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BERGMAN REPRESENTATIVE DOMAINS

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1. Introduction

Let D be a bounded domain in C^n. Let $K_D(z, \bar{t})$ $(z, t \in D)$ be the Bergman kernel function of D.

In this paper, making use of $K_D(z, \bar{t})$ and $T_D(z, \bar{t}) = \dfrac{\partial^2 \log K_D(z, \bar{t})}{\partial z^* \partial t}$, we define relative invariant $T_{D,(p,q)}(z, \bar{t})$ under any pseudo-conformal mapping. Using the relative invariant property of $T_{D,(p,q)}(z, \bar{t})$, we define (p, q)-representative domain, (p, q)-A-representative domain and (p, q)-normal domain. These are generalizations of the Bergman representative domain and normal domain. Moreover we give a necessary and sufficient condition for a domain $\varDelta$ to be a (p, q)-representative domain.

2. Preliminaries

Let D be a bounded domain in C^n. We represent a system of n-holomorphic functions as $w(z)=(w_1(z), \cdots, w_n(z))'$. We define the matrix derivative $\dfrac{dw}{dz}$ of n-dimensional vector function $w(z)=(w_1(z), \cdots, w_n(z))'$ with respect to $z=(z_1, \cdots, z_n)'$ by the formula, denoted by an $n \times n$ matrix $\dfrac{dw(z)}{dz} = \dfrac{\partial w(z)}{\partial z} = \dfrac{\partial}{\partial z} \times w(z)$, where $\dfrac{\partial}{\partial z} = \left(\dfrac{\partial}{\partial z_1}, \cdots, \dfrac{\partial}{\partial z_n}\right)$, $\dfrac{\partial}{\partial z^*} = \left(\dfrac{\partial}{\partial z}\right)^* = \left(\dfrac{\partial}{\partial \bar{z}_1}, \cdots, \dfrac{\partial}{\partial \bar{z}_n}\right)'$.

Vector and matrices marked with the symbol $'$ and $*$ denote the transposed and transposed conjugate vectors or matrices, respectively. We have the following relation

$$dw = \left(\frac{\partial}{\partial z} \times w\right) dz = \frac{dw}{dz}\, dz.$$

A mapping $w(z)$ is called pseudo-conformal in D if the mapping $w(z)$ is one-to one and holomorphic in D.
All integrals appeared in this paper are understood in the sense of Lebesgue.

3. (p, q)-representative domain

Let D be a bounded domain in C^n. Let $K_D(z, \bar{t})$ $(z,\ t \in D)$ be the Bergman kernel function of D. Then it is well-known that if $w=w(z)$ is a pseudo-conformal mapping of a domain D onto D_w, then we have

$$(1)\qquad K_D(z, \bar{t}) = \overline{\left(\det \frac{dw}{dz}\right)}_{z=t} K_{D_w}(w, \bar{\tau}) \left(\det \frac{dw}{dz}\right),$$

where $\tau = w(t)$, $D_w = w(D)$,
also, that if we define

$$T_D(z, \bar{t}) = \frac{\partial^2 \log K_D(z, \bar{t})}{\partial z^* \, \partial t},$$

$T_D(z, \bar{t})$ is relative invariant under pseudo-conformal mapping, that is

$$(2)\qquad T_D(z, \bar{t}) = \left(\frac{dw}{dz}\right)^*_{z=t} T_{D_w}(w, \bar{\tau}) \left(\frac{dw}{dz}\right),$$

where $\tau = w(t)$, $D_w = w(D)$.

Now making use of $K_D(z, \bar{t})$ and $T_D(z, \bar{t})$, we define as follows:

$$K_{D,(p,q)}(z, \bar{t}) = \det\,(K_D^p(z, \bar{t})T_D^q(z, \bar{t}))$$
$$= K_D^{pn}(z, \bar{t}) \det T_D^q(z, \bar{t}) \qquad (p \geqq 2,\ q \geqq 1),$$

$$K_{D,(1,0)}(z, \bar{t}) = K_D(z, \bar{t}),$$

$$T_{D,(p,q)}(z, \bar{t}) = \frac{\partial^2 \log K_{D,(p,q)}(z, \bar{t})}{\partial z^* \, \partial t}$$

Then, we have the following relative invariant $T_{D,(p,q)}(z, \bar{t})$ which plays an important role throughout this paper ([3]).

$$(3)\qquad T_{D,(p,q)}(z, \bar{t}) = \left(\frac{dw}{dz}\right)^*_{z=t} T_{D_w,(p,q)}(w, \bar{\tau}) \left(\frac{dw}{dz}\right),$$

where $\tau = w(t)$, $D_w = w(D)$.

Remarking that $T_{D,(p,q)}(z, \bar{t})$ is relative invariant under any pseudo-conformal mapping, we have the following theorem.

THEOREM 1. *Let $w(z)$ be a pseudo-conformal mapping with the initial conditions $w(t)=\tau$, $\frac{dw(t)}{dz}=E$. Then,*

$$\eta(z)=T^{-1}_{D,(p,q)}(t,\bar{t})\int_t^z T_{D,(p,q)}(z,\bar{t})dz$$
$$=T^{-1}_{D_w,(p,q)}(\tau,\bar{\tau})\int_\tau^w T_{D_w,(p,q)}(w,\bar{\tau})dw$$

is invariant under $w(z)$. Moreover,

$$\eta(t)=0,\ \frac{d\eta(t)}{dz}=T^{-1}_{D,(p,q)}(t,\bar{t})T_{D,(p,q)}(z,\bar{t}).$$

Therefore we call $\Delta_\eta=\eta(D)$ (p,q)-representative domain with center at 0.

PROOF. From the assumption and (3),

$$T^{-1}_{D,(p,q)}(t,\bar{t})\int_t^z T_{D,(p,q)}(z,\bar{t})\,dz$$
$$=\left(\left(\frac{dw}{dz}\right)^*_{z=t}T_{D_w,(p,q)}(\tau,\bar{\tau})\left(\frac{dw}{dz}\right)_{z=t}\right)^{-1}\int_\tau^w\left(\frac{dw}{dz}\right)^*_{=}T_{D_w,(p,q)}(w,\bar{\tau})\frac{dw}{dz}\,dz$$
$$=T^{-1}_{D_w,(p,q)}(\tau,\bar{\tau})\int_\tau^w T_{D_w,(p,q)}(w,\bar{\tau})\,dw.$$

Q. E. D.

REMARK. In the case of $p=1$, $q=0$, Δ_η is the Bergman representative domain ([1]).

THEOREM 2. *A necessary and sufficient condition for a domain Δ to be a (p,q)-representative domain with center at η_0 is*

$$T_{\Delta,(p,q)}(\eta,\bar{\eta}_0)=T_{\Delta,(p,q)}(\eta_0,\bar{\eta}_0)\ \textit{for}\ ^\forall\eta\in\Delta$$

PROOF. By the function $\eta(z)-\eta_0=T^{-1}_{D,(p,q)}(t,\bar{t})\int_t^z T_{D,(p,q)}(z,\bar{t})dz$, D is mapped onto (p,q)-representative domain Δ with center at η_0. Now translate z to η by the pseudo-conformal mapping $\eta=\eta(z)$, then

$$\eta-\eta_0=T^{-1}_{\Delta,(p,q)}(\eta_0,\bar{\eta}_0)\int_{\eta_0}^{\eta}T_{\Delta,(p,q)}(\eta,\bar{\eta}_0)d\eta.$$

Differentiating the above function concering η, we have

$$E_n=T^{-1}_{\Delta,(p,q)}(\eta_0,\bar{\eta}_0)\ T_{\Delta,(p,q)}(\eta,\bar{\eta}_0),$$

i. e., $T_{\Delta,(p,q)}(\eta, \bar{\eta}_0)=T_{\Delta,(p,q)}(\eta_0, \bar{\eta}_0)=$ constant matrix. Conversely if $T_{\Delta,(p,q)}(\eta,\bar{\eta}_0)=T_{\Delta,(p,q)}(\eta_0, \bar{\eta}_0)$, then

$$T^{-1}_{\Delta,(p,q)}(\eta_0, \bar{\eta}_0)\int_{\eta_0}^{\eta} T_{\Delta,(p,q)}(\eta, \bar{\eta}_0)d\eta=\int_{\eta_0}^{\eta} E_n d\eta=\eta-\eta_0. \qquad \text{Q.E.D.}$$

REMARK. In the case of $p=1$, $q=0$, this is the result of Tsuboi ([6]).

Moreover from (1) and (3), we have the following theorem.

THEOREM 3. *(p, q)-representative domain Δ of homogeneous domain D is the Bergman minimal domain with the same center.*

PROOF. Remarking D and Δ are homogeneous,

$$\frac{\det T_{D,(p,q)}(z, \bar{t})}{K_D(z, \bar{t})}=\frac{\det T_{D,(p,q)}(w, \bar{\tau})}{K_D(w,\bar{\tau})}=\frac{\det T_{\Delta,(p,q)}(\eta, \bar{\eta}_0)}{K_\Delta(\eta, \bar{\eta}_0)}$$
$$=\frac{\det T_{\Delta,(p,q)}(\bar{\eta}, \bar{\eta}_0)}{K_\Delta(\bar{\eta}, \bar{\eta}_0)}, \quad (\eta, \bar{\eta}\in\Delta).$$

From Theorem 2, it follows that $K_\Delta(\eta, \bar{\eta}_0)=K_\Delta(\bar{\eta}, \bar{\eta}_0)$. Therefore Δ is minimal domain from the Maschler's theorem ([4]). Q.E.D.

THEOREM 4. *Let D be a homogeneous domain. Then*

$$\frac{\det T_{D,(p,q)}(z, \bar{t})}{K_D(z, \bar{t})}=\textit{constant for } z,\ t\in D.$$

PROOF.

$$\frac{\det T_{D,(p,q)}(z, \bar{t})}{K_D(z, \bar{t})}=\frac{\det T_{D,(p,q)}(w, \bar{\tau})}{K_D(w, \bar{\tau})}$$
$$=\frac{\det T_{\Delta,(p,q)}(\eta, 0)}{K_\Delta(\eta, 0)}=\frac{\det T_{\Delta,(p,q)}(0, 0)}{K_\Delta(0, 0)}=\text{constant},$$

where the first equlity follows from the fact that D is homogeneous, the second equality follows from the fact that $\eta(t)=0$, $\dfrac{d\eta(t)}{dz}=E$, $T_{D,(p,q)}(z, \bar{t})=ET_{\Delta,(p,q)}(\eta, 0)\dfrac{d\eta}{dz}$, $K_D(z, \bar{t})=1\cdot K_\Delta(\eta, 0)\dfrac{d\eta}{dz}$, and the third equality follows from Theorem 2 and Theorem 3. Q.E.D.

Now, changing the initial conditions of $w(z)$ as follows:
$w(t)=\tau, \ \dfrac{dw(t)}{dz}A=A$, where A is a non-zero fixed $n\times m$ matrix $(n>m)$, we have the following theorem.

THEOREM 5. *The following formula is invariant under any pseudo-conformal mapping with the initial conditions* $w(t)=\tau, \ \dfrac{dw(t)}{dz}A=A$.

$$\eta(z)=A(A^*T_{D,(p,q)}(t,\bar{t})A)^{-1}A^*\int_t^z T_{D,(p,q)}(z,\bar{t})dz.$$

Namely,

$$A(A^*T_{D,(p,q)}(t,\bar{t})A)^{-1}A^*\int_t^z T_{D,(p,q)}(z,\bar{t})\,dz$$
$$=A(A^*T_{D_w,(p,q)}(\tau,\bar{\tau})A)^{-1}A^*\int_\tau^w T_{D_w,(p,q)}(w,\bar{\tau})dw.$$

Therefore we call $\Delta=\eta(D)$ $A-(p,q)$*-representative domain.*

PROOF. From (3) and the initial conditions of a pseudo-conformal mapping $w(z)$,

$$A(A^*T_{D,(p,q)}(t,\bar{t})A)^{-1}A^*\int_t^z T_{D,(p,q)}(z,\bar{t})dz$$
$$=A\left(A^*\left(\frac{dw}{dz}\right)^*_{z=t}T_{D_w,(p,q)}(\tau,\bar{\tau})\left(\frac{dw}{dz}\right)_{z=t}A\right)^{-1}A^*\int_\tau^w\left(\frac{dw}{dz}\right)^*_{z=t}T_{D_w,(p,q)}(w,\bar{\tau})\frac{dw}{dz}dz$$
$$=A(A^*T_{D_w,(p,q)}(\tau,\bar{\tau})A)^{-1}A^*\int_\tau^w T_{D_w,(p,q)}(w,\bar{\tau})dw.$$

Q. E. D.

REMARK. In the case of $p=1$, $q=0$, we have so-called A-representative domain and if A is non-singular matrix, then we obtain Bergman representative domain ([2]).

Now we consider the function

$$\frac{d\zeta(z)}{dz}=T_{D,(p,q)}^{-\frac{1}{2}}(t,\bar{t})T_{D,(p,q)}(z,\bar{t}) \quad (z,\ t\in D).$$

From (3), we have

$$d\zeta^*(z)d\zeta(z)=dz^*T^*_{D,(p,q)}(z,\bar{t})\ T^{-1}_{D,(p,q)}(t,\bar{t})\ T_{D,(p,q)}(z,\bar{t})dz$$
$$=dw^*T^*_{D_w,(p,q)}(w,\bar{\tau})T^{-1}_{D_w,(p,q)}(\tau,\bar{\tau})\ T_{D_w,(p,q)}(w,\bar{\tau})dw,$$

where $w=w(z)$ is a pseudo-conformal mapping, $\tau=w(t)$, $D_w=w(D)$.

Namely, $d\zeta^*d\zeta$ is invariant under any pseudo-conformal mapping. Therefore we obtain

$$T^{-\frac{1}{2}}_{D,(p,q)}(t,\bar{t})\ T_{D,(p,q)}(z,\bar{t})=UT^{-\frac{1}{2}}_{D_w,(p,q)}(\tau,\bar{\tau})T_{D_w,(p,q)}(w,\bar{\tau})dw,$$

where U is a constant unitary matrix. Then we have the following theorem.

THEOREM 6. *Let $\zeta=\zeta(z)$ be a pseudo-conformal mapping with the conditions* $\zeta(t)=0$ and $\dfrac{d\zeta(z)}{dz}=T^{-\frac{1}{2}}_{D,(p,q)}(t,\bar{t})T_{D,(p,q)}(z,\bar{t})$, *where* det $T_{D,(p,q)}(z,\bar{t})\neq 0$. *Then with respect to an arbitrary poseudo-conformal mapping $w=w(z)$, $\zeta=\zeta(z)$ and $\Delta=\zeta(D)$ are invariant, neglecting the constant unitary matrices. Therefore we call a unique domain $\Delta=(\zeta(D))$ a (p, q)-normal domain.*

THEOREM 7. *A necessary and sufficient condition for a domain Δ to be a (p, q)-normal domain with center at a fixed point $\zeta_0\in\Delta$ is*

$$T^{\frac{1}{2}}_{\Delta,(p,q)}(\zeta_0,\bar{\zeta}_0)U^*=T_{\Delta,(p,q)}(\zeta,\bar{\zeta}_0)=\textit{constant matrix},$$

where $\zeta_0=\zeta(t)$.

PROOF. The proof of this theorem is almost identical to the proof of Theorem 2. By the function

$$\zeta(z)-\zeta_0=\int_t^z T^{-\frac{1}{2}}_{D,(p,q)}(t,\bar{t})\ T_{D,(p,q)}(z,\bar{t})dz,$$

D is mapped onto Δ which is a (p, q)-normal domain with center at ζ_0. Now translate z to ζ by the pseudo-conformal mapping $\zeta=\zeta(z)$, then

$$\zeta-\zeta_0=\int_t^z T^{-\frac{1}{2}}_{D,(p,q)}(t,\bar{t})\ T_{D,(p,q)}(z,\bar{t})dz$$
$$=U\int_{\zeta_0}^{\zeta} T^{-\frac{1}{2}}_{\Delta,(p,q)}(\zeta_0,\bar{\zeta}_0)\ T_{\Delta,(p,q)}(\zeta,\bar{\zeta}_0)d\zeta,$$

where Δ is a (p, q)-normal domain and U is a constant unitary matrix. Differentiating the above function concering ζ, we obtain

$$E_n = UT^{-\frac{1}{2}}_{\Delta,(p,q)}(\zeta_0, \bar{\zeta}_0)\, T_{\Delta,(p,q)}(\zeta, \bar{\zeta}_0).$$

Conversely if

$$T^{\frac{1}{2}}_{\Delta,(p,q)}(\zeta_0, \bar{\zeta}_0)U^* = T_{\Delta,(p,q)}(\zeta, \bar{\zeta}_0)$$

holds,

$$U\int_{\zeta_0}^{\zeta} T^{-\frac{1}{2}}_{\Delta,(p,q)}(\zeta_0\ \bar{\zeta}_0)\, T_{\Delta,(p,q)}(\zeta,, \bar{\zeta}_0)\, d\zeta = \int_{\zeta_0}^{\zeta} E_n d\zeta = \zeta - \zeta_0. \qquad \text{Q.E.D.}$$

REMARK. In the case of $p=1$, $q=0$, this is the result of Matuura ([5]).

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