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p-BLOCKS AND p-REGULAR CLASSES IN A FINITE GROUP

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Let G be a finite group, p be a prime number and K be an algebraic number field of finite degree which is a splitting field for all subgroups of G. We denote by $\mathfrak{o}$ the ring of $\mathfrak{p}$-integers, where $\mathfrak{p}$ is a prime ideal in K dividing p, and by F the residue class field $\mathfrak{o}/\mathfrak{p}$. Let $C_1, C_2, \cdots, C_r$ be the p-regular classes of G and $D(C_\nu)$ be a defect group of C_ν. We also denote by C_ν the sum of all elements of C_ν in the group ring FG. Let $\chi_1, \chi_2, \cdots, \chi_n$ be the ordinary irreducible characters of G, $B_1, B_2, \cdots, B_t$ be the p-blocks of G and $D(B_\tau)$ be a defect group of B_τ. We denote by $k(B_\tau)$ and $l(B_\tau)$ the numbers of ordinary and modular irreducible characters in B_τ, respectively.

R. Brauer, in [1] and [2], showed the following. *With each B_τ we can associate $l(B_\tau)$ p-regular classes C_ν as follows:*

(i) *Each C_ν is associated with one and only one B_τ.*

(ii) *If we denote by $C_\mu^\tau (\mu=1, 2, \cdots, l(B_\tau))$ the p-regular classes C_ν which are associated with B_τ, then the $\mathfrak{p}$-rank of the $k(B_\tau)\times l(B_\tau)$ matrix $(\chi_i(x_\mu^\tau))$ $(\chi_i \in B_\tau, 1\leqq \mu \leqq l(B_\tau), x_\mu^\tau \in C_\mu^\tau)$ is equal to $l(B_\tau)$.*

We say such a construction a selection of p-regular classes for p-blocks. Let E_τ be the block idempotent of FG corressponding to B_τ. In [8], M. Osima proved that (ii) can be replaced by the following (ii′).

(ii′) *If $C_\mu^\tau (\mu=1, 2, \cdots, l(B_\tau))$ are p-regular classes associated with B_τ, then $C_\mu^\tau E_\tau (\mu=1, 2, \cdots, l(B_\tau))$ are linearly independent.*

After Brauer [3], Iizuka [6], [7] and Osima [8] we describe some results concerning a selection of p-regular classes for p-blocks. Let P be a p-subgroup of G and $X_G(P)$ be the linear subspace of the center $Z(FG)$ of FG spanned by all C_ν with $D(C_\nu) \underset{G}{=} P$. Set

$$V_G(P)=\sum_{S\underset{G}{\leq} P} X_G(S), \quad \tilde{V}_G(P)=\sum_{S\underset{G}{<} P} X_G(S).$$

Then we have

$$V_G(P)=\sum_{\tau} V_G(P)E_\tau, \qquad \tilde{V}_G(P)=\sum_{\tau} \tilde{V}_G(P)E_\tau.$$

Let $m(B_\tau, P)$ be the F-dimension of the factor space $V_G(P)E_\tau/\tilde{V}_G(P)E_\tau$. $\{C_\mu^\tau E_\tau + \tilde{V}_G(P)E_\tau \mid D(C_\mu^\tau) \underset{G}{=} P,\ 1 \leqq \mu \leqq l(B_\tau)\}$ is an F-basis of $V_G(P)E_\tau/\tilde{V}_G(P)E_\tau$ and $m(B_\tau, P)$ is equal te the number of C_μ^τ with $D(C_\mu^\tau) \underset{G}{=} P$. In particular, if P is normal in G, then,

$$(1) \qquad m(B_\tau, P) = \dim_F X_G(P)E_\tau.$$

The Brauer homomorphism ψ of $Z(FG)$ into $Z(FN_G(P))$ induces isomorphisms

$$V_G(P)/\tilde{V}_G(P) \longrightarrow X_{N_G(P)}(P), \quad V_G(P)E_\tau/\tilde{V}_G(P)E_\tau \longrightarrow X_{N_G(P)}(P)\psi(E_\tau).$$

Hence we have

$$(2) \qquad m(B_\tau, P) = \sum_{b \in Bl(N_G(P), B_\tau)} m(b, P),$$

where $Bl(N_G(P), B_\tau)$ is the set of p-blocks b of $N_G(P)$ with $b^G = B_\tau$. If $m(B_\tau, P) \neq 0$, then P is contained in a defect group of B_τ. For $P \underset{G}{=} D(B_\tau)$, we have $m\ (B_\tau, P) = 1$. Let $\{P_1, P_2, \cdots, P_S\}$ be a set of representatives for the classes of conjugate p-subgroups in G. We have

$$(3) \qquad l\,(B_\tau) = \sum_j m(B_\tau, P_j).$$

In the present paper we shall give further results concerning the numbers $m(B_\tau, P_j)$.

Let π be a p-element of G and $C_G(\pi)$ be the centralizer of π in G. Let $Bl(C_G(\pi), B_\tau)$ $(1 \leqq \tau \leqq t)$ be the set of p-blocks b of $C_G(\pi)$ which are associated with B_τ by the Brauer homomorphism, i. e., $b^G = B_\tau$. For each $P_j (1 \leqq j \leqq s)$ let $\mathscr{S}_j(\pi)$ be the set of maximal members of $\{P_j^x \cap C_G(\pi) \mid P_j^x \ni \pi,\ x \in G\}$ and for a p-subgroup Q of $C_G(\pi)$ denote by $J(\pi, Q)$ the set $\{j \mid \mathscr{S}_j(\pi) \ni Q,\ 1 \leqq j \leqq s\}$. The following is our main result.

THEOREM. *Let π be a p-element of G and Q be a p-subgroup of $C_G(\pi)$. For $\tau = 1, 2, \cdots, t$, we have*

$$(i) \quad \sum_{j \in J(\pi, Q)} m(B_\tau, P_j) \leqq \sum_{b \in Bl(C_G(\pi), B_\tau)} m(b, Q),$$

$$(ii) \quad \sum_{\substack{j \in J(\pi, Q) \\ P_j \underset{G}{<} D(B_\tau)}} m(B_\tau, P_j) \leqq \sum_{\substack{b \in Bl(C_G(\pi), B_\tau) \\ D(b) \underset{C_G(\pi)}{\neq} Q}} m(b, Q)$$

PROOF. We may assume $\pi \in Q$. So we have $QC_G(Q) \subseteqq C_G(\pi) \cap N_G(Q) \subseteqq N_G(Q)$. We put $H = C_G(\pi) \cap N_G(Q)$. We can define the Brauer homomorphism ϕ of $Z(FG)$ into $Z(FH)$. $\{\phi(C_\nu) \mid D(C_\nu) \underset{G}{\geqq} Q, 1 \leqq \nu \leqq r\}$ is an F-basis of $\phi(Z_0)$, where $Z_0 = \sum_{\nu=1}^{r} FC_\nu$. Hence $\{\phi(C_\mu^\tau E_\tau) | D(C_\mu^\tau) \underset{G}{\geqq} Q, 1 \leqq \mu \leqq l(B_\tau), 1 \leqq \tau \leqq t\}$ is an F-basis of $\phi(Z_0)$. In particular, $\{\phi(C_\mu^\tau E_\tau) \mid D(C_\mu^\tau) \underset{G}{=} P_j$ for some $j \in J(\pi\ \ Q), 1 \leqq \mu \leqq l(B_\tau)\}$ are linearly independent. By the definition of $J(\pi, Q)$, if $D(C_\mu^\tau) \underset{G}{=} P_j$ $(j \in J(\pi, Q))$ then we see $\phi(C_\mu^\tau) \in X_H(Q)$ and hence $\phi(C_\mu^\tau E_\tau) \in X_H(Q)\phi(E_\tau)$. Therefore we have

$$\sum_{j \in J(\pi, Q)} m(B_\tau, P_j) \leqq \dim_F X_H(Q)\phi(E_\tau).$$

Let $\phi(E_\tau) = \sum_{\rho=1}^{w_\tau} e_{\tau\rho}$ be the decomposition of $\phi(E_\tau)$ into block idempotents of $Z(FH)$. Then $X_H(Q)\phi(E_\tau) = \sum_{\rho=1}^{w_\tau} X_H(Q)e_{\tau\rho}$. By (1), $\dim_F X_H(Q)\phi(E_\tau) = \sum_{\rho=1}^{w_\tau} m(\mathfrak{b}_{\tau\rho}, Q)$, where $\mathfrak{b}_{\tau\rho}$ is the p-block of H corresponding to $e_{\tau\rho}(\rho = 1, 2, \cdots, w_\tau)$. Clearly $(\mathfrak{b}_{\tau\rho}^{C_G(\pi)})^G = \mathfrak{b}_{\tau\rho}^G = B_\tau$. By (2), for $b \in Bl(C_G(\pi), B_\tau)$ with $m(b, Q) \neq 0$, there exists $\rho(1 \leqq \rho \leqq w_\tau)$ such that $b = \mathfrak{b}_{\tau\rho}^{C_G(\pi)}$. Hence, by (2), we have

$$\sum_{\rho=1}^{w_\tau} m(\mathfrak{b}_{\tau\rho}, Q) = \sum_{b \in Bl(C_G(\pi), B_\tau)} m(b, Q).$$

Thus (i) is proved. If $D(C_\mu^\tau) \underset{G}{=} P_j$ and $P_j \underset{G}{<} D(B_\tau)$, then $C_\mu^\tau E_\tau$ is a nilpotent element and hence $\phi(C_\mu^\tau E_\tau)$ is a nilpotent element. For $\mathfrak{b}_{\tau\rho}$ with $D(\mathfrak{b}_{\tau\rho}) = Q$, we have $X_H(Q)e_{\tau\rho} = Fe_{\tau\rho}$. Therefore, if $D(C_\mu^\tau) \underset{G}{=} P_j$ for some $j \in J(\pi, Q)$ and $P_j \underset{G}{<} D(B_\tau)$ then

$$\phi(C_\mu^\tau E_\tau) \in \sum_{\substack{1 \leqq \rho \leqq w_\tau \\ D(\mathfrak{b}_{\tau\rho}) \neq Q}} X_H(Q)e_{\tau\rho}.$$

So we have

$$\sum_{\substack{j \in J(\pi, Q) \\ P_j \underset{G}{<} D(B_\tau)}} m(B_\tau, P_j) \leqq \dim_F \Big(\sum_{\substack{1 \leqq \rho \leqq W_\tau \\ D(\mathfrak{b}_{\tau\rho}) \neq Q}} X_H(Q)e_{\tau\rho} \Big) = \sum_{\substack{1 \leqq \rho \leqq w_\tau \\ D(\mathfrak{b}_{\tau\rho}) \neq Q}} m(\mathfrak{b}_{\tau\rho}, Q).$$

By (2) and the first main theorem on blocks,

$$\sum_{\substack{1 \leqq \rho \leqq w_\tau \\ D(\mathfrak{b}_{\tau\rho}) \neq Q}} m(\mathfrak{b}_{\tau\rho}, Q) = \sum_{\substack{b \in Bl(C_G(\pi), B_\tau) \\ D(b) \underset{C_G(\pi)}{\neq} Q}} m(b, Q).$$

Hence (ii) holds.

COROLLARY. *If, further,* $\pi \in Q$, *then*

$$(i)\quad m(B_\tau, Q) \leqq \sum_{b \in Bl(C_G(\pi), B_\tau)} m(b, Q).$$

Furthermore if Q is not a defect group of B_τ, *then*

$$(ii)\quad m(B_\tau, Q) \leqq \sum_{\substack{b \in Bl(C_G(\pi), B_\tau) \\ D(b) \underset{C_G(\pi)}{\neq} Q}} m(b, Q).$$

In Fujii [5] he gives an equivalent result to (i) in Corollary and (ii) in Corollary implies Theorem in Fujii [4].

$C_G(\pi)$ acts $\mathscr{S}_j(\pi)$ by conjugation; let $s_j(\pi)$ be the number of the orbits. Then we have

PROPOSITION 1. *Under the notations of Theorem the followings hold.*

$$(i)\quad \sum_{j=1}^{s} m(B_\tau, P_j)s_j(\pi) \leqq \sum_{b \in Bl(C_G(\pi), B_\tau)} l(b),$$

$$(ii)\quad \sum_{\substack{1 \leqq j \leqq s \\ P_j \underset{G}{<} D(B_\tau)}} m(B_\tau, P_j)s_j(\pi) \leqq \sum_{b \in Bl(C_G(\pi), B_\tau),} (l(b)-1).$$

PROOF. If $Q_1, Q_2, \cdots, Q_{s'}$ be a set of representatives for the classes of conjugate p-subgroups in $C_G(\pi)$. Then by (i) in Theorem and by (3), we have

$$\sum_{k=1}^{s'} \sum_{j \in J(\pi, Q_k)} m(B_\tau, P_j) \leqq \sum_{k=1}^{s'} \sum_{b \in Bl(C_G(\pi), B_\tau)} m(b, Q_k)$$

$$= \sum_{b \in Bl(C_G(\pi), B_\tau)} l(b).$$

Since $\sum_{k=1}^{s'} \sum_{j \in J(\pi, Q_k)} m(B_\tau, P_j) = \sum_{j=1}^{s} m(B_\tau, P_j)s_j(\pi)$, we have (i). Similarly, since $m(b, D(b))=1$, (ii) in Theorem and (3) yield

$$\sum_{k=1}^{s'} \sum_{\substack{j \in J(\pi, Q_k) \\ P_j \underset{G}{<} D(B_\tau)}} m(B_\tau, P_j) \leqq \sum_{k=1}^{s'} \sum_{\substack{b \in Bl(C_G(\pi), B_\tau) \\ D(b) \underset{C_G(\pi)}{\neq} Q_k}} m(b, Q_k),$$

$$= \sum_{b \in Bl(C_G(\pi), B_\tau)} (l(b)-1).$$

Hence (ii) follows from

$$\sum_{k=1}^{s'} \sum_{\substack{j \in J(\pi, Q_k) \\ P_j \underset{G}{<} D(B_\tau)}} m(B_\tau, P_j) = \sum_{\substack{1 \leq j \leq s \\ P_j \underset{G}{<} D(B_\tau)}} m(B_\tau, P_j) s_j(\pi).$$

If P_j is abelian, then $\mathscr{S}_j(\pi) = \{P_j^x \mid P_j^x \ni \pi, x \in G\}$. Hence we see that if C is the conjuagcy class of G containing π, then $s_j(\pi)$ is equal to the number of $N_G(P_j)$-conjugacy classes of elements of $C \cap P_j$. Then we have the following.

PROPOSITION 2. *Let m_j be the number of $N_G(P_j)$-conjuagcy classes of elements of P_j and $n(B_\tau)$ be the number of conjuagacy classes of subsections associated with B_τ. If $D(B_\tau)$ is abelian, then*

$$k(B_\tau) \geqq \sum_{\substack{1 \leq j \leq s \\ P_j \underset{G}{<} D(B_\tau)}} m(B_\tau, P_j) m_j + n(B_\tau).$$

PROOF. Let $\pi_1, \pi_2, \cdots, \pi_u$ be a set of representatives for the classes of conjugate p-elements in G. Then we have $k(B_\tau) = \sum_{k=1}^{u} \sum_{b \in Bl(C_G(\pi_k), B_\tau)} l(b)$ and we have $n(B_\tau) = \sum_{k=1}^{u} Bl(C_G(\pi_k), B_\tau)^{\#}$, where $Bl(C_G(\pi_k), B_\tau)^{\#}$ deontes the number of elements of $Bl(C_G(\pi_k), B_\tau)$. Therefore, by (ii) in Proposition 1,

$$\begin{aligned} \sum_{k=1}^{u} \sum_{\substack{1 \leq j \leq s \\ P_j \underset{G}{<} D(B_\tau)}} m(B_\tau, P_j) s_j(\pi_k) &\leqq \sum_{k=1}^{u} \sum_{b \in Bl(C_G(\pi_k), B_\tau)} (l(b) - 1) \\ &= k(B_\tau) - n(B_\tau). \end{aligned}$$

As $D(B_\tau)$ is abelin, if $m(B_\tau, P_j) \neq 0$ then P_j is abelian. Hence we have

$$\sum_{k=1}^{u} \sum_{\substack{1 \leq j \leq s \\ P_j \underset{G}{<} D(B_\tau)}} m(B_\tau, P_j) s_j(\pi_k) = \sum_{\substack{1 \leq j \leq s \\ P_j \underset{G}{<} D(B_\tau)}} m(B_\tau, P_j) m_j.$$

This completes the proof.

Let G_p be the set of p-elements of G. We define an equivalence relaltion in G_p as follows. For $x, y \in G_p$, x and y are equivalent if and and only if the cyclic groups $\langle x \rangle$ and $\langle y \rangle$ are conjugate in G. Let $\pi_1, \pi_2, \cdots, \pi_v$ be a complete system of representatives for the equivalence classes. Then by Brauer's permutation lemma B_τ is a union of at least $\sum_{k=1}^{v} \sum_{b \in Bl(C_G(\pi_k), B_\tau)} l(b)$ families of p-conjuagte characters in B_τ. Hence by (i) in Proposition 1, we have the following.

PROPOSITION 3. *Let $f(B_\tau)$ be the number of the families of p-conjugate characters in B_τ. Then we have*

$$(i)\quad f(B_\tau) \geqq \sum_{j=1}^{s} \sum_{k=1}^{v} m(B_\tau, P_j) s_j(\pi_k).$$

In particular, if p^{e_j} is the exponent of P_j then

$$(ii)\quad f(B_\tau) \geqq l(B_\tau) + \sum_{j=1}^{s} m(B_\tau, P_j) e_j.$$

PROOF. (i) follows from (i) in Proposition 1. It is clear that $\sum_{k=1}^{v} s_j(\pi_k) \geqq e_j + 1$. Hence

$$\sum_{j=1}^{s} \sum_{k=1}^{v} m(B_\tau, P_j) s_j(\pi_k) \geqq \sum_{j=1}^{s} m(B_\tau, P_j)(e_j+1)$$
$$= \sum_{j=1}^{s} m(B_\tau, P_j) + \sum_{j=1}^{s} m(B_\tau, P_j) e_j.$$

(ii) follows from (3).

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