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ON THE RELATIVE HYPERBOLICITY OF COMPLEX ANALYTIC SPACES

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Introduction

In this paper we shall give a generalization of the hyperbolicity that was introduced by H. Kaup. Let X and Y be reduced complex analytic spaces and Y' be the one-point compactification of Y, $Y' := Y \cup \{\omega\}$. $\mathrm{Hol}(X, Y)$ and $C(X, Y)$ stand for the set of all holomorphic mappings and the set of all continuous mappings of X into Y respectively. We put $\mathrm{Hol}(X, Y') = \{f \in C(X, Y') : f \in \mathrm{Hol}(X, Y)$ or $f(X) = \{\omega\}\}$. Kaup called an arbitrary complex analytic space Y a hyperbolic space if for every connected complex analytic space X, $\mathrm{Hol}(X, Y')$ is compact with erspect to the compact open topology.

Then a complex analytic space Y is hyperbolic in this sense if and only if Y is taut in the sense of Kobayashi [8] and Wu [11].

Let S be a complex analytic space. If there exists a holomorphic mapping σ_X of X into S, we call (X, σ_X) an S-space and σ_X the projection of X into S. Then we will define the $/S$-hyperbolicity (hyperbolicity over S) of a complex analytic space in the family of all S-spaces. If a complex space S consists of one point, the $/S$-hyperbolicity reduces to the hyperbolicity by Kaup. We shall prove the following properties concerning the $/S$-hyperbolicity.

Let X and Y be $/S$-hyperbolic. Then the fibre product of X and Y over S is /S-hyperbolic. Moreover, a fibre bundle X with the base space S is hyperbolic if and only if X is $/S$-hyperbolic and S is hyperbolic [6], [9].

If a complex analytic space X is $/S$-hyperbolic for an arbitrary complex analytic space S and an arbitrary projection to S, then X is hyperbolic and the converse is also true. Let X be $/S$-hyperbolic. For every point $s \in S$, the fiber $\sigma_X(s)$ is also hyperbolic.

If X is compact, the space $\mathrm{Hol}(X, Y)$ has a canonical complex analytic structure [3]. Moreover if X is compact and hyperbolic, then $\mathrm{Hol}(X, X)$ is a compact complex analytic space. In our case, if Y is compact and $/S$-hyperbolic, then

$\mathrm{Hol}(X, Y)$ is an $/S$-hyperbolic complex analytic space. Moreover, if Y is compact connected and $/S$-hyperbolic, then $\mathrm{Hol}_S(Y, Y)$ is compact and the group $\mathrm{Aut}_S(Y, Y) := \mathrm{Hol}_S(Y, Y) \cap \mathrm{Aut}(Y)$ is finite.

1. Definition of $/S$-hyperbolicity

In this paper, complex analytic spaces are always assumed to be reduced and countable at infinity. Let S and X be arbitrary analytic spaces, σ_X be a holomorphic mapping from X into S. We call a pair (X, σ_X) an $/S$-space. We write X instead of (X, σ_Y) when there is no confusion. In the following, $C(X, Y)$ stands for the set of all continuous mappings of a complex analytic space X into a complex analytic space Y. The set $C(X, Y)$ is a topological space with the compact open topology. Let Y' be the one-point compactification of Y, $Y' := Y \cup \{\omega\}$, where ω is the point at infinity. Y' is assumed to be Y if Y is compact.

DEFINITION 1.1. *Let (X, σ_X) and (Y, σ_Y) be $/S$-spaces. We call a holomorphic mapping f from X to Y a holomorphic mapping from (X, σ_X)* to *(Y, σ_Y) if $\sigma_Y \circ f = \sigma_X$. We denote* $\mathrm{Hol}_S(X,Y) := \{f \in \mathrm{Hol}(X,Y) : \sigma_Y \circ f = \sigma_X\}$ *and* $\mathrm{Hol}_S(X,Y') := \{f \in C(X,Y') : f \in \mathrm{Hol}_S(X, Y)$ or $f(X) = \{\omega\}\}$.

DEFINITION 1.2. *Let (Y, σ_Y), (Z, σ_Z) be $/S$-spaces. We call (Z, σ_Z) an $/S$-subspace of (Y, σ_Y) if there exists a proper injective holomorphic mapping $\tau \in$* $\mathrm{Hol}_S(Z,Y)$.

For the definition of the $/S$-hyperbolicity we prepare the following lemmas.

LEMMA 1.3. *The space* $\mathrm{Hol}_S(X, Y')$ *is compact if and only if, for any compact set $K \subset X$ and any compact set $L \subset Y$, the set $\{f \in \mathrm{Hol}_S(X, Y) : f(K) \cap L \neq \phi\}$ is compact.*

(PROOF) Assume the set $A := \{f \in \mathrm{Hol}_S(X,Y) : f(K) \cap L \neq \phi\}$ is compact for any compact sets K and L. Let $\{K_n\}$ be a sequence of compact sets such that $K_1 \subset K_2 \subset K_3 \subset \cdots \longrightarrow X$ and $K_i \subset \mathring{K}_{i+1}$ for all $i \in N$.

For the convergence of a given sequence $\{f_n\} \subset \mathrm{Hol}_S(X, Y')$, we have the two possibilities:

(I) there exists an integer n_0 such that the set $\{f_n : n \geq n_0\}$ dose not contain the mapping $f(X) = \{\omega\}$;

(II) for any integer n_1, the set $\{f_n : n \geq n_1\}$ contains the mapping $f(X) = \{\omega\}$.

In the case (II), $\{f_n\}$ has a subsequence which converges to the mapping $f(X) = \{\omega\}$, In the case (I) we have one of the following two possibilities:

(I)$'$ for each integer j and each compact set L, there exists an integer n_0 such that $f_n(K_j)\cap L=\phi$ for $n\geq n_0$;

(I)$''$ there exists an integer j_0 and a compact set L_0 such that $f_n(K_{j_0})\cap L_0\neq\phi$, for some n which can be chosen arbitrarily large.

In the case (I)$'$, $\{f_n\}$ converges to the mapping $f(X)=\{\omega\}$. In the case (I)$''$ we have a divergent $\{n_i\}$ such that $f_{n_j}(K_{j_0})\cap L_0\neq\phi$. The sequence $\{f_n\}$ has a subsequence $\{g_j\}$ so that $g_j\longrightarrow f$, $f\in \mathrm{Hol}(X, Y)$. The mapping f belongs to $\mathrm{Hol}_S(X, Y)$ since $\mathrm{Hol}_S(X, Y)$ is closed in $\mathrm{Hol}(X, Y)$. Hence, the space $\mathrm{Hol}_S(X, Y')$ is compact since the sequence $\{f_n\}$ has in any case a subsequence that converges in $\mathrm{Hol}_S(X, Y)$.

Conversely assume the space $\mathrm{Hol}_S(X, Y')$ is compact. Let $\{f_n\}$ be a sequence in A. Since any subsequence of $\{f_n\}$ is not convergent to $f(X)=\{\omega\}$, we may assume $\{f_n\}$ has a subsequence $\{f_{n_j}\}$ which converges in $\mathrm{Hol}(X, Y)$. Let $g_j=f_{n_j}$ and $\lim_{j\to\infty} g_j=f$. There exists $x_j\in K$ such that $g_j(x_j)=y_j\in L$ because $g_j(K)\cap L\neq\phi$. We can take $\lim_{j\to\infty} x_j=x_0$, $x_0\in K$ and $\lim_{j\to\infty} y_j=y_0\in L$ since K and L are compact. So $y_0=\lim_{j\to\infty} g_j(x_j)=f(\lim_{j\to\infty} x_j)=f(x_0)$ and $f(K)\cap L\neq\phi$. Hence $f\in A$; A is compact.

We will define the mapping

$$\Phi:\ X\times \mathrm{Hol}_S(X, Y)\longrightarrow X\times Y$$

by the formula $\Phi(x, f)=(x, f(x))\in X\times Y$ for each $(x, f)\in X\times \mathrm{Hol}_S(X, Y)$. This is called the canonical mapping.

LEMMA 1.4. *Let X be a connected complex space and Φ be the canonical mapping of $X\times \mathrm{Hol}_S(X, Y)$ to $X\times Y$. Then Φ is proper if and only if the set $\{f\in \mathrm{Hol}_S(X, Y)$: $f(K)\cap L\neq\phi\}$ is compact for any compact $K\subset X$ and any compact $L\subset Y$.*

(PROOF) For any compact $K\subset X$ and any compact $L\subset Y$, we put

$$A := \{f\in \mathrm{Hol}_S(X, Y):\ f(X)\cap L\neq\phi\},$$

$$B := \{(x, f):\ (x, f(x))\in K\times L\}.$$

Define the continuous mapping π: $X\times \mathrm{Hol}_S(X, Y)\longrightarrow \mathrm{Hol}_S(X, Y)$ by the formula $\pi(x, f)=f$. Then $\pi(B)=A$ and $\Phi^{-1}(K\times L)=B$. Hence A is compact if Φ is proper.

Conversly, take a sequence $\{p_n\}\subset B$; $p_n=(x_n, f_n)$, $x_n\in K$, $f_n\in \mathrm{Hol}_S(X, Y)$.

Since K is compact, we have a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\lim_{n_j\to\infty} x_{n_j}=x_0$ $\in K$. Since A is compact, we can take a convergent subsequence $\{f_{n'_j}\}$ so that the limit $f \in \mathrm{Hol}_S(X, Y)$. We have $f(x_0) \in L$ and, hence $f(K)\cap L \neq \phi$. So $\lim_{n'_j\to\infty} p_{n_j}=p_0=$ $(x_0, f) \in X\times \mathrm{Hol}_S(X, Y)$ and $(x_0, f(x_0)) \in K\times L$. Hencce $\{p_n\}$ has a subsequence which converges in B. B is compact.

Let X be a connected comlex space and Y be a complex space. By Lemmas 1.3. and 1.4., the following conditions (a), (b) and (c) are equivalent.

(a) $\mathrm{Hol}_S(X, Y')$ is compact with respect to the compact open topology.

(b) For any compact $K\subset X$ and any compact $L\subset Y$,

$$\{f \in \mathrm{Hol}_S(X, Y): f(K)\cap L \neq \phi\}$$

is compact.

(c) The canonical mapping

$$\Phi: X\times \mathrm{Hol}_S(X, Y) \longrightarrow X\times Y$$

is proper.

Now we can define the $/S$-hyperbolicity.

DEFINITION 1.5. *Let (X, σ_X) be a connected $/S$-space and $((Y, \sigma_Y)$ be an $/S$-space. (Y, σ_Y) is called* an *$(X, \sigma_X)/S$-hyperbolic space if (Y, σ_Y) satisfies one of the conditions (a), (b) and (c).*

DEFINITION 1.6. *(Y, σ_Y) is called an $/S$-hyperbolic space if (Y, σ_Y) is (X, σ_X) $/S$-hyperbolic for every connected $/S$-space (X, σ_X).*

If $\sigma_Y(Y)=\{s\}$ for some point $s \in S$ or the projection σ_Y is a constant mapping, then $\mathrm{Hol}_S(X, Y)=\mathrm{Hol}(X, Y)$. Hence our definition reduces the definition of an X-hyperbolicity in Kaup [6] or of tautness in Kobayashi [8] and Wu [11].

If every holomorphic mapping f of X into Y satisfies $\sigma_X(X)\cap\sigma_Y(f(X))=\phi$, in particular if $\sigma_X(X)\cap\sigma_Y(Y)=\phi$, then Y is X-hyperbolic. Trivially S is an X/S-hyperbolic space for all $/S$-space X.

EXAMPLE

Let B be a fibre bundle with the base space S. If the fibre is hyperbolic, then B is $/S$-hyperbolic.

2. Some properties of /S-hyperbolicity

In this section we prove some properties of the /S-hyperbolicity defined in the section **1.**

PROPOSITION 2.1. *Let Y be an/S-space. If Y is /S-hyperbolic, then every /S-subspace Z of Y is also /S-hyperbolic.*

(PROOF) Since Z is an /S-subspace of Y, there exists a proper injective holomorphic mapping τ: $Z \longrightarrow Y$ so that $\sigma_Z = \sigma_Y \circ \tau$. This mapping τ defines a homeomorphism of $\mathrm{Hol}_S(X, Z)$ with $\mathrm{Hol}_S(X, \tau(Z))$. When $\tau(Z)$ is non compact (resp. compact) in Y then, $\mathrm{Hol}_S(X, \tau(Z)')$ (resp. $\mathrm{Hol}_S(X, \tau(Z))$) is closed in $\mathrm{Hol}_S(X, Y')$, Hence $\mathrm{Hol}_S(X, Z')$ is compact and Z is /S-hyperbolic.

COROLLARY 2.2. *Let Z be a complex subspace of Y. If Y is hyperbolic, then Z is also hyperbolic.*

If X is a compact complex space, then $\mathrm{Hol}(X, Y)$ has a complex analytic structure constructed by Douady [3]. For $\mathrm{Hol}_S(X, Y)$,we have the following.

PROPOSITION 2.3. *Let X and Y be /S-spaces. Assume X is compact. Then* $\mathrm{Hol}_S(X, Y)$ *is a complex analytic subvariety of* $\mathrm{Hol}(X, Y)$.

(PROOF) Let σ_Y: $Y \longrightarrow S$ be the projection. Then, for a complex analytic space X

$$\sigma_Y*: \ \mathrm{Hol}(X, Y) \longrightarrow \mathrm{Hol}(X, S)$$

defined by $\sigma_Y*(f) = \sigma_Y \circ f$ for each $f \in \mathrm{Hol}(X, Y)$ is a holomorphic mapping. Since $\sigma_X \in \mathrm{Hol}(X, S)$ and $\sigma_{Y*}^{-1}(\sigma_X) = \mathrm{Hol}_S(X, Y)$, $\mathrm{Hol}_S(X, Y)$ is an analytic subvariety of $\mathrm{Hol}(X, Y)$.

We denote by $\mathrm{Aut}(Y)$ the automorphism group of Y and, if Y is an /S-space, by $\mathrm{Aut}_S(Y)$ the subgroup $\{f \in \mathrm{Aut}(Y)$: $\sigma_Y \circ f = \sigma_Y\}$.

COROLLARY 2.4. *Let Y be a compact /S-space. Then* $\mathrm{Aut}_S(Y)$ *is an analytic svbgroup.*

Let X be an analytic space. If Y is a projective analytic variety, then Hol

(X, Y) is projective algebraic [2]. Hence we have

COROLLARY 2.5. *Let X be an /S-space and Y be a projective algebraic variety over S. Then* $\mathrm{Hol}_S(X, Y)$ *is projective algebraic.*

Let Z be an open subset in complex space X and $i_Z\colon Z \longrightarrow X$ be the injection. We denote the restriction mapping $i_Z^*\colon C(X, Y) \longrightarrow C(Z, Y)$ defined by $i_Z^*(f) = f \circ i_Z$ for $f \in C(X, Y)$.

PROPOSITION 2.6. *Let S be a complex manifold and Y be an /S-space. Then Y is /S-hyperbolic if and only if it is (E^n, f)/S-hyperbolic for all $f \in \mathrm{Hol}(E^n, S)$ and for all $n \in N$.*

(PROOF) It suffices to prove the /S-hyperbolicity of Y when Y is (E^n, f)/S-hyperbolic. Let X be an arbitrary connected /S-space and $\mathscr{F}$ be an arbitrary ultrafilter on $\mathrm{Hol}_S(X, Y')$.

In order to prove the compactness of $\mathrm{Hol}_S(X, Y')$ it is sufficient to prove that for every $x \in X$ we can take an open neighbor hood V of x in X so that $\mathscr{F}|V := i_V^*(\mathscr{F})$ converges. If $\mathscr{F}$ has an infinite number of $f(X) = \{\omega\}$ then it converges to the mapping $f(X) = \{\omega\}$.

Otherwise we devide the argument in two cases.

(I) X is non-singular

For every $x \in X$, we can take a neighborhood V of x in X so that $V \simeq E^n$. Since an arbitrary ultrafilter $\mathscr{F}|V$ converges in $\mathrm{Hol}_S(V, Y')$, i. e. the ultrafilater $\mathscr{F}$ converges in $\mathrm{Hol}_S(X, Y')$.

(II) X is general

For every $x \in X$, we can take an open neighborhood V such that it has a resolution $(\tilde{V}, \rho)$, where ρ is a proper surjective holomorphic mapping $\rho\colon \tilde{V} \longrightarrow V$. Then for arbitrary ultrafilter $\mathscr{F}$, $\rho^*(\mathscr{F}|V)$ is also an ultrafilter bases. Since $\mathrm{Hol}_S(V, Y')$ is compact, $\rho^*(\mathscr{F}|V)$ converges in $\mathrm{Hol}_S(\tilde{V}, Y')$.

Therefore, an ultrafilter $\mathscr{F}|V$ converges in $\mathrm{Hol}_S(V, Y')$.

PROPOSITION 2.7. *If Y it /S-hyperbolic, then every fibre $Y_s := \sigma_Y^{-1}(s)$, $s \in S$ is hyperbolic.*

(PROOF) For any point s, we have $\mathrm{Hol}_S(X \times \{s\}, Y) = \mathrm{Hol}(X, Y_s)$. For any compact sets $K \subset X \times \{s\}$ and $L \subset Y$, we put

$$A := \{f \in \mathrm{Hol}_S(X \times \{s\}, Y)\colon (K) \cap L \neq \phi\}.$$

If we write $K=M\times\{s\}$ for a compact set M in X, we have $A=\{f \in \mathrm{Hol}(X, Y_S): f(M) \cap L_S \neq \phi\}$. Since A is compact. Y_S is X-hyperbolic for any X [6].

PROPOSITION 2.7. *An analytic space* Y *is hyperbolic if and only if it is /S-hyperbolic for any* S.

(PROOF) Consider the case $S=\{s\}$. Since $\mathrm{Hol}_{\{s\}}(X, Y)=\mathrm{Hol}(X, Y)$, we see that Y is hyperbolic if it is /S-hyperbolic. Conversely, if $\mathrm{Hol}_{S_0}(X, Y')$ is not compact for some S_0 and for some (X, σ_X), then $\mathrm{Hol}(X, Y')$ is not compact because $\mathrm{Hol}_{S_0}(X, Y') \subset \mathrm{Hol}(X, Y')$ and $\mathrm{Hol}_{S_0}(X, Y)$ is closed in $\mathrm{Hol}(X, Y)$.

3. Hyperbolicity of fibre product

In this section we will consider the hyperbolicity of the fibre product of /S-hyperbolic spaces.

Let $(X\times Y)'$ be the one point compactification of a product space $X\times Y$. We put $X'=X\cup\{\omega_1\}$, $Y'=Y\cup\{\omega_2\}$ and $(X\times Y)'=X\times Y\cup\{\omega\}$. We define the mapping σ: $X'\times Y'\longrightarrow(X\times Y)'$ by the formula $\sigma(x, y)=(x, y)$ for $(x, x) \in X\times Y$ and $\sigma(x, y)=\omega$ for $(x, y) \in X'\times Y'-X\times Y$. This mapping is surjective and continuous.

PROPOSITION 3.1. *Let X, Y and Z be /S-spaces. If X, Y are Z/S-hyperbolic, then the fibre product $X\times_S Y$ of X and Y over S is Z/S-hyperbolic.*

(PROOF) The mapping σ induces the continuous mapping σ_* of $C(Z, X')\times C(Z, Y')$ into $C(Z, (X\times Y)')$. The restriction

$$\sigma_*: \mathrm{Hol}_S(Z, X)\times\mathrm{Hol}_S(Z, Y)\longrightarrow\mathrm{Hol}(Z, X\times_S Y)$$

is injective and surjective by the definition of fibre product.

Since the mapping σ_* sends the all of $\{\omega_1\}\times\mathrm{Hol}_S(Z, Y)$, $\mathrm{Hol}_S(Z, Y)\times\{\omega_2\}$ and $\{\omega_1\}\times\{\omega_2\}$ to $\{\omega\}$, the image $\sigma_*(\mathrm{Hol}_S(Z, X')\times\mathrm{Hol}_S(Z, Y))$ is equals to $\mathrm{Hol}_S(Z, X\times_S Y)\cup\{\omega\}$, which is $\mathrm{Hol}_S(Z,(X\times_S Y)')$. For an arbitrary ultrafilter $\mathscr{F}$ on $\mathrm{Hol}_S(Z, (X\times_S Y)')$ and an arbitrary $z \in Z$, we can take open neighborhoods U, V_1 and V_2 of z in Z and ultrafilters $\mathscr{F}'$ on $\mathrm{Hol}_S(Z, X')$, $\mathscr{F}''$ on $\mathrm{Hol}_S(Z, Y')$ such that

$$\mathscr{F}|U=\sigma_*((\mathscr{F}'|V_1)\times(\mathscr{F}''|V_2)).$$

The ultrafilter $\mathscr{F}|U$ is converges because ultrafilters $\mathscr{F}'|V_1$ and $\mathscr{F}''|V_2$ are convergent. Hence $\mathrm{Hol}_S(Z,(X\times_S Y)')$ is compact. $X\times_S Y$ is Z/S-hyperbolic.

COROLLARY 3.2. *Let X, Y and Z be complex analytic spaces. If X, Y are Z-hyperbolic, then X×Y is also Z-hyperbolic.*

4. Hyperbolicity of fibre bundle

PROPOSITION 4.1. *Le X be a fibre bundle with the base space S. If the fibre* $X_s:=\sigma_X^{-1}(s)$ *for each* $s\in S$ *is hyperbolic, then X is /S-hyperbolic.*

(PROOF) Let Z be a connected $/S$-space, $\mathscr{F}$ be an urtrafilter on $\mathrm{Hol}_S(Z, X)$ and the mapping σ_X be the projection of X into S. Assume first that $\lim\mathscr{F}(z)=x$, $x\in X$. Since X is a fibre bundle, there is an open neighborhood V such that $V\times X_s\simeq\sigma_X^{-1}(V)$, for every $s\in S$. Put $U:=\sigma^{-1}{}_Z(V)$, then $\mathscr{F}|U\subset\mathrm{Hol}_S(U, V\times X_s)$, V can be assumed to be a hyperbolic subdomain.

Since V, X_s is hyperbolic, the ultrafilter $\mathscr{F}|U$ converges. Then putting $\lim\mathscr{F}|U=f_U$, we have $f_U\in\mathrm{Hol}_S(U, \sigma_X^{-1}(U))$ or $f_U=\omega_U$. For another open neighborhood V' we have the mapping $f_{U'}$ in the same way for the ultrafilter $\mathscr{F}|V'$. Assume $V\cap V'\neq\phi$ and let $\phi_{VV'}$ be the transition mapping, Then $f_U|U\cap U'=\phi_{VV'}(f_{U'}|U\cap U')$. Thus a holomorphic mapping f is defined on $U\cup U'$. So we have a holomorphic mapping f defined on Z which is a limit of $\mathscr{F}$.

THEOREM 4.2. *Let X be a fibre bundle with the base space S. Then X is hyperbolic if and only if it is /S-hyperbolic and S is hyperbolic.*

(PROOF) [6] and Proposition 4.1.

We can prove the next theorem in the same way.

THEOREM 4.3. *Let* (X, σ_X) *be an /S-space. If for every point* $s\in S$, *there exists an open neighborhood U of s such that* $\sigma_X^{-1}(U)$ *is hyperbolic, then X is /S-hyperbolic.*

5. Mappings to /S-hyperbolic space

PROPOSITION 5.1. *Let S be hyperbolic and* (Y, σ_Y) *be /S-hyperbolic. Then there is no nonconstant holomorphic mapping of a complex plane C into Y.*

(PROOF) For a holomorphic mapping $f \in \mathrm{Hol}(C, Y)$, put $\phi=\sigma_Y\circ f$. Since S is hyperbolic, ϕ is a constant mapping [6]. Thus Im $\sigma_Y\circ f=\{s\}$ for some $s \in S$. Hence $f(C)=\sigma_X^{-1}(s)$. By Proposition 2.7. the fibre $\sigma_Y^{-1}(s)$ is hyperbolic. So f is a constant mapping.

Moreover assume Y and S are manifolds, and Y is compact we know that Y is hyperbolic when S is hyperbolic [1], [10]

We will generalize Proposition 5.1. For $/S$-spaces (X, σ_X) and (Y, σ_Y), we put $\Gamma_s(X, Y):=\{f \in \mathrm{Hol}_S(X, Y): f_s$ is constant$\}$, where $f_s:=f\,|\,\sigma_X^{-1}(s)$, $s \in S$.

PROPOSITION 5.2. *If* $\overline{\mathrm{Aut}_S(X)} \cap \Gamma_s(X, X)\neq\phi$ *for some* $s \in S$ *and* Y *is* X/S-*hyperbolic, then* $\mathrm{Hol}_S(X, Y)=\Gamma_s(X, Y)$.

(PROOF) Consider an arbitrary mapping $f \in \mathrm{Hol}_S(X, Y)$ and points $\alpha\neq\beta$, α, $\beta \in X_s$. Since $\overline{\mathrm{Aut}_S(X)} \cap \Gamma_s(X, X) \neq \phi$, there is a sequence $\{\psi_n\}$ in $\mathrm{Aut}_S(X)$ such that $\lim \psi_n=\psi \in \Gamma_s(X, X)$.

Let $\psi(X_s)=\{\gamma\}$, $\gamma \in X_s$. We can take neighborhoods U_n of γ such that $\bigcap_n \overline{U_n}=\{\gamma\}$ and $U_n \supset U_{n+1}\supset\cdots$. For any compact set K in X_s and any integer k, there exists an integer n_0 such that $\psi_n(K) \subset U_K$, for all $n\geq n_0$. Then, putting $\psi_n^{-1}=\phi_n$, $\alpha_n=\psi_n(\alpha)$ and $\beta_n=\psi_n(\beta)$. There is subsequence $\{\phi_{n_k}\}$ of $\{\phi_n\}$ such that $\phi_{n_k}(\beta_{n_k})=\beta$, $\phi_{n_k}(\alpha_{n_k})=\alpha$. Thus putting $g_{n_k}=f\circ\phi_{n_k}$, we can assume $g_{n_k}\longrightarrow g \in \mathrm{Hol}_S(X, Y')$ since Y is X/S-hyperbolic. For an arbitrary compact neighborhood L of γ, there is an integer k_0 so that $g_{n_k}(L) \supset g_{n_k}(U_k)=f\circ\phi_{n_k}(U_k) \supset f(K)$ for all $k>k_0$. Thus $g_{n_k}(L) \cap f(K) \neq \phi$ for all n_k, i. e. $g \in \mathrm{Hol}_S(X, Y)$. Hence $f(\beta)=\lim f\circ\phi_{n_k}(\beta_{n_k})=\lim g_{n_k}(\beta_{n_k})$ and $f(\alpha)=\lim f\circ\phi_{n_k}(\alpha_{n_k})=\lim g_{n_k}(\alpha_{n_k})$. Since $\alpha_{n_k}\longrightarrow\gamma$ and $\beta_{n_k}\longrightarrow\gamma$, we have $f(\alpha)=f(\beta)$. Hence $f \in \Gamma_s(X, Y)$.

REMARK. *The complex plane* C *satisfies the condition* $\overline{\mathrm{Aut}_S(X)} \cap \Gamma_s(X, X)\neq\phi$. *Hence Proposition* 5.2. *implies Propesition* 5.1.

In the case the projection σ_X is constant or $S=\{s\}$ we have

COROLLARY 5.3. *If* $\overline{\mathrm{Aut}(X)} \cap \Gamma(X, X) \neq \phi$ *and* Y *is* X-*hyerbolic, then* $\mathrm{Hol}(X, Y)=\Gamma(X, Y)$.

PROPOSITION 5.4. *Let* Y *be an* $/S$-*hyperbolic space and* X *be a connected complex space. If there exists a proper discrete holomorphic mapping* τ *of* X *into* Y, *then* X *is* $/S$-*hyperbolic.*

(PROOF) Let Z be a connected $/S$-space. Let $\mathscr{F}$ be an ultrafilter an Hol_S (Z, X'). Then $\tau_*(\mathscr{F})$ is an ultrafilter on $\mathrm{Hol}_S(Z\ Y')$. Since Y is $/S$-hyperbolic, $\tau_*(\mathscr{F})$ converges in $\mathrm{Hol}_S(Z, Y)$ or to ω_Y, ω_Y being the point at infinity of Y'. Assume $\tau_*(\mathscr{F})$ converges to a holomorphic mapping $f \in \mathrm{Hol}_S(Z, Y)$. For a point $z \in Z$, put $\lim \mathscr{F}(z)=x$, $x \in X$. Since τ is proper discrete, we can ta ke anopen neighborhood U of x such that, for an arbitrary open neighborhood V of x and an arbitrary connected subset M which is contained in $\tau^{-1}(\tau(V))$, we have $M \cap U \neq \phi$ implies $M \subset V$. Thus, for the open neighborhood U of x, there exists an $F \in \mathscr{F}$ so that $F(z) \in U$. Since F is continuous there exists an open connected neighborhood W so that $F(W) \subset V$.

PROPOSITION 5.5. *Let X be a connected compact $/S$-space. If Y is an $/S$-hyperbolic complex space, then each irreducible component of* $\mathrm{Hol}_S(X, Y)$ *is an analytic cover of a subvariety of Y*.

(PROOF) The canonical mapping Φ: $X\times \mathrm{Hol}_S(X, Y) \longrightarrow X\times Y$ is proper and $\mathrm{Hol}_S(X, Y)$ is a complex space. For any fixed point x_0 in X we see that Φ_{x_0}: $(x_0, f) \longrightarrow (x_0, f(x_0))$ is proper and holomorphic. Identifying $\{x_0\}\times \mathrm{Hol}_S(X, Y)$ with $\mathrm{Hol}_S(X, Y)$ and $\{x_0\}\times Y$ with Y, we see that the mapping Φ_{x_0}: $\mathrm{Hol}_S(X, Y) \longrightarrow Y$ is proper and holomorphic. So the set $\Phi_{x_0}^{-1}(y)=\{f \in \mathrm{Hol}_S(X, Y)\colon f(x_0)=y\}$ is finite [6], [10]. Thus, by the proper mapping theorem, $\Phi_{x_0}(\mathrm{Hol}_S(X, Y))$ is a complex subspace of Y and $\dim \mathrm{Hol}_S(X, Y)=\dim \Phi_{x_0}(\mathrm{Hol}_S(X, Y))$. Let W be one of irreducible components of $\mathrm{Hol}_S(X, Y)$. Then the mapping Φ_{x_0}: $W \longrightarrow \Phi_{x_0}(W)$ is a covering mapping. Since Y is $/S$-hyperbolic, Φ_{x_0}-image of every component of $\mathrm{Hol}_S(X, Y)$ is also $/S$-hyperbolic.

COROLLARY 5.6. *Let X be a connected compact complex space and Y be a hyperbolic space. Then every irreducible component of* $\mathrm{Hol}(X, Y)$ *is an analytic cover over a subvariety of Y.*

(PROOF) Put $S=\{s\}$ in Proposition 5.5.

COROLLARY 5.7. *Let X and Y be compact complex spaces. If Y is hyperbolic, then* $\mathrm{Hol}(X, Y)$ *is hyperbolic. If Y is $/S$-hyperbolic, then* $\mathrm{Hol}_S(X, Y)$ *is $/S$-hyperbolic.*

Let Y be a compact $/S$-space. Assume Y is $/S$-hyperbolic. Then $\mathrm{Hol}_S(Y, Y)$ is compact. Since Y is compact, $\mathrm{Aut}(Y)$ is open in $\mathrm{Hol}(Y, Y)$. $\mathrm{Aut}_S(Y)$ is also open in $\mathrm{Hol}_S(Y, Y)$. Since Y is $/S$-hyperbolic, $\mathrm{Aut}_S(Y)$ is closed in $\mathrm{Hol}_S(Y, Y)$.

Hence $\mathrm{Aut}_S(Y)$ is a union of components of $\mathrm{Hol}_S(Y, Y)$. As a complex subgroup of Aut (Y), $\mathrm{Aut}_S(Y)$ is a complex Lie group.

PROPOSITION 5.8. *Let Y be a connected /S-space. If Y is compact and /S-hyperbolic, then* $\mathrm{Aut}_S(Y)$ *is a finite group.*

(PROOF) Since Y is /S-hyperbolic, $\mathrm{Hol}_S(Y, Y)$ is compact. By Proposition 5.5. the mapping Φ_{x_0}: $\mathrm{Hol}_S(Y, Y) \longrightarrow Y$ is proper, fibre discrete and holomorphic. Then the restriction of Φ_{x_0} onto $\mathrm{Aut}_S(Y)$ defines an analytic covering of $\mathrm{Aut}_S(Y)$ onto $\Phi_{x_0}(\mathrm{Aut}_S(Y))$. But the space $\Phi_{x_0}(\mathrm{Aut}_S(Y))$ is hyperbolic since $Y_{\sigma_{Y(x_0)}}$ is hyperbolic. Then $\mathrm{Aut}_S(Y)$ is hyperbolic, by Proposition 5.5. Moreover, $\mathrm{Aut}_S(Y)$ is compact in $\mathrm{Hol}_S(Y, Y)$.

Hence $\mathrm{Aut}_S(Y)$ is a finite group [4].

PROPOSITION 5.9. *Let Y be a compact and /S-hyperbolic space. Then any surjection in* $\mathrm{Hol}_S(Y, Y)$ *is an automorphism.*

(PROOF) Let $\phi \in \mathrm{Hol}_S(Y, Y)$ be a surjection. There exists a sequence $\{n_k\}$ of integers so that $\lim \phi^{n_k}=f$, $f \in \mathrm{Hol}_S(Y, Y)$. Then f is also a surjection. We can take a sequence $\{n_k\}$ such that $m_k :=n_{k+1}-n_k > 0$, $m_k \longrightarrow +\infty$ and $\lim \phi^{m_k}= g \in \mathrm{Hol}_S(X, Y)$. For an arbitrery point x in X we have $\phi^{m_k}(\phi^{n_k}(x))=\phi^{n_{k+1}}(x)$. Hence $g(f(x))=f(x)$. Since f is a surjection $g=id_Y \in \mathrm{Aut}_S(Y)$. As $\mathrm{Aut}_S(Y)$ is open in $\mathrm{Hol}_S(Y, Y)$, there exists an integer m so that $\phi^m \in \mathrm{Aut}_S(Y)$. Hence $\phi \in \mathrm{Aut}_S(Y)$.

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