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OPTIMALITY OF NON-SEQUENTIAL ESTIMATION RULE OF THE EXPONENTIAL MEAN LIFE IN A LIFE TEST

Dedicated to Professor Kenzo Iizuka on his 60th birthday

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1. Introduction

The data to which statistical methods are applied in reliability usually results from life tests. A life test is one in which a sample of M items from a population of interest is put into an environment as similar as possible to the one which the items will experience in actual use. Since the failures occur in order, it often takes a long time to continue the experiment until the last failure. So usually censoring is used. If the life test is terminated at a specified time, the censoring is called Type I. If the life test is terminated at the time of a particular failure, it is called Type II.

In this paper, an extension of Type II censoring is considered, that is, the number of failures are determined sequentially.

As a failure model, we assume the exponential distribution with the probability density function

$$f(x)=\theta^{-1}exp(-x/\theta),\ x>0,\ \theta>0.$$

A life test is conducted under two different situations. One is the non-replacement case in which a failed item is not replaced by a new item. The other is the replacement case where a failed item is replaced at once by the new item drawn at random from the same population. Epstein and Sobel [4], [5] and Epstein [6] considered the problem of estimating the mean life θ for Type II censoring case. In these papers the number of failures is given beforehand. But it is also important to decide the number. The greater the number of failures taken, the more accurately we can estimate θ. But in the case that the cost of the test time must be considered, the fewer number of failures should be taken. Therefore the decision of the number of failures becomes important in addition to that of estimation rule. We apply a sequential method to this problem, that is, the decision to stop the experiment at the nth failure is based on the failure times of the first through the nth failures.

We adopt a relative squared error loss, $(\hat{\theta}-\theta)^2/\theta^2$, where $\hat{\theta}$ is an estimate of θ, and a cost function proportional to the test time, i. e., cx_n/θ if we stop the experiment at nth failure and x_n is the failure time, where c is some positive constant,

In non-replacement case, we [11] obtained that a non-sequential rule is the best invariant sequential rule. The purpose of this paper is to show that a non-sequential rule is minimax in both replacement and non-replacement cases.

In non-replacment case, Sen [10] treated this problem under different loss and cost structure. In replacement case, Shapiro and Wardrop [9] considered the Bayes sequential rule under the same loss and different cost structure.

Although this problem can be treated by continuous time sequential method, we restrict ourselves to rules stopping the experiment at each failure time. Continuous time sequential rules have been studied by Chen and Wardrop [1] in non-replacement case, and by El-Sayyad and Freeman [3] and Shapiro and Wardrop [10] in replacement case.

In Section 2, we consider the non-replacement case. The replacement case is treated in Section 3. In Section 4, another minimax property of non-sequential rules is considered.

2. Non-replacement case

In this section the non-replacement case is considered. We denote the nth failure time by X_n $(n=1,\dots,M)$.
First assuming the following prior distribution for θ,

$$h(\theta)=\lambda^{\alpha}\theta^{-(\alpha+1)}exp(-\lambda/\theta)/\Gamma(\alpha),\ \theta>0, \tag{2.1}$$

we shall seek the minimum Bayes risk.

Since the density function of $\boldsymbol{X}_n=(X_1,\dots,X_n)$ given θ is

$$f_\theta(x_1,\dots,x_n)=\frac{M!}{(M-n)!\theta^n}\ exp\left[-\{\sum_{i=1}^{n}x_i+(M-n)x_n\}/\theta\right],$$

the posterior density of θ given $\boldsymbol{X}_n$ is

$$h_n(\theta)=(\lambda+S_n)^{n+\alpha}\theta^{-(n+\alpha+1)}exp\{-(\lambda+S_n)/\theta\}/\Gamma(n+\alpha),$$

where $S_n=\sum_{i=1}^{n} X_i+(M-n)X_n$. Therefore the Bayes estimator of θ with respect to the relative squared error loss given $\boldsymbol{X}_n$ is

$$\hat{\theta}_n=(\lambda+S_n)/(n+\alpha+1),$$

and the posterior expected loss and posterior expected cost given $\boldsymbol{X}_n$ become

$$E\{(\hat{\theta}_n-\theta)^2/\theta^2|\boldsymbol{X}_n\}=1/(n+\alpha+1)$$

and

$$E\{cX_n/\theta|\boldsymbol{X}_n\}=c(n+\alpha)X_n/(\lambda+S_n),$$

respectively. Hence the total cost given $\boldsymbol{X}_n$ is

(2.2) $$L_n=(n+\alpha+1)^{-1}+c(n+\alpha)X_n/(\lambda+S_n),\ n=1,\ldots,M,$$

and we set

$$L_0=1/(\alpha+1),$$

which is the Bayes risk of the rule of not having the life test. The Bayes stopping rule minimizes EL_N among all stopping rules N, where the expectation is taken with respect to the marginal distribution of $(X_1,\ldots,X_M)$. For details see Chapter 7 of Ferguson [7].

It is easy to see that

$$E\left\{\frac{X_{n+1}}{\lambda+S_{n+1}}|\boldsymbol{X}_n\right\}=\frac{1}{(M-n)(n+\alpha+1)}+\frac{(n+\alpha)X_n}{(n+\alpha+1)(\lambda+S_n)},$$

so that from (2.2)

$$E\{L_{n+1}|\boldsymbol{X}_n\}=L_n+c(M-n)^{-1}-(n+\alpha+1)^{-1}(n+\alpha+2)^{-1}.$$

Letting

(2.3) $n(\alpha)=\text{Min }\{0\leq n\leq M \text{ such that } c\geq(M-n)\ (n+\alpha+1)^{-1}\ (n+\alpha+2)^{-1}\}$,

the finite version of the monotone case theorem (Theorem 3.3 of Chow ,et al. [2]) shows that the non-sequential rule which stops the experiment at the

$n(\alpha)th$ failure and estimate θ by the Bayes estimate $\hat{\theta}_{n(\alpha)}$ is the Bayes sequential rule. Therefore from (2.2) the minimum Bayes risk is given by

$$V(\alpha)=EL_{n(\alpha)} = (n(\alpha)+\alpha+1)^{-1}+c\sum_{i=1}^{n(\alpha)}(M-i+1)^{-1}. \tag{2.4}$$

Now we have the following theorem.

THEOREM 1. *The non-sequential rule that the experiment is terminated at the n^*th failure and θ is estimated by $S_{n^*}/(n^*+1)$ is minimax, where*

$$n^*=Min\ \{0\leq n\leq M \text{ such that } c\geq(M-n)\ (n+1)^{-1}\ (n+2)^{-1}\}. \tag{2.5}$$

PROOF. It follows from (2.3) and (2.4) that

$$\lim_{\alpha\to 0} V(\alpha)=(n^*+1)^{-1}+c\sum_{i=1}^{n^*}(M-i+1)^{-1}, \tag{2.6}$$

where n^* is defined by (2.5). It is easy to see that the non-sequential rule in the theorem has a constant risk function, which is equal to the right hand side of (2.6). Therefore from the well known result (e. g. Theorem 2 of Ferguson [7], p. 90), the rule is minimax.

REMARK. We [11] showed that the rule in Theorem 1 is the best invariant sequential rule. Therefore Theorem 1 is an extension of the result.

3. Replacement case

In this section we consider the replacemen case. We shall show that a non-sequential rule is minimax by the same method as that in the non-replacement case.

Denoting the nth failure time by X_n $(n=1,2,\ldots)$, the density function of $\boldsymbol{X}_n=(X_1,\ldots,X_n)$ is given by

$$f_\theta(x_1,\ldots,x_n)=(M/\theta)^n exp(-Mx_n/\theta)$$

Assuming the prior distribution given by (2.1), the posterior density of θ given

$\boldsymbol{X}_n$ is

$$h_n(\theta)=(\lambda+Mx_n)^{\ +n}\theta^{-(\alpha+n+1)}\ exp\ \{-(\lambda+Mx_n)/\theta\}/\Gamma(\alpha+n).$$

Therefore the Bayes estimator of θ given $\boldsymbol{X}_n$ is

$$\hat{\theta}_n=(\lambda+Mx_n)/(n+\alpha+1),$$

and the posterior expected loss and posterior expected cost given $\boldsymbol{X}_n$ become

$$E\{(\hat{\theta}_n-\theta)^2/\theta^2|\boldsymbol{X}_n\}=(n+\alpha+1)^{-1}$$

and

$$E\{cX_n/\theta|\boldsymbol{X}_n\}=c(n+\alpha)X_n/(\lambda+MX_n).$$

Hence the total cost at the nth failure given $\boldsymbol{X}_n$ is

(3.1) $$L_n=(n+\alpha+1)^{-1}+c(n+\alpha)X_n/(\lambda+MX_n),\ n=1,2,\dots,$$

and we set

$$L_0=(\alpha+1)^{-1},$$

which is the Bayes risk of the rule not having the life test.

The Bayes stopping rule is defined by the stopping rule which minimizes EL_N among all stopping rules N where the expectation is taken to the marginal distribution of $(X_1, X_2, \dots)$.

It is easy to see that

$$E\left\{\frac{X_{n+1}}{\lambda+MX_{n+1}}\,|\boldsymbol{X}_n\right\}=\frac{1}{M(n+\alpha+1)}+\frac{(n+\alpha)X_n}{(n+\alpha+1)(\lambda+MX_n)},$$

so that from (3.1)

$$E\{L_{n+1}|\boldsymbol{X}_n\}=L_n+cM^{-1}-(n+\alpha+1)^{-1}(n+\alpha+2)^{-1}.$$

Therefore from the monotone case theorem by Chow, et al. [2], we have that the non-sequential rule which stops the experiment at the $n(\alpha)$th failure and estimates θ by $\hat{\theta}_{n(\alpha)}$ is the Bayes sequential rule where

(3.2) $$n(\alpha)=\text{Min}\ \{n\geq 0 \text{ such that } c\geq M(n+\alpha+1)^{-1}(n+\alpha+2)^{-1}\}.$$

By (3.1) the minimum Bayes risk is given by

(3.3) $$\begin{aligned} V(\alpha)&=EL_{n(\alpha)} \\ &=(n(\alpha)+\alpha+1)^{-1}+cn(\alpha)M^{-1}. \end{aligned}$$

Now we shall show the following theorem.

THEOREM 2. *The non-sequential rule which terminates the experiment at the n^*th failure and estimates θ by $MX_{n^*}/(n^*+1)$ is minimax where*

(3.4) $$n^*=Min\ \{n\geq 0 \text{ such that } c\geq M(n+1)^{-1}(n+2)^{-1}\}.$$

PROOF. It follows from (3.2) and (3.3) that

(3.5) $$\lim_{\alpha\to 0} V(\alpha)=(n^*+1)^{-1}+cn^*M^{-1},$$

where n^* is defined by (3.4). It is easy to see that the risk function of the rule in the theorem is constant and equal to the right hand side of (3.5). Hence it is minimax.

4. Another optimality of non-sequential rules

In this section we study another desirable properties of non-sequential rules.

Let us consider the following non-sequential rules; the experiment is stopped at the rth failure and θ is estimated by $\delta=S_r/(r+1)$ in the non-replacement case ($r=0, 1,\ldots, M$, $S_0=0$) and $\delta=MX_r/(r+1)$ in the replacement case ($r=0, 1,\ldots, X_0=0$), respectively.

Difine

$$a_r=E\{(\delta-\theta)^2/\theta^2\}$$

and

$$b_r=E\{X_r/\theta\},$$

which are independent of θ. Then we shall show the following desirable properties of the non-sequential rules.

THEOREM 3. *(i) Let $(N,\tilde{\theta}_N)$ be any sequential rule which satisfies*

$$E\{(\tilde{\theta}_N-\theta)^2/\theta^2\}\leq a_r \text{ for all } \theta,$$

then

$$\underset{\theta}{Sup}\ E\{X_N/\theta\}\geq b_r.$$

(ii) Let $(N,\tilde{\theta}_N)$ be any sequential rule which satisfies

$$E\{X_N/\theta\}\leq b_r \text{ for all } \theta,$$

then

$$\underset{\theta}{Sup}\ E\{\tilde{\theta}_N-\theta)^2/\theta^2\}\geq a_r.$$

PROOF. From (2.5) and (3.4), there exists a positive constant c such that $n^*=r$. Therefore from Theorems 1 and 2,

$$\underset{\theta}{Sup}[\{(\tilde{\theta}_N-\theta)^2/\theta^2\}+cE\{X_N/\theta\}]\geq a_r+cb_r,$$

which implies

$$\underset{\theta}{Sup}E\{(\tilde{\theta}_N-\theta)^2/\theta^2\}-a_r\geq c[b_r-\underset{\theta}{Sup}E\{X_N/\theta\}].$$

The results follow from this inequality.

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