shall derive an estimate of the rate of convergence of the finite element solution. By $\hat{u}_c$ and $\bar{u}_c$ we denote the finite element solutions, which are obtained in §1. By $u(t, x)$ we denote the exact solution. $\hat{u}_P$ and $\bar{u}_P$ denote the projection of u onto the space spanned by $\{\phi_j\}(j\in P_h)$ and $\{\chi_j\}(j\in P_h)$, respectively.

THEOREM 5. *Put* $\varepsilon=u-\hat{u}_c$. *Under Assumption* 1 *and* 2 *we have the following estimate.*

$$\|\varepsilon\|+\|\frac{\partial\varepsilon}{\partial x}\|+\left(\int_0^T\|\varepsilon'\|^2dt\right)^{1/2}\leqq C(h+\sqrt{h}\sqrt{b(h)}), \tag{5.1}$$

$$\lim_{h\to 0} b(h)=0, \tag{5.2}$$

where C *is a constant independent of* h *and* t, *and* $b(h)$ *is given by*

$$b^2(h)=\int_{\hat{u}_c'|_{\Sigma}\neq 0}(\varepsilon')^2|_{\Sigma}\,dt.$$

For the proof of this theorem, we prepare the next two lemmas, which are easily proved.

LEMMA 3. *Let* $\hat{u}$, $\hat{v}$ *and* $\bar{u}$, $\bar{v}$ *be the functions represented by* $\{\phi_j\}$ *and* $\{\chi_j\}$, *respectively. Then we have*

$$(\bar{u}, \bar{v})-(\hat{u}, \hat{v})=\frac{h^2}{6}(\frac{\partial\hat{u}}{\partial x}, \frac{\partial\hat{v}}{\partial x}). \tag{5.3}$$

LEMMA 4. *Let u be the exact solution, then $\partial^2 u'/\partial x^2$ belongs to $L^2(Q)$.*

PROOF OF THEOREM 5. Since u is the exact solution, we have

$$(u',\ \hat{u}'_P-\hat{u}'_c)+(\frac{\partial u}{\partial x},\ \frac{\partial \hat{u}'_P}{\partial x}-\frac{\partial \hat{u}'_c}{\partial x})-\int_\Gamma \frac{\partial u}{\partial n}(\hat{u}'_P-\hat{u}'_c)d\Gamma=0.$$

By the boundary condition the last term is non-negative. Hence we have

$$(5.4)\qquad (u',\ \hat{u}'_P-\hat{u}'_c)+(\frac{\partial u}{\partial x},\ \frac{\partial \hat{u}'_P}{\partial x}-\frac{\partial \hat{u}'_c}{\partial x})\leqq 0.$$

On the other hand, multiplying (1.1) by $u'_j-u'_j(t, jh)$ and summing with respect to j,

$$(\bar{u}'_c,\ \bar{u}'_c-\bar{u}'_P)+(\frac{\partial \hat{u}_c}{\partial x},\ \frac{\partial \hat{u}'_c}{\partial x}-\frac{\partial \hat{u}'_P}{\partial x})-\frac{h}{2}\{u'_0(u'_0-u'(0))+u'_J(u'_J-u'(1))\}$$
$$+\{D_x u_0(u'_0-u'(0))-D_x u_{J-1}(u'_J-u'(1))\}=0.$$

The last term is again non-negative. Therefore we have

$$(5.5)\qquad (\bar{u}'_c,\ \bar{u}'_c-\bar{u}'_P)+(\frac{\partial \hat{u}_c}{\partial x},\ \frac{\partial \hat{u}'_c}{\partial x}-\frac{\partial \hat{u}'_P}{\partial x})-h\alpha(t)\leqq 0,$$

where $\alpha(t)=1/2\{u'_0(u'_0-u'(0))+u'_J(u'_J-u'(1))\}$.

Adding (5.4) to (5.5), and putting $\varepsilon=u-\hat{u}_c$,

$$(5.6)\qquad \|\varepsilon'\|^2+(\varepsilon',\ \hat{u}'_P-u')+\{\|\bar{u}'_c\|^2-\|\hat{u}'_c\|^2\}+\{(\hat{u}'_c,\ \hat{u}'_P)-(\bar{u}'_c,\ \bar{u}'_P)\}$$
$$+\frac{1}{2}\frac{d}{dt}\|\frac{\partial \varepsilon}{\partial x}\|^2+(\frac{\partial \varepsilon}{\partial x},\ \frac{\partial \hat{u}'_P}{\partial x}-\frac{\partial u'}{\partial x})-h\alpha(t)\leqq 0.$$

Integrating with respect to t and using Schwarz's inequality and Lemma 3,

$$(5.7)\qquad \int^t \|\varepsilon'\|^2 dt+\frac{1}{2}\|\frac{\partial \varepsilon}{\partial x}\|^2$$
$$\leqq \|\hat{u}'_P-u'\|_{L^2(Q)}\left(\int_0^t \|\varepsilon'\|^2 dt\right)^{1/2}+\|\frac{\partial \hat{u}'_P}{\partial x}-\frac{\partial u'}{\partial x}\|_{L^2(Q)}\left(\int_0^t \|\frac{\partial \varepsilon}{\partial x}\|^2 dt\right)^{1/2}$$
$$+h^2\|\frac{\partial \hat{u}'_c}{\partial x}\|_{L^2(Q)}\|\frac{\partial \hat{u}'_P}{\partial x}\|_{L^2(Q)}+h\left|\int_0^t \alpha(t)\,dt\right|+\|\frac{\partial \varepsilon}{\partial x}(0)\|^2.$$

The fourth term of the right side of (5.7) is estimated as

$$h\left|\int_0^t \alpha(t)dt\right| \leqq h\left(\int_0^t (\hat{u}_c')^2|_\Sigma\, dt\right)^{1/2}\left(\int_{\hat{u}_c'|_\Sigma \neq 0} (\varepsilon')^2|_\Sigma\, dt\right)^{1/2}$$

$$\leqq Ch\{\|\hat{u}_c'\|_{L^2(Q)} + \|\frac{\partial \hat{u}_c'}{\partial x}\|_{L^2(Q)}\}\ b(h),$$

where $b(h) = \left(\int_{\hat{u}_c'|_\Sigma \neq 0} (\varepsilon')^2|_\Sigma\, dt\right)^{1/2}$.

Hence, by (2.5), Theorem 3, Lemma 4 and Assumption 1, we have

(5.8) $$\int_0^t \|\varepsilon'\|^2\, dt + \|\frac{\partial \varepsilon}{\partial x}\|^2 \leqq Ch\left\{\left(\int_0^t \|\varepsilon'\|^2 dt\right)^{1/2} + \left(\int_0^t \|\frac{\partial \varepsilon}{\partial x}\|^2 dt\right)^{1/2} + h + b(h)\right\},$$

where C is a constant independent of h and t.
Therefore we have

(5.9) $$\|\frac{\partial \varepsilon}{\partial x}\|^2 \leqq Ch\left\{\left(\int_0^t \|\frac{\partial \varepsilon}{\partial x}\|^2\, dt\right)^{1/2} + h + b(h)\right\},$$

and hence by a similar argument to prove the Gronwall's inequality we have

(5.10) $$\|\frac{\partial \varepsilon}{\partial x}\| \leqq C(h + \sqrt{h}\ \sqrt{b(h)}).$$

To estimate ε' substituting (5.10) into (5.8),

$$\int_0^t \|\varepsilon'\|^2\, dt \leqq Ch\left\{\left(\int_0^t \|\varepsilon'\|^2\, dt\right)^{1/2} + h + \sqrt{h}\ \sqrt{b(h)} + b(h)\right\}.$$

By solving this quadratic inequality, we have

(5.11) $$\left(\int_0^t \|\varepsilon'\|^2\, dt\right)^{1/2} \leqq C(h + \sqrt{h}\ \sqrt{b(h)}).$$

Since $\varepsilon(t, x) = \varepsilon(0, x) + \int_0^t \varepsilon'\, dt$, we have

(5.12) $$\|\varepsilon\| \leqq C(h + \sqrt{h}\ \sqrt{b(h)}).$$

(5.2) follows from (3.6). The proof is now complete.

6. Stability of a difference scheme

We devide the time interval $[0, T]$ at the points $\{t_n\}(n=0, 1, \dots, N)$, where $t_0=0$, $t_N=T$. Let $t_{n+1}-t_n=k$, and $\tilde{P}_k$ be the set $\{0, 1, \dots, N-1\}$.

We consider the following difference scheme.

$$(6.1)\qquad D_t u_j^n - D_{xx} u_{j-1}^n = 0 \qquad \text{for all } j \in P_h^o,\ n \in \tilde{P}_k,$$

$$(6.2)\qquad u_j^0 = a_j \qquad \text{for all } j \in P_h,$$

$$(6.3)\qquad \begin{cases} u_0^{n+1} = u_0^n & \text{if } u_0^n > u_1^{n+1} \\ u_0^{n+1} = u_1^{n+1} & \text{if } u_0^n \leqq u_1^{n+1} \end{cases} \quad \text{and} \quad \begin{cases} u_J^{n+1} = u_J^n & \text{if } u_J^n > u_{J-1}^{n+1} \\ u_J^{n+1} = u_{J-1}^{n+1} & \text{if } u_J^n \leqq u_{J-1}^{n+1} \end{cases}$$

where $u_j^n = u(t_n, x_j)$, and the difference operators are defined as follows.

$$D_t u_j^n = \frac{1}{k}(u_j^{n+1} - u_j^n),\ D_x u_j^n = \frac{1}{h}(u_{j+1}^n - u_j^n),\ D_{xx} u_j^n = D_x(D_x u_j^n), \text{ etc.}$$

We show that this scheme is stable. To do this we assume the next condition as a stability criterion of the difference scheme.

ASSUMPTION 3. $\quad \dfrac{k}{h^2} \leqq \dfrac{1}{4}$.

THEOREM 6. *Under Assumption* 1 *and* 3, *we have*

$$(6.4)\qquad h\sum_{j=0}^{J}(u_j^n)^2 + h\sum_{j=0}^{J-1}(D_x u_j^n)^2 + kh\sum_{n=0}^{N-1}\sum_{j=0}^{J}(D_t u_j^n)^2 \leqq C,$$

where C is a positive constant independent of h, k and n.

PROOF. Multiplying (6.1) by $D_t u_j^{n-1}$ and summing with respect to j, we have

$$(6.5)\qquad h\sum_{j=1}^{J-1} D_t u_j^n \cdot D_t u_j^{n-1} - h\sum_{j=1}^{J-1} D_{xx} u_{j-1}^n \cdot D_t u_j^{n-1} = 0.$$

The first term of (6.5) can be rewritten as follows.

$$h\sum_{j=1}^{J-1} D_t u_j^n \cdot D_t u_j^{n-1} = \frac{h}{2}\sum_{j=1}^{J-1}\{(D_t u_j^n)^2 + (D_t u_j^{n-1})^2\} - \frac{h}{2}\sum_{j=1}^{J-1}(D_t u_j^n - D_t u_j^{n-1})^2.$$

By using the boundary condition $(D_x u_0^n)(D_t u_0^{n-1}) = (D_x u_{J-1}^n)(D_t u_J^{n-1}) = 0$, the second term of (6.5) can be rewritten as follows.

$$-h\sum_{j=1}^{J-1} D_{xx}u_{j-1}^n \cdot D_t u_j^{n-1}$$

$$=\frac{h}{k}\sum_{j=0}^{J-1}\{(D_x u_j^n)^2 - D_x u_j^n \cdot D_x u_j^{n-1}\}$$

$$=\frac{h}{2k}\sum_{j=0}^{J-1}\{(D_x u_j^n)^2-(D_x u_j^{n-1})^2\}+\frac{h}{2k}\sum_{j=0}^{J-1}(D_x u_j^n - D_x u_j^{n-1})^2.$$

Therefore (6.5) is written as

$$(6.6)\qquad h\sum_{j=1}^{J-1}\{(D_t u_j^n)^2+(D_t u_j^{n-1})^2\}+\frac{h}{k}\sum_{j=0}^{J-1}\{(D_x u_j^n)^2-(D_x u_j^{n-1})^2\}$$
$$+h\left[\frac{1}{k}\sum_{j=0}^{J-1}(D_x u_j^n - D_x u_j^{n-1})^2-\sum_{j=1}^{J-1}(D_t u_j^n - D_t u_j^{n-1})^2\right]=0.$$

Since $D_t u_j^n = (D_x u_j^n - D_x u_{j-1}^n)/h$, the last term of (6.6) is estimated from below as

$$\frac{1}{k}\sum_{j=0}^{J-1}(D_x u_j^n - D_x u_j^{n-1})^2-\sum_{j=1}^{J-1}(D_t u_j^n - D_t u_j^{n-1})^2$$

$$=k\sum_{j=0}^{J-1}(D_{xt}u_j^{n-1})^2-\frac{1}{h^2}\sum_{j=1}^{J-1}\{(D_x u_j^n - D_x u_{j-1}^n)-(D_x u_j^{n-1}-D_x u_{j-1}^{n-1})\}^2$$

$$\geqq k\left(1-\frac{4k}{h^2}\right)\sum_{j=0}^{J-1}(D_{xt}u_j^{n-1})^2.$$

By Assumption 3 the last term of (6.6) is non-negative. Also, by the boundary condition, we have $(D_t u_0^n)^2 \leqq (D_t u_1^n)^2$ and $(D_t u_J^n)^2 \leqq (D_t u_{J-1}^n)^2$. Therefore, we finally have

$$(6.7)\qquad \frac{h}{2}\sum_{j=0}^{J}\{(D_t u_j^n)^2+(D_t u_j^{n-1})^2\}+\frac{h}{k}\sum_{j=0}^{J-1}\{(D_x u_j^n)^2-(D_x u_j^{n-1})^2\}\leqq 0.$$

Now, multiplying (6.7) by k and summing with respect to n,

$$(6.8)\qquad kh\sum_{n=0}^{n}\sum_{j=0}^{J}(D_t u_j^n)^2 + h\sum_{j=0}^{J-1}(D_x u_j^n)^2 \leqq 2h\sum_{j=0}^{J-1}(D_x u_j^0)^2 = 2h\sum_{j=0}^{J-1}(D_x a_j)^2.$$

By the smoothness of $a(x)$, the right side of (6.8) is bounded by a constant C independent of h, k and n.

By Schwarz's inequality and $u_j^n = u_j^0 + k\sum_{n=0}^{n-1} D_t u_j^n$, it follows that

$$h\sum_{j=0}^{J}(u_j^n)^2 \leqq 2h\sum_{j=0}^{J}(u_j^0)^2 + 2h\sum_{j=0}^{J}(k\sum_{n=0}^{n-1}D_t u_j^n)^2 \leqq C.$$

This completes the proof.

THEOREM 7. *Under Assumption* 1, 2 *and* 3, *we have*

$$kh\sum_{n=0}^{N-2}\sum_{j=1}^{J-1}(D_{tt}u_j^n)^2 + h\sum_{j=0}^{J-1}(D_{xt}u_j^n)^2 \leqq C, \tag{6.9}$$

where C is a constant independent of h, k and n.

To prove this theorem, we prepare the next lemma.

LEMMA 5. *Under the same assumption as in Theorem* 7, *we have*

$$h\sum_{j=}^{J-1}(D_{xt}u_j^0)^2 \leqq C. \tag{6.10}$$

PROOF. By the difference equation (6.1), we have

$$h\sum_{j=1}^{J-2}(D_{xt}u_j^0)^2 = h\sum_{j=1}^{J-2}(D_{xxx}u_{j-1}^0)^2 = h\sum_{j=1}^{J-2}(D_{xxx}a_{j-1})^2.$$

Since $a(x)$ is sufficiently smooth, this term is bounded by a constant independent of h. To estimate $h(D_{xt}u_0^0)^2$, since $(D_t u_0^0)^2 \leqq (D_t u_1^0)^2$, we first have

$$(D_{xt}u_0^0)^2 = \{(D_t u_1^0 - D_t u_0^0)/h\}^2 \leqq \frac{4}{h^2}(D_t u_1^0)^2$$

$$= \frac{4}{h^2}(D_{xx}u_0^0)^2 = \frac{4}{h^2}\{a''(0) + O(h)\}^2.$$

By Assumption 2 $a''(0) = 0$, and hence $h(D_{xt}u_0^0)^2 = O(h)$. Similarly we have $h(D_{xt}u_{J-1})^2 = O(h)$. The proof is complete.

PROOF OF THEOREM 7. From the difference equation (6.1), we have

$$D_{tt}u_j^n - D_{xxt}u_{j-1}^n = 0 \qquad \text{for all } j \in P_h^o,\ n \in \tilde{P}_k. \tag{6.11}$$

Now put $v_j^n = D_t u_j^n$. Multiplying (6.11) by $D_t v_j^{n-1}$ and summing with respect to j, we have

$$h\sum_{j=1}^{J-1} D_t v_j^n \cdot D_t v_j^{n-1} - h\sum_{j=1}^{J-1} D_{xx} v_{j-1}^n \cdot D_t v_j^{n-1} = 0 \tag{6.12}$$

The second term of (6.12) can be rewritten as follows

$$-h\sum_{j=1}^{J-1} D_{xx} v_{j-1}^n \cdot D_t v_j^{n-1}$$

$$= \frac{h}{k}\sum_{j=0}^{J-1} \{(D_x v_j^n)^2 - D_x v_j^n \cdot D_x v_j^{n-1}\} + D_x v_0^n \cdot D_t v_0^{n-1} - D_x v_{J-1}^n \cdot D_t v_J^{n-1}.$$

By the boundary condition $(D_x u_0^n)(D_t u_0^{n-1}) = 0$, we have

$$D_x v_0^n \cdot D_t v_0^{n-1} = \frac{-1}{k^2} (D_x u_0^{n+1} \cdot D_t u_0^{n-1} + D_x u_0^n \cdot D_t u_0^n) \geqq 0.$$

Samely $-(D_x v_{J-1}^n)(D_t v_J^{n-1})$ is also non-negative.

By the same way as (6.5)—(6.7), we have

$$h\sum_{j=1}^{J-1} \{(D_t v_j^n)^2 + (D_t v_j^{n-}\ )^2\} + \frac{h}{k}\sum_{j=0}^{J-1} \{(D_x v_j^n)^2 - (D_x v_j^{n-1})^2\} \leqq 0. \tag{6.13}$$

Multiplying (6.13) by k and summing with respect to n, we have

$$kh\sum_{n=0}^{n}\sum_{j=0}^{J-1} (D_t v_j^n)^2 + h\sum_{j=0}^{J-1} (D_x v_j^n)^2 \leqq h\sum_{j=0}^{J-1} (D_x v_j^0)^2. \tag{6.14}$$

By Lemma 5, the right side of (6.14) is bounded by a constant independent of h, k and n. This completes the proof.

7. Convergence of the difference scheme

We obtained a rate of convergence of the finite element solution to the exact solution in §5. In this section we shall derive an estimate of the error between the finite element solution and the finite difference solution obtained in the previous section. We extend the discrete solution $\{u_j^n\}$ to the function defined in the whole space Q as follows.

$$\left.\begin{aligned} \hat{u}_d(t, x) &= \sum_{j \in P_h} u_j^n \phi_j(x) \\ \bar{u}_d(t, x) &= \sum_{j \in P_h} u_j^n \chi_j(x) \end{aligned}\right\} \quad \text{in the strip } t_{n-1}^+ < t \leqq t_n^+,$$

where $t_n^+=(n+1/2)k$.
We use the difference operators defined as follows

$$\tilde{D}_x\psi(t, x)=\frac{1}{h}\{\psi(t, x+h)-\psi(t, x)\}, \quad \tilde{D}_t\psi(t, x)=\frac{1}{k}\{\psi(t+k, x)-\psi(t, x)\},$$

$$\tilde{D}_{\bar{t}}\psi(t, x)=\frac{1}{k}\{\psi(t, x)-\psi(t-k, x)\}, \text{ etc.}$$

Let $[\ ,\ ]$ and $|||\cdot|||$ be the inner product and norm of $L^2(h/2, 1-h/2)$, respectively.

THEOREM 8. *Put* $\bar{\varepsilon}=\hat{u}_c-\hat{u}_d$. *Under Assumption* 1, 2 *and* 3, *we have*

$$\|\bar{\varepsilon}\|+\left\|\frac{\partial\bar{\varepsilon}}{\partial x}\right\|\leqq C\sqrt{k}, \tag{7.1}$$

where C is a constant independent of h, k and t.

PROOF. (i) In the case that $t\leqq t_0^+$. Multiplying (1.1) by $u_j'-D_tu_j^0$, summing with respect to j and using the boundary condition, we have

$$[\bar{u}_c', \bar{u}_c'-\tilde{D}_t\bar{u}_d]+\left(\frac{\partial\hat{u}_c}{\partial x}, \frac{\partial\hat{u}_c'}{\partial x}-\tilde{D}_t\left(\frac{\partial\hat{u}_d}{\partial x}\right)\right)\leqq 0. \tag{7.2}$$

Next, multiplying (6.1) by $D_tu_j^0-u_j'$ and summing with respect to j, we have

$$\begin{aligned}[\tilde{D}_t\bar{u}_d, \tilde{D}_t\bar{u}_d-\bar{u}_c']&+\left(\frac{\partial\hat{u}_d}{\partial x}, \tilde{D}_t\left(\frac{\partial\hat{u}_d}{\partial x}\right)-\frac{\partial\hat{u}_c'}{\partial x}\right)\\ &+\{D_xu_0^0(D_tu_0^0-u_0')-D_xu_{J-1}^0(D_tu_J^0-u_J')\}=0.\end{aligned} \tag{7.3}$$

Here, if $D_xu_0^0\neq 0$ then we have $a_0>a_1>a_2$ by taking h sufficiently small. Since

$$u_1^1=\frac{k}{h^2}a_0+\left(1-\frac{2k}{h^2}\right)a_1+\frac{k}{h^2}a_2<a_0=u_0^0,$$

we have $D_tu_0^0=0$. Also if $D_xu_{J-1}^0\neq 0$ then $D_tu_J^0=0$, and hence the last term of (7.3) is non-negative. Therefore we have

$$[\tilde{D}_t\bar{u}_d, \tilde{D}_t\bar{u}_d-\bar{u}_c']+\left(\frac{\partial\hat{u}_d}{\partial x}, \tilde{D}_t\left(\frac{\partial\hat{u}_d}{\partial x}\right)-\frac{\partial\hat{u}_c'}{\partial x}\right)\leqq 0. \tag{7.4}$$

Adding (7.2) to (7.4), we have

(7.5) $$\||\bar{u}_c' - \tilde{D}_t\bar{u}_d\||^2 + \left(\frac{\partial \hat{u}_c}{\partial x} - \frac{\partial \hat{u}_d}{\partial x}, \frac{\partial \hat{u}_c'}{\partial x} - \tilde{D}_t\left(\frac{\partial \hat{u}_d}{\partial x}\right)\right) \leqq 0.$$

Integrating this from 0 to t_0^+ and using the facts $\hat{u}_d(t_0^+, x) = \hat{u}_d(0, x) = \hat{u}_c(0, x)$, we have

(7.6) $$\int_0^{t_0^+} \||\bar{u}_c' - \tilde{D}_t\bar{u}_d\||^2 dt + \frac{1}{2}\left\|\frac{\partial \bar{\varepsilon}}{\partial x}(t_0^+)\right\|^2$$

$$\leqq \int_0^{t_0^+} \left(\frac{\partial \hat{u}_c}{\partial x} - \frac{\partial \hat{u}_c}{\partial x}(0), \tilde{D}_t\left(\frac{\partial \hat{u}_d}{\partial x}\right)\right) dt \leqq Ck^2.$$

(ii) In the case that $t > t_0^+$. Multiplying (1.1) by $u_j' - D_t u_j^{n-1}$ and summing with respect to j, we have

$$\left(\bar{u}_c', \sum_{j=1}^{J-1}\{u_j' - D_t u_j^{n-1}\}\chi_j\right) + \left(\frac{\partial \hat{u}_c}{\partial x}, \sum_{j=1}^{J-1}\{u_j' - D_t u_j^{n-1}\}\frac{d\phi_j}{dx}\right) = 0 \quad \text{in } t_{n-1}^+ < t \leqq t_n^+.$$

By the same way as in the case (i), we have

(7.7) $$[\bar{u}_c', \bar{u}_c' - \tilde{D}_{\bar{t}}\bar{u}_d] + \left(\frac{\partial \hat{u}_c}{\partial x}, \frac{\partial \hat{u}_c'}{\partial x} - \tilde{D}_{\bar{t}}\left(\frac{\partial \hat{u}_d}{\partial x}\right)\right) \leqq 0.$$

On the other hand, multiplying (6.1) by $D_t u_j^{n-1} - u_j'$ and summing with respect to j, we have

$$h\sum_{j=1}^{J-1} D_t u_j^n (D_t u_j^{n-1} - u_j') - h\sum_{j=1}^{J-1} D_{xx} u_{j-1}^n (D_t u_j^{n-1} - u_j') = 0 \quad \text{in } t_{n-1}^+ < t \leqq t_n^+.$$

Now, $h\sum_{j=1}^{J-1} D_t u_j^n \cdot D_t u_j^{n-1} - h\sum_{j=1}^{J-1} D_{xx} u_{j-1}^n \cdot D_t u_j^{n-1}$ is estimated from below:

$$\geqq \frac{h}{2}\sum_{j=1}^{J-1}\{(D_t u_j^n)^2 + (D_t u_j^{n-1})^2\} + \frac{h}{2k}\sum_{j=0}^{J-1}\{(D_x u_j^n)^2 - (D_x u_j^{n-1})^2\},$$

as shown in §6.

Therefore we have

(7.8) $$[\tilde{D}_t\bar{u}_d, \frac{1}{2}\tilde{D}_t\bar{u}_d - \bar{u}_c'] + \frac{1}{2}\||\tilde{D}_{\bar{t}}\bar{u}_d\||^2 - \left(\frac{\partial \hat{u}_d}{\partial x}, \frac{\partial \hat{u}_c'}{\partial x}\right)$$

$$+ \frac{1}{2k}\left\{\left\|\frac{\partial \hat{u}_d}{\partial x}(t_n^+)\right\|^2 - \left\|\frac{\partial \hat{u}_d}{\partial x}(t_{n-1}^+)\right\|^2\right\} \leqq 0.$$

Adding (7.7) to (7.8) and using the equality

$$[\bar{u}_c', \bar{u}_c' - \tilde{D}_{\bar{t}}\bar{u}_d] + [\tilde{D}_t\bar{u}_d, \frac{1}{2}\tilde{D}_t\bar{u}_d - \bar{u}_c'] + \frac{1}{2} ||| \tilde{D}_{\bar{t}}\bar{u}_d |||^2$$

$$= \frac{1}{2} ||| \bar{u}_c' - \tilde{D}_t\bar{u}_d |||^2 + \frac{1}{2} ||| \bar{u}_c' - \tilde{D}_{\bar{t}}\bar{u}_d |||^2,$$

we have

$$\frac{1}{2} ||| \bar{u}_c' - \tilde{D}_t\bar{u}_d |||^2 + (\frac{\partial \hat{u}_c}{\partial x}, \frac{\partial \hat{u}_c'}{\partial x} - \tilde{D}_{\bar{t}}\left(\frac{\partial \hat{u}_d}{\partial x}\right)) - (\frac{\partial \hat{u}_d}{\partial x}, \frac{\partial \hat{u}_c'}{\partial x}) + \frac{1}{2k}\{\| \frac{\partial \hat{u}_d}{\partial x}(t_n^+)\|^2 - \| \frac{\partial \hat{u}_d}{\partial x}(t_{n-1}^+)\|^2\} \leqq 0. \tag{7.9}$$

Integrating this from t_{n-1}^+ to t_n^+, we have

$$\int_{t_{n-1}^+}^{t_n^+} ||| \bar{u}_c' - \tilde{D}_t\bar{u}_d |||^2 dt + \| \frac{\partial \bar{\varepsilon}}{\partial x}(t_n^+)\|^2 - \| \frac{\partial \bar{\varepsilon}}{\partial x}(t_{n-1}^+)\|^2 \leqq 2(\int_{t_{n-1}^+}^{t_n^+} \{\frac{\partial \hat{u}_c}{\partial x} - \frac{\partial \hat{u}_c}{\partial x}(t_{n-1}^+)\}\, dt, \tilde{D}_{\bar{t}}\left(\frac{\partial \hat{u}_d}{\partial x}\right)(t_n^+)) . \tag{7.10}$$

By using Schwarz's inequality, the right side of (7.10) is estimated as

$$\leqq k^{3/2} \| \tilde{D}_{\bar{t}}\left(\frac{\partial \hat{u}_d}{\partial x}\right)(t_n^+)\| \left(\int_{t_{n-1}^+}^{t_n^+} \| \frac{\partial \hat{u}_c'}{\partial x} \|^2 dt\right)^{1/2} \leqq Ck^2.$$

Therefore we have

$$\int_{t_{n-1}^+}^{t_n^+} ||| \bar{u}_c' - \tilde{D}_t\bar{u}_d |||^2 dt + \| \frac{\partial \bar{\varepsilon}}{\partial x}(t_n^+)\|^2 - \| \frac{\partial \bar{\varepsilon}}{\partial x}(t_{n-1}^+)\|^2 \leqq Ck^2. \tag{7.11}$$

Combining (7.6) and (7.11), we have successively

$$\int_0^{t_n^+} ||| \bar{u}_c' - \tilde{D}_t\bar{u}_d |||^2 dt + \| \frac{\partial \bar{\varepsilon}}{\partial x}(t_n^+)\|^2 \leqq C\cdot \Sigma k^2 \leqq CTk. \tag{7.12}$$

Therefore by using the equality

$$\frac{\partial \bar{\varepsilon}}{\partial x}(t) = \int_{t_n^+}^{t} \frac{\partial \hat{u}_c'}{\partial x}\, dt + \frac{\partial \bar{\varepsilon}}{\partial x}(t_n^+) \qquad \text{for } t \in (t_{n-1}^+, t_n^+],$$

we have $\| \frac{\partial \bar{\varepsilon}}{\partial x} \| \leqq C\sqrt{k}$.

Noticing

$$(\bar{u}_c - \bar{u}_d)(t_n^+) = (\bar{u}_c - \bar{u}_d)(0) + \int_0^{t_{n-1}^+} (\bar{u}_c' - \tilde{D}_t \bar{u}_d)\, dt$$

$$+ \int_{t_{n-1}^+}^{t_n^+} \bar{u}_c'\, dt - \int_0^{t_0^+} \tilde{D}_t \bar{u}_d\, dt,$$

we have $\|| (\bar{u}_c - \bar{u}_d)(t_n^+) \|| \leqq C\sqrt{k}$.

Also we have

$$h(u_0 - u_0^n)^2 \leqq 2h^3 (D_x u_0^n - D_x u_0)^2 + 2h(u_1 - u_1^n)^2 \leqq Ck.$$

Samely we have $h(u_J - u_J^n)^2 \leqq Ck$. Summarizing the results, we have $\|(\bar{u}_c - \bar{u}_d)(t_n^+)\| \leqq C\sqrt{k}$. Finaly, by using the equality

$$\bar{u}_c(t) = \int_{t_n^+}^{t} \bar{u}_c'\, dt + \bar{u}_c(t_n^+),$$

we have $\|\bar{u}_c - \bar{u}_d\| \leqq C\sqrt{k}$ for any t. The proof is complete.

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