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ISOTOPY GROUPS OF BOUNDED 2-MANIFOLDS

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1. INTRODUCTION.

Let M be a closed, orientable 2-manifold and let M_n denote the bounded manifold obtained by removing from M the interiors of n disjoint closed disks $D_1, \cdots, D_n$ with boundaries $C_1, \cdots, C_n$. For $0 \leq r \leq n$, let $H^r(M_n)$ denote the group of all orientation-preserving homeomorphisms h of M_n with the property that h is the identity on C_i for $i \leq r$ and $h(C_i)=C_i$ for $i > r$. If $H^r(M_n)$ is given the compact-open topology, then the arc-component of the identity, denoted $H_0^r(M_n)$, is a normal subgroup of $H^r(M_n)$. Following the notation and terminology of [5], the quotient group $H^r(M_n)/H_0^r(M_n)$ is denoted $\Pi_0(H^r(M_n))$ and is called the isotopy group of $H^r(M_n)$.

In [5], $\Pi_0(H^r(M_n))$ is investigated for the case M equal to the 2-sphere. This paper considers the structure of $\Pi_0(H^r(M_n))$ for manifolds other than the 2-sphere. In Section 2 it is shown that $\Pi_0(H^r(M_n))$ can be obtained as an extension of thee free abelian group on r generators by the group $\Pi(H^0(M_n))$. In Section 3 the relationship between $\Pi_0(H^0(M_n))$ and the group of all isotopy classes of orientation preserving homeomorphisms of M is determined. In particular, this relationship is used to show how a presentation of $\Pi_0(H^r(M_n))$ can be obtained for all r and n in the case of M equal to the torus.

2. SPIN HOMEOMORPHISMS.

Given an integer r with $0 \leq r \leq n$, let $B_r = \bigcup_{i=1}^{r} C_i$. Note that two homeomorphisms f and g in $H^r(M_n)$ represent the same element in $\Pi_0(H^r(M_n))$ provided f is isotopic to g by an isotopy which is the identity on B_r, i.e., f is isotopic to g (rel B_r). Let Ψ_r: $\Pi_0(H^r(M_n)) \rightarrow \Pi_0(H^0(M_n))$ be the natural homeomorphism which sends the isotopy class (rel B_r) of a homeomorphism in $H^r(M_n)$ to the isotopy class that this homeomorphism represents in $\Pi_0(H^0(M_n))$. It follows from Proposition 3.22 (ii) of [6] that any orientation preserving homeomorphism f of M_n with $f(C_i)=C_i$ for $1 \leq i \leq n$ is isotopic to a homeomorphism which is the identity on C_i for each i. In particular, each equivalence class in $\Pi_0(H^0(M_n))$ has a representative from $H_r(M_n)$, so Ψ_r is an epimorphism. The rest of this section is concerned with finding representatives for the kernal of Ψ_r.

Let $K^r(M_n)$ denote the kernal of Ψ_r and let A_k be a collar neighborhood of C_k for $1 \leq k \leq n$ with $A_i \cap A_j = \emptyset$ if $i \neq j$.

LEMMA 2.1. *Every element of $K^r(M_n)$ can be represented by a homeomorphism which is the identity on $M_n - \bigcup_{k=1}^{n} \mathrm{Int}\, A_k$.*

PROOF. Let h be a homeomorphism which represents a non-trivial element of $K^r(M_n)$. That is, h is isotopic to the identity, but not by an isotopy which keeps B_r pointwise fixed. Without loss of generality we can assume h is the identity on $\partial M_n = \bigcup_{k=1}^{n} C_k$. Let h_t be an isotopy which takes h to the identity. Using the "unwinding" technique of Proposition 3.22 (ii) of [6], it is possible to extend $h_t^{-1}/\partial M_n$ to an isotopy g_t of M_n which takes the identity to a homeomorphism which is the identity on $X - \bigcup_{k=1}^{n} \mathrm{Int}\, A_k$. The isotopy $g_t h_t$ is then an isotopy (rel ∂M_n) which takes h to a homeomorphism of the type given in the statement of the lemma.

DEFINITION 2.1. For each k, $1 \leq k \leq n$, let S_k^p: $M_n \rightarrow M_n$ be defined by letting S_k^p restricted to the annulus A_k be the homeomorphism given in [8] which spins one component of ∂A_k p-times while holding the other boundary component fixed and by letting S_k^p restricted to $M_n - A_k$ be the identity. S_k^p will be referred to a "*spin homeorphism*" of M_n.

THEOREM 2.1. *If M is not the 2-sphere, then $k^r(M_n)$ is the free abelian group on r-generators.*

PROOF. It will be shown that every element of $K^r(M_n)$ has a unique representation as a product of spin homeomorphisms of the form $S_1^{p_1} \cdots S_r^{p_r}$. A consequence of Theorem 7.2 of [3] is that every homeomorphism of A_k which is the identity on ∂A_k is isotopic (rel ∂A_k) to S_k^p / A_k for some p. Since $A_i \cap A_j = \emptyset$ for $i \neq j$, this means that any homeomorphism of M_n which is the identity on $M_n - \bigcup_{k=1}^{n} \mathrm{Int}\, A_k$ is isotopic (rel ∂M_n) to a product of homeomorphism of the form $S_1^{p_1} \cdots S_n^{p_n}$. Thus by Lemma 2.1 every element of $K^r(M_n)$ can be represented as a product of spin homeomorphism. Moreover, since $S_k^{p_k}$ is isotopic to the identity (rel B_r) for $k > r$, it follows that every element of $K^r(M_n)$ has a representation of the form $S_1^{p_1} \cdots S_r^{p_r}$.

To show the representation is unique, it suffices to show that if $S_1^{p_1} \cdots S_r^{p_r}$ is isotopic (rel B_r) to the identity, then $p_i = 0$ for $1 \leq i \leq r$. On the contrary, assume this product is isotopic (rel B_r) to the identity, but $p_k \neq 0$ for some k. Let α be a curve which represents a generator of $\Pi_1(M, q)$ where q εC_k and α is chosen so that $\alpha \cap A_j = \emptyset$ for $j \neq k$ and $\alpha \cap D_k = \{q\}$. Let β_k be the curve based q which wraps once around C_k in the direction of the spin corresponding to $S_k^{p_k}$. In the free group $\Pi_1(M_n, q)$, $S_k^{p_k}(\alpha)$ represents the same element as the curve $\beta_k^{-p_k}$ α $\beta_k^{p_k}$. On the other hand, since α is outside the support of $S_i^{p_i}$ for $i \neq k$, it follows that $S_1^{p_1} \cdots S_r^{p_r}(\alpha) = S_k^{p_k}(\alpha)$. But $S_1^{p_1} \cdots S_r^{p_r}$ is isotopic (rel B_r) to the identity, so the curve $S_k^{p_k}(\alpha)$ must be isotopic (rel q) to α. Thus $S_k^{p_k}(\alpha)$ must also represent the same elemet as α the free group $\Pi_1(M_n, q)$. This contradiction establishes the theorem.

It should be noted that Theorem 2.1 is also true for M equal to the 2-sphere (see Theorem 3.7 of [5]) provided the condition $n \geqslant 3$ is added. For the remaining cases involving $M = S^2$ we have: $K^1(M_1) \cong 1$ (by Alexander's trick), $K^2(M_2) \cong Z$ (by Theorem 7.2 of [3]), and $K^1(M_2) \cong 1$ (since the spin homeomorphism about C_1 are all isotopic (rel C_1) to the identity).

3. Isotopy and Homeotopy Groups.

In this section it will be shown how the group $\Pi_0(H^r(M_n))$ is related to the homeotopy group of M, i. e. the group of all isotopy classes of homeomorphisms of M.

Definition 3.1 Let $F_n = \{p_1, \cdots, p_n\}$ where p_k is a point in the interior of the disk D_k for each k. Define $H(M, F_n)$ to be the group of isotopy classes (rel F_n) of orientation preserving homeomorphisms of M which are the identity on F_n.

Let Φ: $\Pi_0(H^0(M_n)) \rightarrow H(M, F_n)$ be the homeomorphism which sends the isotopy class of a homeomorphism f of M_n to the isotopy class (rel F_n) of the homeomrphism $\bar{f}$ of M which is obtained by taking the "cone" of f. In more detail, if for each i, $1 \leq i \leq n$, e_i is a homeomorphism of D_i to the unit disk in R^2

taking p_i to the origin, then $\bar{f}(x)=f(x)$ for $x\varepsilon M_n$ and

$\bar{f}(e_i^{-1}(te_i(x)+(1-t)e_i(p_i)))=e_i^{-1}(te_i(f(x))+(1-t)e_i(p)))$ for $x\varepsilon C_i$.

THEOREM 3.1. *If M is not the 2-sphere, then the following sequence is exact.*

$$1 \to Z^r \to \Pi_0(H^r(M_n)) \overset{g}{\to} H(M, F_n) \to 1, \text{ where } g=\Phi\Psi_r.$$

PROOF. By Theorem 6 of [7], Φ is an isomorphism from $\Pi_0(H^0(M_n))$ to $H(M, F,_n)$ and by Theorem 2.1, ker Ψ_r=ker $\Psi_r\Phi=Z^r$.

The above theorem shows that the isotopy group $\Pi_0(H^r(M_n)$ can be obtained as an extension of Z^r by $H(M, F_n)$. In turn $H(M, F_n)$ is part of the short exact sequence:

$$1 \to \Pi_1(M-F_{n-1}, p_n) \to H(M, F_n) \overset{d}{\to} H(M, F_{n-1}) \to 1,$$

where d sends the isotopy class (rel F_n) of a homeomorhism of M to the isotopy class (rel F_{n-1}) of this homeomorphism. The representation of an element in the kernal of d as an element in $\Pi_1(M-F_{n-1}, p_n)$ is obtained by taking the curve formed by tracing the path of p_n during the isotopy (rel F_{n-1}) which takes a representative homeomorphism of an element in the kernal of d to the identity.

Thus if we let $H(M)$ denote the homeotopy group of M and $\bar{H}(M)$ the subgroup of $H(M)$ consisting of all isotopy classes of orientation preserving homeomorphisms of M, then $H(M, F_n)$ may be built up from $\bar{H}(M)$ by repeatedly extending $\Pi_1(M-F_k, p_{k+1})$ by $H(M, F_k)$ for $k=1, \cdots, n-1$ (see [2]).

For the case $M=S^2$, $\bar{H}(M)$ and $H(M, F_n)$ for $n \leq 2$ are all the trivial group. A presentation of $H(M, F_n)$ for $n \geq 3$ is given in [8]. This presentation combined with Theorem 3.1 of [5] yields a presentation of $\Pi_0(H^r(M_n))$ for all r and n with $2 \leq r \leq n$.

For the case M equal to the torus, presentations of $H(M)$ and $\bar{H}(M)$ are given in [1], [2] and [9]. In [1] a slight modification of the inductive technique described above is used to obtain a presentation of the group of all isotopy classes (rel F_n) of homeomorphisms of M which are the identity on F_n. As is mentioned in the first paragraph of Section 4 of [1], this presentation can be used to obtain a presentation of the more restricted group of all isotopy classes (rel F_n) of orientation preserving homeomorphisms, i. e. $H(M, F_n)$, by simplifying all calculations to use only orientation preserving maps in the base group $H(M)$. An analysis of the resulting presentation of $H(M, F_n)$ shows that the generators of this group may be represented by homeomorphisms which commute with the spin homeomorphisms and, in fact the short exact sequence given in Theorem 3.1 splits. Thus the presentation of $H(M, F_n)$ obtained from [1] and the standard presentation of Z^r can be combined as a direct sum to yield a presentation of Π_0 $(H^r(M_n))$ for any r and n with $r \leq n$.

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