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CLASSIFICATION OF INVARIANT COMPLEX STRUCTURES ON $SL(3, R)$

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§1. Introduction

By an invariant complex structure on a real Lie group we mean such a structure that all left multiplications are biholomorphic transformations of the group. The invariant complex structure on a compact semisimple Lie group or, more generally, on a compact homogeneous space is fully described by H. C. Wang [7]. In the noncompact case it is known that there exists always an invariant complex structure if the group is reductive and of even dimension [3]. But the problem of classification of all invariant complex structures on a noncompact group is very complicated. In the previous paper [4], we treated the group $GL(2, R)$. It was seen that the group $GL(2, R)$ is always Stein with any invariant complex structure.

The purpose of this paper is to classify such structures on $SL(3, R)$. The reason to take $SL(3, R)$ into consideration here is that it is simple to handle by the lowdimensionality and it has, the author believes, the general nature that is shared by the real semisimple group having no compact Cartan subgroup.

We prove that the invariant complex structures on $SL(3, R)$ are classified into one series and two discrete ones (Theorem 1). The structure appearing in the series is general in the sense that every reductive group of even dimension possesses structures of this kind. And it is easy to describe this structure. It is not Stein (Theorem 2). But, unfortunately, the author does not know yet what kind of properties are characteristic for the remaining two cases.

§2. Preliminaries

Let G be a connected Lie group and $\mathfrak{g}$ be its Lie algebra identified with the set of all left invariant vector fields on G.

DEFINITION 1. *A complex structure on G is called invariant if this makes every* *left multiplication a biholomorphic transformation of G.*

Let J be a complex structure tensor on G. It is an endomorphism of the tangent space of G and satisfies

$$J^2 = -1, \tag{1}$$
$$[X, Y]+J[JX, Y]+J[X, JY]-[JX, JY]=0. \tag{2}$$

If J is invariant, it is determined by the value of J at the unit, which we denote also by J. For a given J, let $\mathfrak{m}$ be the set in the complexification $\mathfrak{g}^C$ of $\mathfrak{g}$ defined by

$$\mathfrak{m} = \{X+iJX;\ \ X \in \mathfrak{g}\}.$$

Denote by σ the complex conjugation of $\mathfrak{g}^C$ with respect to $\mathfrak{g}$. Then, using (1) and (2), we have

PROPOSITION 1. *$\mathfrak{m}$ is a complex subalgebra and satisfies*

$$\mathfrak{m}+\sigma(\mathfrak{m})=\mathfrak{g}^C;\ \mathfrak{m} \cap \sigma(\mathfrak{m})=\{0\}. \tag{3}$$

We call a complex subalgebra satisfying (3) an *invariant complex subalgebra* with respect to σ. Trivially every invariant complex subalgebra defines a complex structure on the real form $\mathfrak{g}$.

DEFINITION 2. *Two invariant complex structures J_1 and J_2 are said to be equivalent if there exists an automorphism x of $\mathfrak{g}$ so that $xJ_1=J_2x$.*

In terms of invariant complex subalgebras the equivalence in this sense is stated as follows.

PROPOSITION 2. *Let $\mathfrak{m}_1$ and $\mathfrak{m}_2$ be invariant complex subalgebras corresponding to complex structures J_1 and J_2 respectively. Then J_1 is equivalent to J_2 if and only if there exists an automorphism x of $\mathfrak{g}^C$ such that $x\sigma=\sigma x$ and $x\mathfrak{m}_1=\mathfrak{m}_2$.*

With these definitions the classification of invariant complex structures is reduced to the classification of equivalence classes of invariant complex subalgebras. For the later use we cite the following

PROPOSITION 3. [1]. *Any maximal proper subalgebra of a complex semisimple Lie algebra is either parabolic or semisimple.*

§ 3. Classification of invariant complex subalgebras of $\mathfrak{sl}(3, C)$

1. Let $\mathfrak{g}=\mathfrak{sl}(3, \boldsymbol{R})$. The complexification $\mathfrak{g}^C$ is $\mathfrak{sl}(3, C)$. Let $\mathfrak{h}^C$ be some Cartan subalgebra of $\mathfrak{g}^C$. The root space consists of six roots. We denote by α, β the two simple positive roots and by $\gamma=\alpha+\beta$ the third positive root. For a root δ, e_δ denotes the corresponding root vector. Let $H_\delta=[e_\delta, e_{-\delta}]$ for positive δ. Root vectors are supposed to be chosen so as they form a Weyl basis.

Let $\mathfrak{m}$ be a complex subalgebra of $\mathfrak{g}^C$ of dimension four. Let $\mathfrak{n}$ be a proper maximal subalgebra containing $\mathfrak{m}$. Then, by Proposition 3, $\mathfrak{n}$ is semisimple or parabolic. But the algebra $\mathfrak{sl}(3, C)$ cannot contain a semisimple subalgebra of dimension greater than three, hence $\mathfrak{n}$ must be parabolic. Let $\mathfrak{p}_\alpha=\mathfrak{h}^C+C\{e_\alpha, e_\beta, e_\gamma, e_{-\alpha}\}$ and $\mathfrak{p}_\beta=\mathfrak{h}^C+C\{e_\alpha, e_\beta, e_\gamma, e_{-\beta}\}$. Since any parabolic subalgebra contains a σ-stable Cartan subalgebra [9, Theorem 2.6], we may assume, taking an appropriate ordering of roots, $\mathfrak{n}$ is conjugate to one of $\mathfrak{p}_\alpha$ and $\mathfrak{p}_\beta$ under an inner automorphism of $\mathfrak{g}$. Hence, we may assume $\mathfrak{m}\subset\mathfrak{p}_\alpha$ or $\mathfrak{m}\subset\mathfrak{p}_\beta$.

2. It is known that $\mathfrak{g}=\mathfrak{sl}(3, \boldsymbol{R})$ has two conjugate classes of Cartan subalgebras [6]. We take, as their representatives, $\mathfrak{h}_1$ and $\mathfrak{h}_2$ defined as follows. Let E_{ij} be the matrix with 1 in the (i, j)-th component and 0 in the others. Let

$$
\begin{aligned}
&h_1=E_{11}-E_{22},\ h_2=E_{22}-E_{33},\\
&H_1=(i/2)(E_{21}-E_{12}),\ H_2=(-i/2)(E_{11}+E_{22}-2E_{33}),
\end{aligned}
\tag{4}
$$

and set $\mathfrak{h}_1=\boldsymbol{R}\{h_1, h_2\}$ and $\mathfrak{h}_2=\boldsymbol{R}\{iH_1, iH_2\}$. The latter is a fundamental Cartan subalgebra [8]. In order to classify invariant complex subalgebras, we can assume $\mathfrak{h}^C$ is one of $\mathfrak{h}_1^C$ and $\mathfrak{h}_2^C$ by the definition of equivalences (definition 3). We call $(C1)$ the case $\mathfrak{h}^C=\mathfrak{h}_1^C$ and $(C2)$ the case $\mathfrak{h}^C=\mathfrak{h}_2^C$. To do the calculation, let us fix root vectors. In the case $(C1)$

$$
\begin{aligned}
&H_\alpha=h_1,\ H_\beta=h_2,\ e_\alpha=E_{12},\ e_\beta=E_{23},\ e_\gamma=E_{13}\\
&e_{-\alpha}=E_{21},\ e_{-\beta}=E_{32},\ e_{-\gamma}=E_{31},
\end{aligned}
\tag{5}
$$

and in the case $(C2)$

$$
\begin{aligned}
&H_\alpha=H_1+iH_2,\ H_\beta=H_1-iH_2,\ e_\alpha=(1/\sqrt{2})(E_{13}+iE_{23}),\\
&e_\beta=(1/\sqrt{2})(E_{31}+iE_{32}),\ e_\gamma=(1/2)(i(E_{12}+E_{21})+(E_{11}-E_{22})),\\
&e_{-\alpha}=(1/\sqrt{2})(E_{31}-iE_{32}),\ e_{-\beta}=(1/\sqrt{2})(E_{13}-E_{23}),\\
&e_{-\gamma}=(1/2)(E_{11}-E_{22})-i(E_{12}+E_{21})).
\end{aligned}
\tag{6}
$$

These vectors are taken so as the Cartan matrix is $\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$. In fact the value of roots are given by

(7) $\alpha(H_\alpha)=2,\ \alpha(H_\beta)=-1,\ \beta(H_\alpha)=-1,\ \beta(H_\beta)=2.$

For the sake of calculations afterwards we list the bracket relations between root vectors.

(8) $[e_\alpha, e_\beta]=e_\gamma,\ [e_\beta, e_{-\gamma}]=e_{-\alpha},\ [e_{-\gamma}, e_\alpha]=e_{-\beta},$
$[e_{-\alpha}, e_\gamma]=e_\beta,\ [e_\gamma, e_{-\beta}]=e_\alpha,\ [e_{-\alpha}, e_{-\beta}]=-e_{-\gamma}.$

In the case $(C1)$ the basis vectors are all real and, in the case $(C2)$, the conjugation σ is given as follows:

(9) $\sigma(e_\alpha)=e_{-\beta},\ \sigma(e_\beta)=e_{-\alpha},\ \sigma(e_\gamma)=e_{-\gamma},\ \sigma(H_\alpha)=-H_\beta.$

3. In this subsection we carry out the calculation to classify invariant complex subalgebras. The result is put together in Proposition 6 and equivalences will be established in the next section.

First note that, in the case $(C1)$, any complex subalgebra contained in $\mathfrak{p}_\alpha$ or $\mathfrak{p}_\beta$ do not satisfy the condition (3) in Proposition 1, since $\mathfrak{p}_\alpha+\sigma(\mathfrak{p}_\alpha)$ is a proper subspace of $\mathfrak{g}^C$ due to the fact that root vectors are chosen to be real. Hence we have

PROPOSITION 4. *There is no invariant complex subalgebra in the case* $\mathfrak{h}^C=\mathfrak{h}_1^C$.

In the following we treat the case $\mathfrak{h}^C=\mathfrak{h}_2^C$. Let x be the involutive automorphism of $\mathfrak{g}$ defined by $x(X)=-{}^tX$ for $X\in\mathfrak{g}$. In our situation x satisfies

(10) $x(H_\alpha)=H_\beta,\ x(e_{\pm\alpha})=-e_{\pm\beta},\ x(e_{\pm\gamma})=-e_{\pm\gamma}.$

Hence $x(\mathfrak{p}_\alpha)=\mathfrak{p}_\beta$. This means

PROPOSITION 5. *Any invariant complex subalgebra contained in* $\mathfrak{p}_\beta$ *is equivalent to one of invariant complex subalgebras contained in* $\mathfrak{p}_\alpha$.

So we have only to consider the case $\mathfrak{m}\subset\mathfrak{p}_\alpha$. Let $\mathfrak{p}^C$ denote the space spanned by all root vectors and $\mathfrak{q}_\alpha=\mathfrak{p}^C\cap\mathfrak{p}_\alpha=C\{e_\alpha, e_\beta, e_\gamma, e_{-\alpha}\}$. Set $\mathfrak{m}_0=\mathfrak{m}\cap\mathfrak{h}^C$ and $\mathfrak{m}_1=\mathfrak{m}\cap\mathfrak{p}^C$. By (9), both $\mathfrak{h}^C$ and $\mathfrak{p}^C$ are σ-stable. Hence, in order for $\mathfrak{m}$ to satisfy the codition (3), it is necessary that dim $\mathfrak{m}_0=0$ or 1 and dim $\mathfrak{m}_1=2$ or 3. So we devide the calculation into three cases, namely

(A) dim $\mathfrak{m}_1=3$,
(B) dim $\mathfrak{m}_1=2$ and dim $\mathfrak{m}_0=0$,
(C) dim $\mathfrak{m}_1=2$ and dim $\mathfrak{m}_0=1$.

Case (A). As a basis of $\mathfrak{m}_1$ we take linearly independent vectors X, Y and Z. Since $\sigma(e_\beta)=e_{-\alpha}$, $\mathfrak{m}_1$ does not contain $C\{e_{-\alpha}, e_\beta\}$. Hence we have two subcases according to the types of elements in $\mathfrak{m}_1$. Namely, (A-1) $\mathfrak{m}_1\equiv C\{e_\alpha, e_{-\alpha}, e_\gamma\}$ (mod. e_β) or (A-2) $\mathfrak{m}_1\equiv C\{e_\alpha, e_\beta, e_\gamma\}$ (mod. $e_{-\alpha}$). In the case (A-1), $\mathfrak{m}_1$ contains $X=e_{-\alpha}+ae_\beta$ and $Y=e_\gamma+be_\beta$; $a, b\in C$. Then it also contains $[X, Y]=e_\beta$, and $e_{-\alpha}=X-ae_\beta$. Hence $\mathfrak{m}_1\supset C\{e_{-\alpha}, e_\beta\}$, which is a contradiction.

Let us consider the case (A-2). In this case $\mathfrak{m}$ is spanned by vectors $U=h+ae_{-\alpha}(h\in\mathfrak{h}^C)$, $X=e_\alpha+be_{-\alpha}$, $Y=e_\beta+ce_{-\alpha}$ and $Z=e_\gamma+de_{-\alpha}$. Then $[U, X]=\alpha(h)(e_\alpha-be_{-\alpha})-aH_\alpha$. Assume $a\neq0$, then h must be a multiple of H_α. Hence we may set $U=H_\alpha+ae_{-\alpha}$. Since $\alpha(H_\alpha)=2$, $[U, X]=2X-aU$. We have $a^2=4b$. On the other hand $[Y, Z]=ce_\beta$ means $c=0$. Then from $[X, Y]=e_\gamma$, we have $d=0$. Hence $\mathfrak{m}$ is

1) $C\{H_\alpha+ae_{-\alpha}, e_\alpha+(a^2/4)e_{-\alpha}\ e_\beta, e_\gamma\}$, $a\neq0$.

If a=0, $[U, X]=\alpha(h)\ (e_\alpha-be_{-\alpha})$. Hence $\alpha(h)=0$ or $b=0$. And $\mathfrak{m}$ is one of

2) $C\{H_\alpha+2H_\beta, e_\alpha+ae_{-\alpha}, e_\beta, e_\gamma\}$, $a\in C$,

3) $C\{h, e_\alpha, e_\beta, e_\gamma\}$, $h\in\mathfrak{h}^C$.

Case (B). Since dim $\mathfrak{m}_1=2$ we take two independent vectors X and Y in $\mathfrak{m}_1$. Since dim $\mathfrak{m}=4$, $\mathfrak{m}$ contains, other than X and Y, two more independent vectors which can be written as $U=H_\alpha+u$ and $V=H_\beta+v$ for $u, v\in\mathfrak{q}_\alpha$. According to types of X and Y we have the following six subcases: (B-1) $\mathfrak{m}_1\equiv C\{e_\alpha, e_{-\beta}\}$ (mod. e_β, e_γ), (B-2) $\mathfrak{m}_1\equiv C\{e_\alpha, e_\gamma\}$ (mod. $e_{-\alpha}, e_\beta$), (B-3) $\mathfrak{m}_1\equiv C\{e_\alpha, e_\beta\}$ (mod. $e_{-\alpha}, e_\gamma$), (B-4) $\mathfrak{m}_1\equiv C\{e_{-\alpha}, e_\beta\}$ (mod. e_α, e_γ), (B-5) $\mathfrak{m}_1\equiv C\{e_{-\alpha}, e_\gamma\}$ (mod. e_α, e_β) and (B-6) $\mathfrak{m}_1\equiv C\{e_\beta, e_\gamma\}$ (mod. $e_\alpha, e_{-\alpha}$).

(B-1): $\mathfrak{m}$ contains $U=H_\alpha+u$, $V=H_\beta+v$ for $u, v\in\mathfrak{q}_\alpha$ and $X=e_\alpha+ae_\gamma+ce_\beta$, $Y=e_{-\alpha}+be_\beta+de_\gamma$. Then $[X, Y]=H_\alpha+be_\gamma-ae_\beta$. This must equal to U, since it contains H_α. Then $[X, U]=-2e_\alpha-2ae_\gamma+ce_\beta$. This is necessarily equal to $-2X$. Hence $c=0$. From the equality $[Y, U]=2e_{-\alpha}+2be_\beta-de_\gamma$ we have $d=0$. Let $V=H_\beta+ee_\beta+fe_\gamma$. Then $[U, V]=(2a-e)e_\beta+(f-b)e_\gamma$ means $f=b$ and $e=2a$. Therefore $\mathfrak{m}$ is $C\{H_\alpha+be_\gamma-ae_\beta, H_\beta+2ae_\beta+be_\gamma, e_\alpha+ae_\gamma, e_{-\alpha}+be_\beta\}$. But it is seen, using (9), that this algebra does not satisfy the condition (3).

(B-2): $\mathfrak{m}_1$ contains $X=e_\alpha+be_{-\alpha}+de_\beta$ and $Y=e_\gamma+ce_{-\alpha}+ae_\beta$. The constant c may be chosen to be zero. Otherwise, this case reduces to (B-1). Then $[X, Y]=ae_\gamma+be_\beta$, hence $b=a^2$. We can assume $X=e_\alpha+a^2e_{-\alpha}+de_\beta$ and $Y=e_\gamma+ae_\beta$. Let $U=H_\alpha+ce_{-\alpha}+be_\beta$ and $V=H_\beta+ee_{-\alpha}+fe_\beta$. By the equality $[U, Y]=e_\gamma+(c-a)e_\beta$, we have $c=2a$, and by $[V, Y]=e_\gamma+(2a+e)e_\beta$, $e=-a$. Then we have $[U,X]=3(ab-d)e$ (mod. U, X, Y). Hence $d=ab$. By $[U, V]=-(f+2b)e_\beta$, $f=-2b$. Therefore $\mathfrak{m}$ is

4) $C\{H_\alpha+2ae_{-\alpha}+be_\beta,\ H_\beta-ae_{-\alpha}-2be_\beta,\ e_\alpha+a^2e_{-\alpha}+abe_\beta,\ e_\gamma+ae_\beta\}$.

(B-3) to (B-5): Similar considerations show that these cases either lead to a contradiction or reduces to another cases.

(B-6): We can assume $\mathfrak{m}$ contains $X=e_\beta$ and $Y=e_\gamma$. Let $U=H_\alpha+ae_\alpha+de_{-\alpha}$ and $V=H_\beta+ce_\alpha+be_{-\alpha}$. Then $[U, V]=(2c+a)e_\alpha-(d+2a)e_{-\alpha}+((ab-cd)H_\alpha$ means $[U, V]=(ab-cd)U$. Solving this equation we get $ad+1=0$ and $c=a(ab-1)$. Here, $ab-cd=2ab-1\neq 0$ by the assumption dim $\mathfrak{m}_0=0$. Hence we have

5) $C\{H_\alpha+ae_\alpha-(1/a)e_{-\alpha},\ H_\beta+a(ab-1)e_\alpha+be_{-\alpha},\ e_\beta,\ e_\gamma\}$.

Case (*C*). Our assumptions are dim $\mathfrak{m}\cap\mathfrak{h}^C=1$ and dim $\mathfrak{m}\cap\mathfrak{q}_\alpha=2$. We can take as a basis of the algebra $\mathfrak{m}$, four vectors H, $U=H'+Z$, X and Y where $H, H'\in\mathfrak{h}^C$ which are linearly independent and $X, Y, Z\in\mathfrak{q}_\alpha$; X and Y are linearly independent and $Z\neq 0$. Then $[H, H'+Z]=[H, Z]$, $[H, X]$ and $[H, Y]$ belong to $\mathfrak{q}_\alpha$. These are linearly dependent since dim $\mathfrak{m}_1=2$. Hence, choosing Z appropriately, we may assume $[H, Z]=0$ or $[H, X]=0$. Since ker $\alpha=C\{H_\alpha+2H_\beta\}$, ker $\beta=C\{2H_\alpha+H_\beta\}$ and ker $\gamma=C\{H_\alpha-H_\beta\}$ and since $[H, ae_\alpha+be_{-\alpha}+ce_\beta+de_\gamma]=a\alpha(H)e_\alpha-b\alpha(H)e_{-\alpha}+c\beta(H)e_\beta+d\gamma(H)e_\gamma$, H must be one of $H_\alpha+2H_\beta$, $2H_\alpha+H_\beta$ and $H_\alpha-H_\beta$ up to a non-zoro costant. Therefore there arises the following six cases to consider. (C-1) $H=H_\alpha+2H_\beta$, $U=H_\alpha+ae_\alpha+be_{-\alpha}$, (C-2) $H=2H_\alpha+H_\beta$, $U=H_\beta+ae_\beta$, (C-3) $H=H_\alpha-H_\beta$, $U=H_\alpha+H_\beta+ae_\gamma$, (C-4) $H=H_\alpha+2H_\beta$, $X=ae_\alpha+be_{-\alpha}$, (C-5) $H=2H_\alpha+H_\beta$, $X=e_\beta$ and (C-6) $H=H_\alpha-H_\beta$, $X=e_\gamma$. The first three cases correspond to the assumption $[H, Z]=0$. The other three correspond to $[H, X]=0$.

(C-1): X and Y are of the form $W=ce_\alpha+de_{-\alpha}+ee_\beta+fe_\gamma$. $[H, W]=3(ee_\beta+fe_\gamma)$ implies X and Y are one of $ce_\alpha+de_{-\alpha}$ and $ee_\beta+fe_\gamma$. So we have three subcases: (1) $X=c_1e_\alpha+d_1e_{-\alpha}$, $Y=c_2e_\alpha+d_2e_{-\alpha}$, (2) $X=e_1e_\beta+f_1e_\gamma$, $Y=e_2e_\beta+f_2e_\gamma$, (3) $X=ce_\alpha+de_{-\alpha}$, $Y=ee_\beta+fe_\gamma$. In the subcase (1) we may assume $X=e_\alpha$ and $Y=e_{-\alpha}$. Then $[X, Y]=H_\alpha$ also belongs to $\mathfrak{m}$, which contradicts to dim $\mathfrak{m}_0=1$. In the subcase (2) we may assume $X=e_\beta$ and $Y=e_\gamma$. Then $\mathfrak{m}$ is

6) $C\{H_\alpha+2H_\beta,\ H_\alpha+ae_\alpha+be_{-\alpha},\ e_\beta,\ e_\gamma\}$ for $|a|+|b|\neq 0$.

Let us consider the subcase (3). In this case $[U, X]=2(ce_\alpha-de_{-\alpha})+(ad-bc)H_\alpha$. If $ad-bc=0$, then $ce_\alpha-de_{-\alpha}$ belongs to $\mathfrak{m}$. Hence $c=0$ or $d=0$. When $c=0$, we may assume $X=e_\alpha$ and $U=H_\alpha+be_{-\alpha}$. The equality $[X, Y]=ee_\gamma$ shows $e=0$ and $Y=e_\gamma$. Now $[U, Y]=e_\gamma+be_\beta$ implies $b=0$. Hence U belongs to $\mathfrak{m}$, which contradicts to dim $\mathfrak{m}_0=1$. When $d=0$, the similar argument leads to a contradiction. Therefore $ad-bc\neq 0$ and $[U, X]=(ad-bc)U$. Hence $2c=(ad-bc)a$ and $-2d=(ad-bc)b$. We have $ab=-1$ and $c=a^2d$. Since $d=0$ implies $c=0$, we may assume $d=1/a$ and $c=a$. Then $[X, Y]=aee_\gamma+(f/a)e_\beta$. Comparing this with Y, we have $f=\pm ae$. Hence $Y=e_\beta\pm ae_\gamma$. When $Y=e_\beta+ae_\gamma$, $[U, Y]=-2(e_\beta-ae_\gamma)$ by $ab=-1$. Since $a\neq 0$, this cannot occur. Now we have $Y=e_\beta-ae_\gamma$ and the algebra

7) $C\{H_\alpha+2H_\beta,\ H_\alpha+ae_\alpha-(1/a)e_{-\alpha},\ ae_\alpha+(1/a)e_{-\alpha},\ e_\beta-ae_\gamma\}$.

(C-2): In this case, for $W=ce_\alpha+de_{-\alpha}+ee_\beta+fe_\gamma$, $[H, W]=3(ce_\alpha-de_{-\alpha}+fe_\gamma)$. Hence X and Y are one of $ce_\alpha+fe_\gamma$ and $e_{-\alpha}$. We have two subcases: (1) $X=ce_\alpha+fe_\gamma$ and $Y=e_{-\alpha}$, (2) $X=e_\alpha$ and $Y=e_\gamma$. In the case (1), $[U, X]=-ce_\alpha+(f-ac)e_\gamma$ implies $c(2f-ac)=0$. If $c=0$, then $\mathfrak{m}$ contains $X=fe_\gamma$, $Y=e_{-\alpha}$ and $[X, Y]=-fe_\beta$, which is a contradiction. Hence $2f=ac$. Therefore $\mathfrak{m}$ is $C\{2H_\alpha+H_\beta,\ H_\beta+ae_\beta\ 2e_\alpha+ae_\gamma,\ e_{-\alpha}\}$, for $a\neq 0$. But this does not satisfy the condition (3). In the case (2) we have

8) $C\{2H_\alpha+H_\beta,\ H_\beta+ae_\beta,\ e_x,\ e_\gamma\}$, $a\neq 0$.

(C-3): For $W=ce_\alpha+\mathrm{d}e_{-\alpha}+ee_\beta+fe_\gamma$, $[H, W]=3(ce_\alpha-de_{-\alpha}-ee_\beta)$. Hence we have two subcases: (1) $X=de_{-\alpha}+ee_\beta$ and $Y=e_\alpha$, (2) $X=e_{-\alpha}$ and $Y=e_\beta$. In the case (1), $[U, X]=-de_{-\alpha}-(e+ad)e_\beta$ implies $d=0$. Here $X=e_\beta$. Then $\mathfrak{m}$ contains $[X, Y]=-e_\gamma$, which implies dim $\mathfrak{m}_0=2$ and a contradiction. In the case (2) $\mathfrak{m}$ is $C\{H_\alpha-H_\beta,\ H_\alpha+H_\beta+ae_\gamma,\ e_{-\alpha},\ e_\beta,\}$ $a\neq 0$. But this does not satisfy the condition (3).

(C-4) to (C-6): By the same reasoning used for (C-1) to (C-3) we see that these cases are led to a contradiction or reduces to other cases.

From above considerations we have eight possibilities: **1**) to **8**). We rearrange these algebras and introduce notations as follows:

$$
\begin{array}{lll}
 & \mathrm{I}_\lambda & C\{h(\lambda),\ e_\alpha,\ e_\beta,\ e_\gamma\} \text{ for } \lambda\in C\cup\{\infty\}, \text{ where } h(\lambda)=H_\alpha+\lambda H_\beta \text{ and} \\
 & & h(\infty)=H_\beta, \\
 & \mathrm{II}_a & C\{H_\alpha+2H_\beta,\ e_\alpha+ae_{-\alpha},\ e_\beta,\ e_\gamma\}, \\
 & \mathrm{III}_a & C\{H_\alpha+2H_\beta,\ H_\alpha+ae_\alpha-(1/a)e_{-\alpha},\ e_\beta-ae_\gamma,\ ae_\alpha+(1/a)e_{-\alpha}\},\ a\neq 0, \\
(11) & \mathrm{IV}_{a,b} & C\{H_\alpha+2H_\beta,\ H_\alpha+ae_\alpha+be_{-\alpha},\ e_\beta,\ e_\gamma\}, \\
 & \mathrm{V}_a & C\{H_\alpha+ae_{-\alpha},\ e_\alpha+(a^2/4)e_{-\alpha},\ e_\beta,\ e_\gamma\}, \\
 & \mathrm{VI}_a & C\{2H_\alpha+H_\beta,\ H_\beta+ae_\beta,\ e_\alpha,\ e_\gamma\},\ a\neq 0, \\
 & \mathrm{VII}_{a,b} & C\{H_\alpha+ae_\alpha-(1/a)e_{-\alpha},\ H_\beta+a(ab-1)e_\alpha+be_{-\alpha},\ e_\beta,\ e_\gamma\},\ a\neq 0, \\
 & \mathrm{VIII}_{a,b} & C\{H_\alpha+2ae_{-\alpha}+be_\beta,\ H_\beta-ae_{-\alpha}-2be_\beta,\ e_\alpha+a^2e_{-\alpha}+abe_\beta,\ e_\gamma+ae_\beta\}.
\end{array}
$$

PROPOSITION 6. *Every invariant complex subalgebra is equivalent to one of the classes listed in* (11). *Parameters are subject to* $|\lambda|\neq 1$ *for* I_λ, $|(ab-1)|\neq 1$ *for* $\mathrm{VII}_{a,b}$ *and* $b\neq\bar{a}$ *for* $\mathrm{VIII}_{a,b}$.

PROOF. It remains to check the condition (3). It is easily seen using (9) and we get conditions above.

REMARK 1. We have identities between these algebras: $\mathrm{V}_0=\mathrm{I}_\lambda$, $\mathrm{II}_0=\mathrm{I}_{1/2}$, $\mathrm{IV}_{0,b}=\mathrm{II}_b$, $\mathrm{VIII}_{0,b}=\mathrm{VI}_{-2b}$. So we understand in the sequel $a\neq 0$ for II_a, V_a, $\mathrm{IV}_{a,b}$ and $\mathrm{VIII}_{a,b}$.

§ 4. Equivalences

We will examine possible equivalences or inequivalences between invariant complex subalgebras I_λ to VIII.

Let us recall the automorphism x defined in the previous section by $x(X)=-{}^tX$ for $X\in\mathfrak{g}^C$.

PROPOSITION 7. *The automorphism* x *gives the equivalence of* I_λ *with* $\mathrm{I}_{1/\lambda}$ *and the equivalence of* VI_a *with* $\mathrm{IV}_{-a,0}$.

PROOF. By definition $x(h(\lambda))=\lambda h(1/\lambda)$ and $x(h(\infty))=h(0)$. Since the space of positive root vectors is stable under x, we have the first equivalence. The automorphism x sends VI_a to $C\{H_\alpha+2H_\beta,\ H_\alpha-ae_\alpha,\ e_\beta,\ e_\gamma\}$ which is $\mathrm{IV}_{-a,0}$. So the second equivalence.

We define another automorphism y_k for a nonzero constant k. It fixes every element of $\mathfrak{h}^C$ and transforms root vectors as $y_k(e_\alpha)=ke_\alpha$, $y_k(e_\beta)=(1/\bar{k})e_\beta$, $y_k(e_\gamma)=(k/\bar{k})e_\gamma$, $y_k(e_{-\alpha})=(1/k)e_{-\alpha}$, $y_k(e_{-\beta})=\bar{k}e_{-\beta}$ and $y_k(e_{-\gamma})=(\bar{k}/k)e_{-\gamma}$. Then y_k is a real automorphism. We have

PROPOSITION 8. *There exist equivalences*:

(1) $\mathrm{II}_a=\mathrm{II}_1$, (2) $\mathrm{III}_a=\mathrm{III}_1$, (3) $\mathrm{IV}_{a,b}=\mathrm{IV}_{1,ab}(a\neq 0)$, (4) $V_a=V_2$,
(5) $\mathrm{VII}_{a,b}=\mathrm{VII}_{1,ab}(a\neq 0)$, (6) $\mathrm{VIII}_{a,b}=\mathrm{VIII}_{1,b/\bar{a}}(a\neq 0)$.

PROOF. The automorphism y_k sends $e_\alpha \pm e_{-\alpha}$ to $ke_\alpha \pm (1/k)e_{-\alpha}$, and $e_\beta - e_\gamma$ to $(1/\bar{k})\ (e_\beta - ke_\gamma)$. This implies $y_a(\mathrm{III}_1)=\mathrm{III}_a$. Similarly we can see other equivalences making use of y_k.

Due to this proposition we will introduce simplified notations:

(12) $\mathrm{II}=\mathrm{II}_1$, $\mathrm{III}=\mathrm{III}_1$, $\mathrm{IV}_\mu=\mathrm{IV}_{1,\mu}$, $\mathrm{V}=\mathrm{V}_2$, $\mathrm{VI}=\mathrm{VI}_1$, $\mathrm{VII}_\lambda=\mathrm{VII}_{1,1-(1/\lambda)}$,
$\mathrm{VII}_\infty=\mathrm{VII}_{1,1}$, $\mathrm{VIII}_\kappa=\mathrm{VIII}_{1,\kappa}$.

Parameters λ, κ are subject to conditions $|\lambda|\neq 1$, $\kappa\neq 1$ respectively.

In order to see further equivalences we will prepare a lemma. Let $\mathfrak{a}$ be a $2n$-dimensional real Lie algebra and $\mathfrak{a}^C$ be its complexification. σ denotes the complex conjugation with respect to $\mathfrak{a}$. Let $\mathfrak{b}$ and $\mathfrak{b}'$ be complex subalgebras of dimension n such that $\mathfrak{b}+\sigma\mathfrak{b}=\mathfrak{a}^C$ and $\mathfrak{b}'+\sigma\mathfrak{b}'=\mathfrak{a}^C$. Let $\{X_i\}$ ($\{X_i'\}$) be a basis of $\mathfrak{b}$ ($\mathfrak{b}'$) and c_{ij}^k (c'^k_{ij}) be structure constants: $[X_i, X_j]=\Sigma c_{ij}^k X_k$ ($[X_i', X_j']=\Sigma c'^k_{ij}X_k'$). Define constants d_{ij}^k, e_{ij}^k (d'^k_{ij}, e'^k_{ij}) by $[X_i, \sigma X_j]=\Sigma d_{ij}^k X_k+\Sigma e_{ij}^k \sigma X_k$ ($[X_i', \sigma X_j']=\Sigma d'^k_{ij}X_k'+\Sigma e'^k_{ij}\sigma X_k'$). Then we have

LEMMA 1. *Assume* $c_{ij}^k=c'^k_{ij}$, $d_{ij}^k=d'^k_{ij}$ *and* $e_{ij}^k=e'^k_{ij}$. *Then there exists an automorphism of* $\mathfrak{a}$ *such that its extension to* $\mathfrak{a}^C$ *defines an isomorphism of* $\mathfrak{b}$ *with* $\mathfrak{b}'$.

PROOF. It is enough to see that the mappig ϕ defined by $\phi(X_i)=X_i'$ and $\phi(\sigma X_i)=\sigma X_i'$ is an automorphism. Since $\phi\sigma=\sigma\phi$, ϕ is real.

In order to apply this lemma we need to fix a basis of the algebra. We do this in the following way.

I_λ $U=h(\lambda)$, $X=e_\alpha$, $Y=e_\beta$, $Z=e_\gamma$,
II $U=H_\alpha+2H_\beta$, $X=e_\alpha+e_{-\alpha}$, $Y=e_\beta$, $Z=e_\gamma$,
III $U=(1/2)\ (H_\alpha+2H_\beta-3(e_\alpha+e_{-\alpha}))$, $X=(1/2)\ (H_\alpha+2H_\beta+e_\alpha+e_{-\alpha})$
$Y=-H_\alpha-e_\alpha+e_{-\alpha}+e_\beta-e_\gamma$, $Z=H_\alpha+e_\alpha-e_{-\alpha}+e_\beta-e_\gamma$,
IV_μ $(1+\mu\neq 0)$ $U=H_\alpha+2H_\beta+3e_\beta$, $X=H_\alpha+e_\alpha+\mu e_{-\alpha}+e_\gamma$, $Y=\mu' e_\beta$,
$Z=\mu' e_\gamma$, where $\mu'^2=1+\mu$,
(13) IV_{-1} $U=H_\alpha+2H_\beta+3e_\beta$, $X=H_\alpha+e_\alpha-e_{-\alpha}-e_\beta+e_\gamma$, $Y=e_\beta$, $Z=-e_\beta+e_\gamma$,

V $\quad U=H_\alpha+2e_{-\alpha}+(1/4)e_\beta,\ X=(1/2)\ (e_\alpha-e_{-\alpha}-H_\alpha-e_\beta-e_\gamma),$
$\quad Y=2e_\beta,\ Z=e_\beta+e_\gamma,$

VII$_\lambda$ $\quad U=\lambda H_\beta+H_\alpha-(2-\lambda)e_{-\alpha}-(1-2\lambda)e_\beta\ (\lambda\neq\infty)$ or $H_\alpha+e_{-\alpha}+2e_\beta(\lambda=\infty),$
$\quad X=H_\alpha+e_\alpha-e_{-\alpha}-e_\beta+e_\gamma,\ Y=e_\beta,\ Z=e_\gamma-e_\beta,$

VIII$_\kappa$ $\quad U=H_\beta+2H_\alpha+3e_{-\alpha}-3Y,\ X=-H_\beta+e_{-\alpha}+2\kappa e_\beta-Z,$
$\quad Y=\kappa'(e_\alpha-H_\alpha-e_{-\alpha}-e_\beta-e_\gamma),\ Z=\kappa'(e_\alpha-H_\alpha-e_{-\alpha}+(1-2\kappa)(e_\beta+e_\gamma)),$
where $\kappa'=1/(2(\bar{\kappa}-1))$.

The bracket relations with respect to these bases are listed:

(14)

	$[U, X]$	$[U, Y]$	$[U, Z]$	$[X, Y]$	$[X, Z]$	$[Y, Z]$
I$_\lambda$, VII$_\lambda$	$(2-\lambda)X$	$(2\lambda-1)Y$	$(\lambda+1)Z$	Z	0	0
I$_\infty$, VII$_\infty$	$-X$	$2Y$	Z	Z	0	0
V	$2X$	$-Y$	Z	Z	0	0
II, III, IV$_\mu(\mu\neq-1)$ VIII$_\kappa$	0	$3Y$	$3Z$	Z	Y	0
IV$_{-1}$	0	$3Y$	$3Z$	Z	0	0

For the use later we also list the bracket relations of bases vectors and their σ-conjugates. We denote by A^σ the σ-conjugate of A.

(15)

	$[U, U^\sigma]$	$[U, X^\sigma]$	$[U, Y^\sigma]$	$[U, Z^\sigma]$	$[X, X^\sigma]$	$[Y, Y^\sigma]$
I$_\lambda$	0	$(1-2\lambda)X^\sigma$	$(\lambda-2)Y_\sigma$	$-(\lambda+1)Z^\sigma$	0	0
II	0	$-3X^\sigma+6Y$	0	$-3Z^\sigma$	$Z-Z^\sigma$	0

	$[X, Y^\sigma]$	$[X, Z^\alpha]$	$[Y, Z^\sigma]$	$[Z, Z^\sigma]$
I$_\lambda$	$(U+\lambda U^\sigma)/(1-\lambda\bar{\lambda})$	$-X^\sigma$	Y^σ	$((1-\bar{\lambda})U-(1-\lambda)U^\sigma)/(1-\lambda\bar{\lambda})$
II	$-(U+2U^\sigma)/2$	$-X^\sigma+Y$	Y^σ	$(U-U^\sigma)/3$

For the algebra III we have

(16) $[U, U^\sigma]=(3/16)\ \{-2(U-U^\sigma)-3(Y-Y^\sigma)-9(Z-Z^\sigma)\},$
$[U, X^\sigma]=(3/16)\ \{-2-3U^\sigma-8X-16X^\sigma+7Y-Y^\sigma-3Z-3Z^\sigma\},$
$[U, Y^\sigma]=3Y/2,\ [U, Z^\sigma]=-U+U^\sigma-3(Y+Y^\sigma+Z+3Z^\sigma)/4,$

$[X, X^\sigma] = \{-(10/3)(U-U^\sigma)-8(X-X^\sigma)-9(Y-Y^\sigma)+5(Z-Z^\sigma)\}/16,$
$[X, Y^\sigma] = (1/3)(U+2U^\sigma)-Y^\sigma-Z/2-Z^\sigma,$
$[X, Z^\sigma] = (2U-7U^\sigma)-X^\sigma/2+3(Y-Y^\sigma)/4-(Z+Z^\sigma)/4,$
$[Y, Y^\sigma]=0, [Y, Z^\sigma]=2(Y^\sigma-Z^\sigma),$
$[Z, Z^\sigma] = -(2/3)(U-U^\sigma)+2(X-X^\sigma)+(Y-Y^\alpha)+(Z-Z^\sigma).$

PROPOSITIN 9. *There exist following equivalences.*
(a) $\mathrm{IV}^\mu(\mu\neq-1)=\mathrm{II}$, (b) $\mathrm{IV}_{-1}=\mathrm{I}_2$, (c) $\mathrm{V}=\mathrm{I}_0$,
(d) $\mathrm{VII}_\lambda=\mathrm{I}_\lambda$, $|\lambda|\leqq\infty$, (e) $\mathrm{VIII}_\kappa=\mathrm{II}$.

PROOF. All equivalences are proved by Lemma 1, because, with respect to the bases (13), the structure constants of the algebras themselves and the constants of the bracket relations of the algebras with their conjugates are the same as those of the corresponding algebras. We omit the calculation of these constants since it is routine and tediously lengthy.

Now we have reduced the examination to the consideration of the algebras I_λ, II and III. Let $\mathfrak{m}$ be one of these algebras. Denote by $[\mathfrak{m}, \mathfrak{m}]$ the commutator of $\mathfrak{m}$. We easily see by the bracket relations (14) that dim $[\mathfrak{m}, \mathfrak{m}]$ is 3 for $\mathfrak{m}=\mathrm{I}_\lambda$ when $\lambda\neq2, 1/2$, but it is 2 for remaining algebras. So we have

PROPOSITIN 10. *The algebra* I_λ, $\lambda\neq2, 1/2$, *is not equivalent to any of* $\mathrm{I}_2=\mathrm{I}_{1/2}$, II *and* III.

Moreover we have

PROPOSITIN 11. *The algebras* I_λ *and* I_μ *are not equivalent unless* $\lambda=1/\mu$ *or* $\lambda=\mu$.

PROOF. It is sufficient to see the algebras I_λ and I_μ are not isomorphic as abstract Lie algebras unless $\lambda=1/\mu$ or $\lambda=\mu$. This is seen by a simple computation using the table (14).

We have also

PROPOSITIN 12. *The algebra* I_2 *is not equivalent to* II *nor to* III.

PROOF. By computation using the table (14) we can see the algebra I_2 is not isomorphic to any of II and III as an abstract Lie algebra.

Next we will see the inequivalence between II and III. For that purpose we prepare

LEMMA 2. *Let* $\mathfrak{m}=C\{U, X, Y, Z\}$ *be a 4-dimensional complex Lie algebra with bracket relations*: $[U, X]=0$, $[U, Y]=3Y$, $[U, Z]=3Z$, $[X, Y]=Z$, $[X, Z]=Y$ *and* $[Y, Z]=0$. *Then any automorphism* ϕ *of* $\mathfrak{m}$ *has the following form*: $\phi(U)=U+3\kappa Y+3\nu Z$, $\phi(X)=\varepsilon(X+\nu Y+\kappa Z)$, $\phi(Y)=\lambda Y+\mu Z$ and $\phi(Z)=\varepsilon(\mu Y+\lambda Z)$, *where* $\varepsilon=\pm 1$ *and* $\kappa, \lambda, \mu, \nu \in C$, $\lambda^2 \neq \mu^2$.

PROOF. By calculation.

PROPOSITIN 13. *The algebra* II *is not equivalent to the algebra* III.

PROOF. If there exists a real automorphism x of $\mathfrak{g}^C$ such that $x\text{III}=\text{II}$, then, identifying II with III as Lie algebras by the bases chosen (14), we see $x|_{\text{III}}$ is an automorphism of III. Write $\phi=x|_{\text{III}}$. The possibility of ϕ is known in Lemma 2. Hence the problem is to look out for an automorphism ϕ of III such that a mapping x from $\mathfrak{g}^C$ to $\mathfrak{g}^C$ defined by $x=\phi$ on III and by $x=\sigma\phi\sigma$ on σIII becomes a real automorphism of $\mathfrak{g}^C$ sending III to II. In order to do this it is necessary to know the bracket relations of vectors in III and vectors in σIII, which are listed in (16). We rewrite a necessary part of these relations: $[Y, Y^\sigma]=0$, $[Y, Z^\sigma]=2(Y^\sigma-Z^\sigma)$. Assume there exists a required automophism ϕ. Set $Y_1=\phi(Y)$ and $Z_1=\phi(Z)$. By Lemma 2, $Y_1=\lambda Y+\mu Z$ and $Z_1=\varepsilon(\mu Y+\lambda Z)$ for some constants $\varepsilon, \lambda, \mu$. Since Y_1 and Z_1 must satisfy the relations corresponding to II, $[Y_1, Y_1^\sigma]=0$ and $[Y_1, Z_1^\sigma]=Y_1^\sigma$. From the identity $[Y_1, Y_1^\sigma]=\varepsilon\lambda\bar{\mu}[Y, Z^\sigma]+\varepsilon\bar{\lambda}\mu[Y^\sigma, Z]+\mu\bar{\mu}[Z, Z^\sigma]$, we see $\mu=0$. Then $[Y_1, Z_1^\sigma]=2\varepsilon\lambda Y_1^\sigma-2\lambda Z_1^\sigma$. But this cannot be equal to Y_1^σ. Hence we have proved that there is no such automorphism. This implies the inequivalence of III with II.

Putting together all of considerations in this section we have

THEOREM 1. (1) *Every invariant complex structure on* $G=SL(3, \boldsymbol{R})$ *is equivalent to one of structures given by the invariant complex subalgebras* $\text{I}_\lambda(|\lambda|<1)$, II *and* III.

(2) *These structures are not equivalent to each other.*

§ 5. Remarks on the complex structure

In Theorem 1 we obtained the infinitesimal classification of the invariant complex structures on $G=SL(3, \boldsymbol{R})$. The next problem is to investigate the structure of the complex manifold G with these invariant complex structures. We will give few remarks on this problem.

First we consider the right action. Let $\mathfrak{r}$ be the set $\{A \in \mathfrak{g};\ [adA, J]=0\}$. This is a subalgebra and every element of the corresponding subgroup acts on G holomorphically both on left and on right. $\mathfrak{r}$ is equal to the set $\{A \in \mathfrak{g};\ [A, \mathfrak{m}] \subset \mathfrak{m}\}$, where $\mathfrak{m}$ is the invariant complex subalgebra defining J. Recall $\mathfrak{h}_2$ denotes the fundamental Cartan subalgebra of $\mathfrak{g}$. We see easily, using the bracket relations (14), (15) and the definition of the basis (6),

PROPOSITIN 14. *The algebra* $\mathfrak{r}$ *is equal to* $\mathfrak{h}_2$ *for the case* I_λ *and is* $\{0\}$ *for the cases* II *and* III.

Let H be the fundamental Cartan supgroup of G corresponding to $\mathfrak{h}_2$. It is closed, connected and abelian, since it is fundamental. Let us consider the case I_λ. Write J_λ the corresponding structure tensor. Since $\mathfrak{h}_2^C$ is invariant under J_λ, H is a closed complex submanifold. H is C^* for any λ. Let $\mathfrak{p}=\mathfrak{g}\cap(\sum_{\delta\in\Delta} e_\delta)$. Then $\mathfrak{g}=\mathfrak{h}_2+\mathfrak{p}$ and $[\mathfrak{h}_2, \mathfrak{p}]\subset\mathfrak{p}$. Since $J_\lambda e_\delta=-ie_\delta$ for $\delta\in\Delta^+$ and $J_\lambda e_\delta=ie_\delta$ for $-\delta\in\Delta^+$, the set $\mathfrak{p}$ is J_λ-stable. Hence, from the fact that $[\mathrm{ad}\,\mathfrak{h}_2, J_\lambda]=0$, we see that the natural projection of G to the quotient G/H is holomorphic and G is a holomorphic C^*-bundle over G/H. But the space G/H is, as is well-known, one of open G-orbits in the flag manifold G^C/B^C, since H is fundamental; where B^C is the Borel subgroup. Since the Borel subalgebra corresponding to B^C is $\mathfrak{h}^C+\mathrm{I}_\lambda$, the complex structure on G/H is nothing but the structure induced from the structure of G^C/B^C as an open orbit.

Let $\mathfrak{k}$ be the maximal compact subalgebra, namely $\mathfrak{so}(3)$ in our case. A base of $\mathfrak{k}$ is given by the set $\{i(H_\alpha+H_\beta),\ e_\alpha+e_{-\beta}+e_\beta+e_{-\alpha},\ i(e_\alpha-e_{-\beta}-e_\beta+e_{-\alpha})\}$. $\mathfrak{k}$ satisfies $\mathfrak{k}=\mathfrak{k}\cap\mathfrak{h}_2+\mathfrak{k}\cap\mathfrak{p}$ and the part $\mathfrak{k}\cap\mathfrak{p}$ is J_λ-stable. Hence the projection of the maximal compact subgroup K on G/H turns out to be the complex submanifold which is compact. This means that the manifold is not Stein. On the other hand we know that if a C^*-bundle is Stein, then its base is also Stein. This is a special case of a theorem in [2; Theoréme 5]. Hence we have

THEOREM 2. *The complex manifold* $SL(3, \mathbf{R})$ *with the structure determined by* I_λ *is not Stein for any* λ.

REMARK 2. This theorem can be generalized to any noncompact semisimple Lie algebra with the invariant complex structure of such a type as this I_λ[5].

REMARK 3. As for the algebra II or III the author has not yet succeeded to describe the structure of G as a complex manifold. But we can see from the next proposition that this manifold has no invariant fibering as I_λ.

PROPOSITION 15. *Let* $\mathfrak{m}$ *be* II *or* III. $\mathfrak{m}$ *has no decomposition such that* $\mathfrak{m}=\mathfrak{a}+\mathfrak{b}$ *for a nontrivial proper subalgebra* $\mathfrak{a}$ *and a vector subspace* $\mathfrak{b}$ *satisfying* $\mathfrak{a}\cap\mathfrak{b}=0$ *and* $[\mathfrak{a}, \mathfrak{b}]\subset\mathfrak{b}$, $[\mathfrak{a}^\sigma, \mathfrak{b}]\subset\mathfrak{b}$.

We can prove this proposition using the bracket relations (14), (15) and (16). The computations needed are routine and omitted.

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