Kumamoto J. Sci. (Math.)
Vol. 16, 1~13 March (1984)

STABILITY OF SPATIO-TEMPORAL OSCILLATIONS OF DIFFUSIVE LOTKA-VOLTERRA SYSTEM

Dedicated to Professor Kenzo Iizuka on his 60th birthday

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(Received September 1, 1983)

In [2] K. Kishimoto, M. Mimura and K. Yoshida gave some examples of diffusive Lotka-Volterra system with three or more species which exibits stable spatio-temporal oscillatory phenomena. We employed there the bifurcation technique and evaluated the "stability constant K" which was defined by S.-N. Chow and J. Mallet-Paret in [1]. Since the calculation of K in [2] was long, we only gave the final result. Thus in this note we not only give a detailed calcu-ation of K but also show the existence of the center manifold, from which the Hopf bifurcation theorem follows easily.

The system which we consider in this note is the diffusive Lotka-Volterra system parameterized by α in one dimensional interval;

(0.1)
$$\dot{u}_i(t,x)=\sigma_i(\alpha)(u_i)_{xx}(t,x)+(r_i+\sum_{j=1}^{d}a_{ij}u_j(t,x))u_i(t,x)$$
$$0<x<\pi,\ t>0,\ i=1,\cdots,d$$

subject to

(0.2)
$$(u_i)_x(t,0)=(u_i)_x(t,\pi)=0,$$

where $\cdot\ =d/dt$. Here $u_i(t,x)$ is the density of the i-th species at time t and at position x. The diffusion coefficients $\sigma_i(\alpha)$ are positive constants. The diagonal coefficients a_{ii} are nonpositive constants, which reflects intraspecific competition, while the off-diagonal coefficients $a_{ij}(i\neq j)$ are real constants. This means that the interspecific relations may be competitive, cooperative, prey-predator type or combinations.

1. Decomposition of the system

Assume that (0.1) has a constant equilibrium solution

$$\bar{U}=\text{col}\,(\bar{u}_1, \bar{u}_2, \ldots, \bar{u}_d),\ \bar{u}_j > 0,$$

where col $(\cdots)$ means a column vector. Then, (0.1) is written as

$$\dot{u}_i=\sigma_i(\alpha)(u_i)_{xx}+\sum_{j=1}^{d} a_{ij}(u_j-\bar{u}_j)u_i,$$

which is rewritten as

$$\dot{U}=D(\alpha)U_{xx}+A(\bar{U})U+N(U), \tag{1.1}$$

if u_i is replaced by $u_i-\bar{u}_i$ and we use the vector notation : $U=\text{col}\,(u_1, \ldots, u_d)$, $D(\alpha)$ is the $d\times d$ diagonal matrix whose (i,i)-components are $\sigma_i(\alpha)$, $A(\bar{U})$ is the $d\times d$ matrix whose (i,j)-components are $-a_{ij}\bar{u}_i$ and

$$N(U)=\text{col}\,(n_1(u_1, \ldots, u_d), \ldots, n_d(u_1, \ldots, u_d))$$

with

$$n_i(u_1, \ldots, u_d)=\sum_{j=1}^{d} a_{ij}u_iu_j.$$

Let us now consider the eigenvalue problem for the linear part of (1.1) with zero Neumann boundary condition;

$$\begin{cases} \lambda V(x)=D(\alpha)V_{xx}(x)+A(\bar{U})V(x) \\ V_x(0)=V_x(\pi)=0 \end{cases} \tag{1.2}$$

If we take the Fourier expansion

$$V(x)=\sum_{n=0}^{\infty} \Phi_n \cos nx$$

with constant vectors

$$\Phi_n=\text{col}\,(\Phi_{n1}, \ldots, \Phi_{nd}),$$

then the eigenvalue problem (1.2) is equivalent to the one

$$\lambda\Phi_n=(A(U)-n^2D(\alpha))\Phi_n,\ n=0,1, \ldots . \tag{1.3}$$

Throughout this note we assume that

(**H.1**) *for some* $n=n_0$ *there exists a real number* $\delta>0$ *such that, for any* α *with* $-\delta<\alpha<\delta$, *the eigenvalue problem* (1.3) *has a unique pair of complex conjugate eigenvalues* $\lambda(\alpha)$ *and* $\overline{\lambda(\alpha)}$ *such that*

$$\lambda(\alpha)=\mu(\alpha)+i\omega(\alpha)=\omega_0>0,$$

$$\mu(0)=0,\ \mu'(0)\neq 0,$$

where $\mu'(\alpha)$ *is the derivative with respect to* α,

(**H.2**) *all the eigenvalues except for* $\lambda(\alpha)$ *and* $\overline{\lambda(\alpha)}$ *have strictly negative real part for any* n.

Under these assumption we have the spectral decomposition

$$\boldsymbol{L}^2([0,\pi])=\boldsymbol{P}\oplus\boldsymbol{Q},$$

where $\boldsymbol{L}^2([0,\pi])=(L^2([0,\pi]))^d$ and $\boldsymbol{P}$ it the two-dimensional eigenspace corresponding to the eigenvalues $\lambda(\alpha)$ and $\overline{\lambda(\alpha)}$.

In what follows we decompose the system (1.1) into the one restricted to $\boldsymbol{P}$ and the other. Let $\boldsymbol{\Phi}_{n_0}$ be the eigenvector corresponding to $\lambda(\alpha)$ and put

$$\boldsymbol{\Phi}=\mathrm{col}\ (\Phi_1,\dots,\Phi_d)=\mathrm{Re}\ \boldsymbol{\Phi}_{n_0}\ \text{and}\ \boldsymbol{\Psi}=\mathrm{col}\ (\Psi_1,\dots,\Psi_d)=\mathrm{Im}\ \boldsymbol{\Phi}_{n_0}.$$

Then $\boldsymbol{\Phi}$ and $\boldsymbol{\Psi}$ are linearly independent but not necessarily orthogonal.
Let us put

$$U(t,x)=z(t)\boldsymbol{\Phi}_{n_0}\cos n_0x+\overline{z(t)\boldsymbol{\Phi}_{n_0}}\cos n_0x+W(t,x) \tag{1.4}$$

with a complex valued scalar function

$$z(t)=u(t)+iv(t)$$

and define the projection $\boldsymbol{\Pi}$ from $\boldsymbol{L}^2([0,\pi])$ to $\boldsymbol{P}$ by

$$\boldsymbol{\Pi}[V(\cdot)]=\Pi_1(V)\boldsymbol{\Phi}\cos n_0x+\Pi_2[V]\boldsymbol{\Psi}\cos n_0x$$

with

$$\Pi[V]=e_1<\int_0^\pi V(x)\cos n_0dx,\ \boldsymbol{\Phi}>+e_2<\int_0^\pi V(x)\cos n_0dx,\boldsymbol{\Psi}>,$$

$$\Pi_2[V]=e_2<\int_0^\pi V(x)\cos n_0 x đx,\ \Phi>+e_3<\int_0^\pi V(x)\cos n_0 x đx,\ \Psi>,$$

where $<\cdot,\ \cdot>$ stands for the scalar product in $\boldsymbol{R}^d$,

$$đx=\frac{2}{\pi}dx,$$

$$e_1=<\Psi,\Psi>/(<\Phi,\Phi><\Psi,\Psi>-<\Phi,\Psi>^2),$$
$$e_2=-<\Phi,\Psi>/(<\Phi,\Phi><\Psi,\Psi>-<\Phi,\Psi>^2),$$
$$e_3=<\Phi,\Phi>/(\Phi,\Phi><\Psi,\Psi>-<\Phi,\Psi>^2).$$

Insert (1.4) into (1.1) and apply $\boldsymbol{\Pi}$ to the both sides. Then we have

$$(1.6)\quad\begin{cases}\dot{u}(t)=\mu(\alpha)u(t)-\omega(\alpha)v(t)+\dfrac{1}{2}\Pi_1[N(U)],\\ \dot{v}(t)=\omega(\alpha)u(t)+\mu(\alpha)v(t)-\dfrac{1}{2}\Pi_2[N(U)],\\ \dot{W}=D(\alpha)W_{xx}+A(\bar{U})+[1-\boldsymbol{\Pi}][N(U)].\end{cases}$$

2. The center manifold and the Hopf bifurcation

Let $\boldsymbol{H}^2([0,\pi])$ be the space of real vector valued L^2 functions whose derivatives of order up to two belong to $\boldsymbol{L}^2([0,\pi])$ and put $\boldsymbol{H}_N^2([0,\pi])=\{V\in\boldsymbol{H}^2([0,\pi]);\ V_x(0)=V_x(\pi)=0\}$. Then by the same way as in [3] we have

THEOREM 1. *Let k be an arbitrary fixed positive integer. Then there exist a zero neighborhood $\mathscr{U}\times I$ in $R^2\times(-\delta,\delta)$ and a k times continuously differentiable function G on $\mathscr{U}\times I$ with values in $Q\cap\boldsymbol{H}_N^2$ satisfying the following conditions:*

i) $G(0,0,\alpha)=0,\ (\partial G/\partial x_1)(0,0,0)=(\partial G/\partial x_2)(0,0,0)=0$

ii) for any $\alpha\in I$, let

$$\mathscr{M}(\alpha)=\{V\in\boldsymbol{H}_N^2;\ V(x)=u\Phi\cos n_0x+v\Psi\cos n_0x+G(u,v,\alpha),\ (u,v)\in\mathscr{U}\}.$$

Then $\mathscr{M}(\alpha)$ is locally invariant in the sense that if, for any $V\in\mathscr{M}(\alpha)$ such that $(\Pi_1[V],\Pi_2[V])\in\mathscr{U}$, the solution $(u(t),\ v(t))$ of

$$\dot{u}(t)=\mu(\alpha)u(t)-\omega(\alpha)v(t)+\frac{1}{2}\Pi_1[N(U_G)]$$

$$\dot{v}(t)=\omega(\alpha)u(t)+\mu(\alpha)v(t)-\frac{1}{2}\Pi_2[N(U_G)]$$

with

$$u(0)=\frac{1}{2}\Pi_1[V],\ v(0)=-\frac{1}{2}\Pi_2[V],$$

$$U_G(t,x)=z(t)\Phi_{n_0}\cos n_0x+\overline{z(t)}\ \bar{\Phi}_{n_0}\cos n_0x+G(u(t),\ v(t),\ \alpha),$$

$$z(t)=u(t)+iv(t)$$

stays in $\mathscr{U}$, *then* $U_G(t,x)$ *defined as above with the solution* $(u(t),\ v(t))$ *is the unique solution of* (1.1) *with zero Neumann boundary condition.*

$\mathscr{M}(\alpha)$ *is locally attractive, that is, if the solution* $U(t,\ x)$ *of* (1.1) *with zero Neumann boundary condition satisfies* $(\Pi_1[U(t,\cdot)],\ \Pi_2[U(t,\cdot)])\in\mathscr{U}$ *for* $0\leqq t<T$, *then there exist positive constants* C *and* γ *independent of* t *such that*

$$\|U(t,\cdot)-\Pi[U(t,\cdot)]-G(u(t),\ v(t),\ \alpha)\|_{H^2}\leqq Ce^{-\gamma t}\|U(0,\cdot)\|_{H^2},$$

where $u(t)=\Pi_1[U(t,\cdot)]$ *and* $v(t)=\Pi_2[U(t,\cdot)]$.

From Theorem 1 we have, easily,

THEOREM 2. *Under the assumption* **(H.1)** *and* **(H.2)** *the spatially inhomogeneous Hopf bifurcation occures a* $\alpha=0$.

3. Calculation of stability costant K

As is stated in Introduction, we make use of S. -N. Chow and J. Mallet-Paret's theory in order to investigate the stability of the Hopf bifurcation. To do so let us remember their theory.

Let us write, in general, an evolution equation

$$z=A(\alpha)z+F(z,\alpha) \tag{3.1}$$

$$F(z,\alpha)=0(|z|^2)$$

in a certain Banach space $\boldsymbol{X}$, where $A(\alpha)$ is a closed operator from $\boldsymbol{X}$ to $\boldsymbol{X}$ with domain $\boldsymbol{Y}\subset\boldsymbol{X}$, $\boldsymbol{Y}$ being a Banach space continuously and densely contained in $\boldsymbol{X}$. Assume that

$$F\colon \boldsymbol{Y}\times(-\delta,\delta)\to\boldsymbol{X}$$

is sufficiently smooth, and further assume that $A(\alpha)$ has only eigenvalues which have the following properties;

i) A pair of complex conjugate eigenvalues $\lambda(\alpha)$ and $\overline{\lambda(\alpha)}$ such that

$$\lambda(\alpha)=\mu(\alpha)+i\omega(\alpha),\ \omega(0)=\omega_0>0$$
$$\mu(0)=0,\ \mu'(0)\neq 0.$$

ii) (**H.2**) holds

Then we have, as usual, the spectral decomposition

$$\boldsymbol{X}=\boldsymbol{P}\oplus\boldsymbol{Q},$$

where $\boldsymbol{P}$ is the two- dimensional eigenspace of $A(\alpha)$ corresponding to $\lambda(\alpha)$ and $\overline{\lambda(\alpha)}$. By making use of this decomposition we rewrite as (3.1)

$$\text{(3.2)}\qquad \begin{cases} \dot{\boldsymbol{x}}=A_P(\alpha)\boldsymbol{x}+\boldsymbol{F}_P(\boldsymbol{x},\boldsymbol{y},\alpha),\\ \dot{\boldsymbol{y}}=A_Q(\alpha)\boldsymbol{y}+\boldsymbol{F}_Q(\boldsymbol{x},\boldsymbol{y},\alpha),\end{cases}$$

where $\boldsymbol{z}=\boldsymbol{x}+\boldsymbol{y}\in\boldsymbol{P}\oplus\boldsymbol{Q}$ and $A_P(\alpha)$ (resp. $A_Q(\alpha)$) and $\boldsymbol{F}_P(\boldsymbol{x},\boldsymbol{y},\alpha)$ (resp. $\boldsymbol{F}_Q(\boldsymbol{x},\boldsymbol{y},\alpha)$) are restrictions of $A(\alpha)$ and $F(\boldsymbol{x},\boldsymbol{y},\alpha)$ to $\boldsymbol{P}$ (resp. $\boldsymbol{Q}$). Let us denote the matrix representation of $A_P(\alpha)$ by

$$\begin{bmatrix}\mu(\alpha) & -\omega(\alpha)\\ \omega(\alpha) & \mu(\alpha)\end{bmatrix}.$$

Expanding (3.2) in the Taylor series we have

$$\dot{x}_1=\mu(\alpha)x_1-\omega(\alpha)x_2+\sum_{j=2}^{\infty}\boldsymbol{B}_1^j(\boldsymbol{x},\boldsymbol{y},\alpha),$$
$$\dot{x}_2=\omega(\alpha)x_2+\mu(\alpha)x_2+\sum_{j=2}^{\infty}\boldsymbol{B}_2^j(\boldsymbol{x},\boldsymbol{y},\alpha),$$
$$\dot{\boldsymbol{y}}=A_Q(\alpha)\boldsymbol{y}+\sum_{j=2}^{\infty}\boldsymbol{B}_Q^j(\boldsymbol{x},\boldsymbol{y},\alpha)$$
$$=A_Q(\alpha)\boldsymbol{y}+J(\alpha)\boldsymbol{x}^2+N(\alpha)\boldsymbol{x}\boldsymbol{y}+E(\alpha)\boldsymbol{y}^2+\Gamma_3(\boldsymbol{x},\alpha)\boldsymbol{y}^3+\dots,$$

which, in polar coordinates $x=(r\cos\theta, r\sin\theta)$, becomes

$$(3.3)\quad \begin{cases} \dot{r}=F_1(\theta,\alpha)y^2+r\{\mu(\alpha)+G_2(\theta,y,\alpha)y\}+r^2C_3(\theta,y,\alpha)+r^3C_4(\theta,y,\alpha)+\dots \\ \dot{\theta}=\omega_0+rD_3(\theta,y,\alpha)+r^2D_4(\theta,y,\alpha)+.. \\ \dot{y}=\text{as above but with } x=(r\cos\theta,\ r\sin\theta). \end{cases}$$

Then let us define K by

$$K=K^*+K^{**},$$

$$K^*=\frac{1}{2\pi}\int_0^{2\pi}\{C_4(\theta,0,0)-\frac{1}{\omega_0}C_3(\theta,0,0)D_3(\theta,0,0)\}d\theta,$$

$$K^{**}=\frac{1}{2\pi}\int_0^{2\pi}w^*(\theta)J(0)(\cos\theta,\ \sin\theta)^2d\theta,$$

where $w^*(\theta)$ is the unique 2π-periodic solution of

$$(3.4)\qquad G_2(\theta,0,0)+\dot{w}^*(\theta)\omega_0+w^*(\theta)A_Q(0)=0.$$

As is stated in [1], note that for each $\alpha\in(-\delta,\delta)$, $J(\alpha)$ is a bilinear form in the x-space $\boldsymbol{R}^2$ taking values in the y-space; in the above definition $J(0)$ acts on the point $(\cos\theta,\ \sin\theta)\in\boldsymbol{R}^2$. Since $G_2(\theta,0,0)$ arises as a coefficient of y in the differential equation involving $\dot{r}$, $G_2(\theta,0,0)$ for each θ is linear functional on y. Also note that the property $K\neq0$ depends on the differential equation at $\alpha=0$.

THEOREM (S. -N. Chow and J. Mallet-Paret). *Suppose that there exists a center manifold taking value in* $Q\cap Y$. *If* $\mu'(0)K<0$, *then the Hopf bifurcation is stable.*

Now let us write (1.6) in polar coordinates as

$$(3.5)\quad \begin{cases} \dot{r}=\mu(\alpha)r+\frac{1}{2}\{\Pi_1[N(U)]\cos\theta-\Pi_2[N(U)]\sin\theta\}, \\ \dot{\theta}=\omega(\alpha)-\frac{1}{2r}\{\Pi_1[N(U)]\sin\theta+\Pi_2[N(U)]\cos\theta\}, \\ \dot{W}=\text{as in (1.6) but with } (u,v)=(r\cos\theta,\ r\sin\theta). \end{cases}$$

Then,

$$C_3(\theta,0,0)=C_4(\theta,0,0)=0$$

and so

$$K^*=0.$$

In order to calculate K^{**} we first determine $J(0)$ $(\cos\theta,\sin\theta)^2$ and $G_2(\theta,0,0)$ W. But these are easily obtained as

(3.6) $J(0)(\cos\theta,\ \sin\theta)^2=\mathrm{col}\ (J_1(0)(\cos\theta,\ \sin\theta)^2,\dots,\ J_d(0)(\cos\theta,\ \sin\theta)^2)$

with

(3.7) $J_j(0)(\cos\theta,\ \sin\theta)^2=4\sum_{k=1}^{d} a_{jk}(\Phi_j\cos\theta-\Psi_j\sin\theta)\ (\Phi_k\cos\theta-\Psi_k\sin\theta)\ \cos^2 n_0 x$

and

(3.8) $G_2(\theta,0,0)W$

$$=\sum_{j=1}^{d}\sum_{k=1}^{d} a_{jk}\{(e_1\Phi_j+e_2\Psi_j)\ \{\cos\theta-(e_2\Phi_j+e_3\Psi_j)\ \sin\theta\}.$$

$$\cdot\{(\Phi_j\ \cos\theta-\Psi_j\ \sin\theta)\int_0^{\pi} w_k(x)\ \cos^2 n_0 x dx$$

$$+(\Phi_k\ \cos\theta-\Psi_k\ \sin\theta)\int_0^{\pi} w_j(x)\ \cos^2 n_0 x dx\}.$$

As in [1] let us write $G_2(\theta,0,0)$ as a Fourier series:

$$G_2(\theta,0,0)=\sum_{n=-\infty}^{\infty} g_n e^{in\theta},\quad g_n\in(Q\cap H_N^2)^*.$$

By expanding $w^*(\theta)$ as a Fourier series

$$w^*(\theta)=\sum_{n=-\infty}^{\infty} w_n e^{in\theta},$$

inserting this into (3.4) and equating coefficients, we arrive at

$$w_n=-g_n(A_Q+in\omega_0)^{-1},$$

where A_Q is the restriction of $D(0)(d/dx)^2+A(\bar{U})$ to Q. Then

(3.9) $$K^{**}=\frac{1}{2\pi}\int_0^{2\pi}\sum e^{in\theta}w_n\ [J(0)\ (\cos\theta,\ \sin\theta)^2]d\theta$$

$$=-\frac{1}{2\pi}\int_0^{2\pi}\sum e^{in\theta}g_n(A_Q+in\omega_0)^{-1}\ [J(0)\ (\cos\theta,\ \sin\theta)^2]\ d\theta.$$

Since

$$\cos\theta=\frac{e^{i\theta}+e^{-i\theta}}{2} \text{ and } \sin\theta=\frac{e^{i\theta}-e^{-i\theta}}{2i},$$

we have

$$(3.10) \qquad g_0(W)=\frac{1}{2}\sum_{j=1}^{d}\sum_{k=1}^{d} a_{jk}(e_1\Phi_j+e_2\Psi_j)\int_0^\pi(\Phi_j w_k+\Phi_k w_j)\cos^2 n_0x dx$$
$$+\frac{1}{2}\sum_{j=1}^{d}\sum_{k=1}^{d} a_{jk}(e_2\Phi_j+e_3\Psi_j)\int_0^\pi(\Psi_j w_k+\Psi_k w_j)\cos^2 n_0x dx,$$

$$(3.11) \qquad g_1(W)=\overline{g_{-1}(W)}=0,$$

$$(3.12) \qquad g_2(W)=\overline{g_{-2}(W)}$$
$$=\frac{1}{4}\sum_{j=1}^{d}\sum_{k=1}^{d} a_{jk}(e_1\Phi_j+e_2\Psi_j)\int_0^\pi(\Phi_j w_k+\Phi_k w_j)\cos^2 n_0x dx$$
$$+\frac{i}{4}\sum_{j=1}^{d}\sum_{k=1}^{d} a_{jk}(e_2\Phi_j+e_3\Psi_j)\int_0^\pi(\Phi_j w_k+\Phi_k w_j)\cos^2 n_0x dx$$
$$+\frac{i}{4}\sum_{j=1}^{d}\sum_{k=1}^{d} a_{jk}(e_1\Phi_j+e_2\Psi_j)\int_0^\pi(\Psi_j w_k+\Psi_k w_j)\cos^2 n_0x dx$$
$$-\frac{1}{4}\sum_{j=1}^{d}\sum_{k=1}^{d} a_{jk}(e_2\Phi_j+e_3\Psi_j)\int_0^\pi(\Psi_j w_k+\Psi_k w_j)\cos^2 n_0x dx$$

and

$$(3.14) \qquad g_n(W)=\overline{g_{-n}(W)}=0 \qquad \text{for } n=3,4,\ldots.$$

Next we determine

$$F(\theta,x)=(A_Q+in\omega_0)^{-1}(J(0)(\cos\theta,\sin\theta)^2) \qquad \text{for } n=0,\ \pm 2,$$

that is, find F of

$$(A_Q+in\omega_0)F=J(0)(\cos\theta,\sin\theta)^2,$$

which is equivalent to

$$(3.14) \quad \begin{cases} D(0)F_{xx}+in\omega_0 F=A(\bar{U})F=J(0)(\cos\theta,\sin\theta)^2 \\ F_x(\theta,0)=F_x(\theta,\pi)=0, \end{cases}$$

because $J(0)(\cos\theta, \sin\theta)^2 \in Q\cap H_N^2([0,\pi])$. From (3.7) we have

$$J_j(0)(\cos\theta, \sin\theta)^2 = a_j(\theta) + a_j(\theta)\cos 2n_0x,$$

where

$$a_j(\theta) = 2\sum_{k=1}^{d} a_{jk}(\Phi_j\cos\theta - \Psi_j\sin\theta)(\Phi_k\cos\theta - \Psi_k\sin\theta).$$

By expanding F as

$$F = \sum_{k=0}^{\infty} F_k \cos kx,$$

inserting this into (3.14) and equating coefficients, we have, for $n=0, \pm 2$,

$$\begin{aligned} F &= (A_Q + in\omega_0)^{-1}J(0)(\cos\theta, \sin\theta)^2 \\ &= (A(\bar{U}) + in\omega_0)^{-1}\boldsymbol{a}(\theta) + (A(\bar{U}) + in\omega_0 - 4n_0^2D(0))^{-1}\boldsymbol{a}(\theta)\cos 2n_0x, \end{aligned} \tag{3.15}$$

where

$$\boldsymbol{a}(\theta) = \mathrm{col}(a_1(\theta), \ldots, a_d(\theta)).$$

Consequently we have

$$\begin{aligned} K^{**} &= \frac{1}{2\pi}\int_0^{2\pi} g_0\,[A(\bar{U})^{-1}\boldsymbol{a}(\theta) + A(\bar{U}) - 4n_0^2D(0))^{-1}\boldsymbol{a}(\theta)\cos 2n_0x]\,d\theta \\ &- \frac{1}{\pi}\int_0^{2\pi} \mathrm{Re}\; e^{2\theta i}\{g_2[(A(\bar{U}) + 2i\omega_0)^{-1}\boldsymbol{a}(\theta) \\ &+ (A(\bar{U}) + 2i\omega_0 - 4n_0^2D(0))^{-1}\boldsymbol{a}(\theta)\cos 2n_0x]\}\,d\theta = I_1 + I_2. \end{aligned}$$

Since

$$\frac{1}{2\pi}\int_0^{2\pi} a_j(\theta)d\theta = \sum_{k=1}^{d} a_{jk}(\Phi_j\Phi_k + \Psi_j\Psi_k),$$

by putting

$$\boldsymbol{b} = \mathrm{col}(b_1, \ldots, b_d) \text{ with } b_j = \sum_{k=1}^{d} a_{jk}(\Phi_j\Phi_k + \Psi_j\Psi_k),$$

$$\boldsymbol{b}^{(1)}=A(\bar{U})^{-1}\boldsymbol{b} \text{ and } \boldsymbol{b}^{(2)}=(A(\bar{U})-4n_0^2D(0))^{-1}\boldsymbol{b},$$

we have

$$\begin{aligned}I_1&=-g_0[\boldsymbol{b}^{(1)}]-g_0[\boldsymbol{b}^{(2)}\cos 2n_0x]\\&=-\frac{1}{2}\sum_{j=1}^{d}\sum_{k=1}^{d}a_{jk}(e_1\Phi_j+e_2\Psi_j)(\Phi_jb_k^{(1)}+\Phi_kb_j^{(1)})\\&\quad-\frac{1}{2}\sum_{j=1}^{d}\sum_{k=1}^{d}a_{jk}(e_2\Phi_j+e_3\Psi_j)(\Psi_jb_k^{(1)}+\Psi_kb_j^{(1)})\\&\quad-\frac{1}{4}\sum_{j=1}^{d}\sum_{k=1}^{d}a_{jk}(e_1\Phi_j+e_2\Psi_j)(\Phi_jb_k^{(2)}+\Phi_kb_j^{(2)})\\&\quad-\frac{1}{4}\sum_{j=1}^{d}\sum_{k=1}^{d}a_{jk}(e_2\Phi_j+e_3\Psi_j)(\Psi_jb_k^{(2)}+\Psi_kb_j^{(2)}).\end{aligned}$$

Since

$$\frac{1}{\pi}\int_0^{2\pi}e^{2\theta i}a_j(\theta)\,d\theta=\sum_{k=1}^{d}a_{jk}(\Phi_j-i\Psi_j)(\Phi_k-i\Psi_k),$$

by putting

$$\boldsymbol{c}=\operatorname{col}(c_1,\cdots,c_d) \text{ with } c_j=\sum_{k=1}^{d}a_{jk}(\Phi_j-i\Psi)(\Phi_k-i\Psi_k),$$

$$\boldsymbol{c}^{(1)}=(A\bar{U})+2i\omega_0)^{-1}\boldsymbol{c} \text{ and } \boldsymbol{c}^{(2)}=(A(\bar{U})+2i\omega_0-4n_0^2D(0))^{-1}\boldsymbol{c},$$

we have

$$\begin{aligned}I_2&=-\operatorname{Re}g_2[\boldsymbol{c}^{(1)}]-\operatorname{Re}g_2[\boldsymbol{c}^{(2)}\cos 2n_0x]\\&=-\frac{1}{4}\operatorname{Re}\sum_{j=1}^{d}\sum_{k=1}^{d}a_{jk}\,[\{(e_1\Phi_j+e_2\Psi_j)+i(e_2\Phi_j+e_3\Psi_j)\}(\Phi_jc_k^{(1)}+\Phi_kc_j^{(1)})\\&\quad+i\{(e_1\Phi_j+e_2\Psi_j)+i(e_2\Phi_j+e_3\Psi_j)\}(\Psi_jc_k^{(1)}+\Psi_kc_j^{(1)})]\\&\quad-\frac{1}{8}\operatorname{Re}\sum_{j=1}^{d}\sum_{k=1}^{d}a_{jk}[\{(e_1\Phi_j+e_2\Psi_j)+i(e_2\Phi_j+e_3\Psi_j)\}(\Phi_jc_k^{(2)}+\Phi_{kj}^{(2)})\\&\quad+i\{(e_1\Phi_j+e_2\Psi_j)+i(e_2\Phi_j+e_3\Psi_j)\}(\Psi_jc_k^{(2)}+\Psi_kc_j^{(2)})].\end{aligned}$$

Consequently we have

THEOREM 3.

$$K = -\frac{1}{4}\sum_{j=1}^{d}\sum_{k=1}^{d} a_{jk}(e_1\Phi_j + e_2\Psi_j)\ \{2(\Phi_j b_k^{(1)} + \Phi_k b_j^{(1)}) + (\Phi_j b_k^{(2)} + \Phi_k b_j^{(2)})\}$$

$$-\frac{1}{4}\sum_{j=1}^{d}\sum_{k=1}^{d} a_{jk}(e_2\Phi_j + e_3\Psi_j)\ \{2(\Psi_j b_k^{(1)} + \Psi_k b_j^{(1)}) + (\Psi_j b_k^{(2)} + \Psi_k b_j^{(2)})\}$$

$$-\frac{1}{4}\operatorname{Re}\sum_{j=1}^{d}\sum_{k=1}^{d} a_{jk}\{(e_1\Phi_j + e_2\Psi_j) + i\,(e_2\Phi_j + e_3\Psi_j)\}\cdot$$

$$\cdot\{(\Phi_j c_k^{(1)} + \Phi_k c_j^{(1)}) + i(\Psi_j c_k^{(1)} + \Psi_k c_j^{(1)})\}$$

$$-\frac{1}{8}\operatorname{Re}\sum_{j=1}^{d}\sum_{k=1}^{d} a_{jk}\ \{(e_1\Phi_j + e_2\Psi_j) + i(e_2\Phi_j + e_3\Psi_j)\}\cdot$$

$$\cdot\{(\Phi_j c_k^{(2)} + \Phi_j c_j^{(2)}) + i(\Psi_j c_k^{(2)} + \Psi_k c_j^{(2)})\},$$

where

$$e_1 = <\Psi,\Psi>/(<\Phi,\Phi><\Psi,\Psi> - <\Phi,\Psi>^2),$$
$$e_2 = -<\Phi,\Phi>/(<\Phi,\Phi><\Psi,\Psi> - <\Phi,\Psi>^2),$$
$$e_3 = <\Phi,\Phi>/(<\Phi,\Phi><\Psi,\Psi> - <\Phi,\Psi>^2),$$
$$\boldsymbol{b}^{(1)} = A(\bar{U})^{-1}\boldsymbol{b}, \qquad \boldsymbol{b}^{(2)} = (A(\bar{U}) - 4n_0^2 D(0))^{-1}\boldsymbol{b}$$

with

$$\boldsymbol{b} = \mathrm{col}\ (b_1, \cdots, b_d), \qquad b_j = \sum_{k=1}^{d} a_{jk}(\Phi_j\Phi_k + \Psi_j\Psi_k)$$

and

$$\boldsymbol{c}^{(1)} = (A(\bar{U}) + 2i\omega_0)^{-1}\boldsymbol{c},\ \boldsymbol{c}^{(2)} = (A(\bar{U}) + 2i\omega_0 - 4n_0^2 D(0))^{-1}\boldsymbol{c}$$

with

$$\boldsymbol{c} = \mathrm{col}\ (c_1, \cdots, c_d),\ c_j = \sum_{k=1}^{d} a_{jk}(\Phi_j - i\Psi_j)(\Phi_k - i\Psi_k).$$

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