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ON AFFINE ISOPERIMETRIC INEQUALITY FOR A STRONGLY CONVEX CLOSED HYPERSURFACE IN THE UNIMODULAR AFFINE SPACE A^{n+1}

To Professor Kenzo Iizuka on his sixtieth birthday

Takeshi SASAKI

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Introduction

In this paper we prove an affine isoperimetric inequality for a strongly convex closed hypersurface in the unimodular affine space A^{n+1}. Here A^{n+1} is the affine space endowed with an invariant volume form.

Let M be a smooth immersed hypersurface in A^{n+1} which is locally strongly convex. Then we can always endow M with the Riemannian metric called the affine metric. In §1 we will recall the definition of this metric and some terminologies, and will give relevant known results.

Further let M be embedded and closed, i. e. the boundary of a convex body. We denote by $V(M)$ the volume of this convex body and by $A(M)$ the surface area of M with respect to the affine metric. Both are invariant under unimodular affine transformations. Then in § 2 we will prove

A. $\quad A(M)^{n+2} \leqq (n+1)^n \omega_n^2 V(M)^n.$

Here ω_n is the volume of the unit n-sphere. The proof which we will give here is for almost part the same as that for $n=2$ given by W. Blaschke [1].

In § 3 we will study the case that the equality is attained in A. This is reduced to the study of a variational problem of the surface area and accomplished making use of computations due to E. Calabi [4]. We reproduce some of these computations in terms of the moving frame. The result is

B. *The equality holds only when the affine mean curvature is positively constant.*

Then by a theorem of [3] on an elliptic affine hypersphere and by a theorem of [10] on a hypersurface with constant affine mean curvature, we can see

THEOREM 1. *Let M be a strongly convex closed hypersurface in the unimodular* *affine space* A^{n+1}. *Then the affine area* $A(M)$ *and the volume* $V(M)$ *of the convex body bounded by* M *satisfies an inequality* $A(M)^{n+2} \leqq (n+1)^n \omega_n^2 V(M)^n$ *and the equality holds if and only if* M *is a hyperquadric.*

§ 1. Affine metric on a hypersurface

In this section we will recall fundamental definitions and properties in the affine geometry of a (locally) strongly convex hypersurface in the unimodular affine space. Since this material is not so familiar, we are going to describe it in a rather detailed manner. For proofs and other properties, see [1], [6], [4], [9].

The unimodular affine $(n+1)$-space A^{n+1} is an affine space endowed with a unimodular structure, with the distinguished element $(.,\ldots,.)$ giving the determinant of $n+1$ vectors. The family $\{x; e_1, \ldots, e_{n+1}\}$ of a point x and $n+1$ vectors e_i is called an affine frame when it satisfies

$$(e_1, \ldots, e_{n+1}) = 1. \tag{1.1}$$

Let G be the unimodular affine group. Then any affine frame is carried to another frame by an element of G. The dual frame $\{\omega^1, \ldots, \omega^{n+1}\}$ is given by

$$dx = \sum \omega^\alpha e_\alpha. \tag{1.2}$$

The change of the frame is written as

$$de_\alpha = \sum \omega_\alpha^\beta e_\beta. \tag{1.3}$$

Then ω^α, ω_α^β are the Maurer-Cartan forms of the group G. The structure equations of A^{n+1} are

$$\begin{cases} \sum \omega_\alpha^\alpha = 0, \\ d\omega_\alpha = \sum \omega_\beta \wedge \omega_\beta^\alpha, \\ d\omega_\alpha^\beta = \sum \omega_\alpha^\gamma \wedge \omega_\gamma^\beta. \end{cases} \tag{1.4}$$

Our object here is to study properties of a hypersurface invariant under the group G. Given a hypersurface M, we choose a frame whose first n vectors e_1, $\ldots$, e_n span the tangent space of M at x. Then

(1.5) $$\omega^{n+1}=0.$$

This gives $d\omega^{n+1}=\sum\omega^i\wedge\omega_i^{n+1}=0$ and we have

(1.6) $$\omega_i{}^{n+1}=\sum h_{ij}\omega^j;\ h_{ij}=h_{ji}.$$

Here we use the index range as $1\leqq i, j, \ldots \leqq n$ and $1\leqq \alpha, \beta, \ldots \leqq n+1$. We also use the summation convention when no fear of ambiguity. Let us assume M is locally strongly convex. Then the appropriate choice of the orientation implies $h=(h_{ij})>0$. Put

(1.7) $$H=det(h_{ij})>0.$$

Then the quadratic form

(1.8) $$II=H^{-1/n+2}h_{ij}\omega^i\omega^j$$

is called the second fundamental form of M. It is sasily verified that this form is invariant under the group G. Hence this defines a Riemannian metric on M called the *affine metric*. The $(n+1)$-*th* vector e_{n+1} of the frame is until now not specified. But we can see that there is a unique way to choose e_{n+1} that the quation

(1.9) $$(n+2)\omega_{n+1}^{n+1}+d\log H=0$$

is satisfied. In this case e_{n+1} is called the *affine normal* at x and the vector

(1.10) $$\nu=H^{1/n+2}e_{n+1}$$

is called the *affine normal vector*. The latter is affinely defined and invariant. As for its geometrical meaning see § 43 in [1]. Moreover one has

(1.11) $$\Delta x=n\nu,$$

which also shows that ν is uniquely defined. We define the derivatives h_{ijk} of h_{ij} by

(1.12) $$h_{ijk}\omega^k=dh_{ij}+h_{ij}\omega_{n+1}^{n+1}-h_{ik}\omega_j^k-h_{jk}\omega_k^i,$$

where (h_{ijk}) is symmetric in all indices. The quantity

(1.13) $$F = |h_{ijk}|^2 = h_{ijk} h_{lmn} h^{il} h^{jm} h^{kn}$$

is called the *Fubini-Pick invariant*. Here (h^{ij}) is the inverse of h. F measures the deviation of the hypersurface from the hyperquadric: F vanishes if and only if M is a hyperquadric ([9]). When e_{n+1} is the affine normal, (h_{ijk}) satisfy the special relation

(1.14) $$h^{ij} h_{ijk} = 0$$

which is called the *apolarity condition*. Let $\bar{\omega}_i^j$ be the Riemannian connection form of the affine metric. It is determined by properties $d\omega^i = \omega^i \wedge \bar{\omega}_j^i$ and $dh_{ik} = h_{ij}\bar{\omega}_k^j + h_{kj}\bar{\omega}_i^j$. In fact we have

(1.15) $$\bar{\omega}_i^j - \omega_i^j = \frac{1}{2} h^{jk} h_{ikm} \omega^m.$$

We next take a derivation of (1.9) and get

(1.16) $$\omega_{n+1}^i \wedge \omega_i^{n+1} = 0,$$

which gives

(1.17) $$\omega_{n+1}^i = -l^{ik} \omega_k^{n+1}; \ l^{ik} = l^{ki}.$$

Let us lower indices using h_{ij}: $l_{ij} = h_{ik} l^{km} h_{mj}$, and define the quadratic form

(1.18) $$III = l_{ij} \omega^i \omega^j,$$

that is called the *third fundamental form* of M. This form is also invariant under a unimodular transformation which keeps the affine normal invarinat. As is easily imagined, the form III plays the role of the second fundamental form in the hypersurface theory in the Riemannian geometry. In fact the curvature tensor R_{ijkl} of the metric II is given, *under the assumption* $H=1$, by

(1.19) $$R_{ijkl} = \frac{1}{2} \{l_{jl} h_{ik} - l_{il} h_{jk} - l_{jk} h_{il} + l_{ik} h_{jl}\} + \\ + \frac{1}{4} (h_{jkm} h_{iln} - h_{jlm} h_{ikn}) h^{mn}.$$

This is the Gauss equation. Codazzi-Minardi equations are

(1.20) $$h_{ijk,m} - h_{ijm,k} = l_{ik} h_{jm} + l_{jk} h_{im} - l_{im} h_{jk} - l_{jm} h_{ik},$$

and

(1.21) $$l_{jk,m}-l_{jm,k}=\frac{1}{2}(h_{ijm}l_{kn}-h_{ijk}l_{mn})h^{ni},$$

Moreover (h_{ijk}) satisfies a Ricci identity:

(1.22) $$h_{ijk,pq}-h_{ijk,qp}=h_{mjk}R_i{}^m{}_{qp}+h_{imk}R_j{}^m{}_{qp}+h_{ijm}R_k{}^m{}_{qp}.$$

These formulas (1.19)−(1.22) are proved similarly to those formulas in the Riemannian geometry.

Let $k_1, \dots, k_n$ be eigenvalues of (l_{ij}) with respect to II. The elementary symmetric functions of k_i are the scalar invariants. We list only two of them for later use.

(1.23) $$L=\frac{1}{n}\Sigma k_i=\frac{1}{n}\operatorname{Trace}(l_{ij}),$$

$$L_2=\frac{2}{n(n-1)}\Sigma_{i<j}k_ik_j.$$

L is called the *affine mean curvature*. Its vanishing is the condition that the first variation of the affine area is trivial (Proposition 4). And a hypersurface with $L=0$ is called the *affine maximal* hypersurface ([7], [4]).

Let us consider the condition

(1.24) $$III=K\,II,$$

which may be called the affine umbilic property. The hypersurface with this property is called an *affine hypersphere*. The scalar function K is known to be constant and equal to the affine mean curvature L by the Codazzi-Minardi equation (1.21). Further one can see, using (1.11), the condition (1.24) is equivalent to

(1.25) $$\nu=\begin{cases}-L(x-a) & L\neq 0\\ a & L=0,\end{cases}$$

where a is a constant vector. This is a geometrical description of hyperspheres. According as $L>0$, $=0$ or <0, the hypersphere is called of type elliptic, parabolic or hyperbolic. For these matters see [11], [3], [6], [12]. We here cite one of structure theorems for later use:

THEOREM A. ([3], [11]) *The only complete affine hyperspheres of elliptic or parabolic types are hyperquadrics.*

At the last of this section we prepare some other notations. Let dA be the surface element given by

$$dA = H^{1/n+2}\omega^1 \wedge \ldots \wedge \omega^n. \tag{1.26}$$

We introduce a function p on M by

$$p(x) = -(e_1, \ldots, e_n, x), \tag{1.27}$$

which is called the *affine support function* of M. It is positive when the origin of A^{n+1} is chosen in the concave side of M. Assume M bounds a convex body. Then the volume $V(M)$ of this convex body is given by

$$V(M) = \frac{1}{n+1}\int_M p\, dA. \tag{1.28}$$

When A^{n+1} is equipped with an euclidean structure so that the determinant $(\ldots)$ comes from the euclidean inner product, then $V(M)$ turns out to be the euclidean volume.

EXAMPLE. ([7]) Let f be a strongly convex function of $x' = (x^1, \ldots, x^n)$ and $M = \{(x', f(x'))\}$ be the graph of f. Then, setting $\omega^i = dx^i$ and $e_i = (0, \ldots, 0, \underset{i}{1}, 0, \ldots, 0, f_i)$, $e_{n+1} = (0, \ldots, 0, 1)$, we can easily see $de_i = f_{ij}\omega^j e_{n+1}$ and hence

$$II = (\det f_{ij})^{-1/n+2} f_{ij} dx^i dx^j. \tag{1.29}$$

The area element is given by

$$dA = (\det f_{ij})^{1/n+2}\, dx^1 \ldots dx^n. \tag{1.30}$$

§ 2. Affine isoperimetric inequality

In the book [1], W. Baschke has given isoperimetric inequalities of several types for the surface in A^3. One of them which seems to be fundamental is the following

THEOREM ([1]). *For any closed strongly convex surface M in A^3, the affine area $A(M)$ and the volume $V(M)$ of the convex body bounded by M satisfies an inequality*

$$A(M)^2 \leqq 12\pi\, V(M)$$

and the equality holds if and only if M is an ellipsoid.

In the following we will give a proof of Theorem 1 in Introduction which is a generalization of this inequality in higher dimension. The proof of the inequality part A, which is the aim of this section, is almost the same as that of Blaschke. The equality part B will be treated in the next section.

Throughout this section M will be a smooth strongly convex closed hypersurface.

Take an affine hyperplane A^n in A^{n+1} and fix a coordinate system $(x^1, \dots, x^{n+1})$ such that $(x^1, \dots, x^n)$ is a coordinate system of A^n. Let p be the projection along x^{n+1}: $A^{n+1} \longrightarrow A^n$. Let $D=p(M)$ and $C=p^{-1}(\partial D)\cap M$. Then C is a hypersurface in M and $M-C$ is composed of two connected parts whose closures we denote by $M_\pm$. Each of $M_\pm$ is written as a graph over D:

$$(2.1) \qquad \begin{aligned} M_+ &= \{x^{n+1} = -f(x^1, \dots, x^n)\} \\ M_- &= \{x^{n+1} = g(x^1, \dots, x^n)\}. \end{aligned}$$

Define a new function h by

$$(2.2) \qquad h(x) = \frac{1}{2}(f(x)+g(x))$$

and set

$$(2.3) \qquad \begin{aligned} M'_+ &= \{x^{n+1} = -h(x)\} \\ M'_- &= \{x^{n+1} = h(x)\}. \end{aligned}$$

Then as is easily seen, the set $M' = M'_+ \cup M'_-$ is a smooth closed convex hypersurface over D and this is called a Steiner symmetrization of M. By definition we can see

(2.4) $$V(M')=V(M).$$

PROPOSITION 1. *Let notations be as above. Then*

(2.5) $$A(M')\geqq A(M).$$

The equality holds only when C is planar and $f-g$ is a linear function.

PROOF. By the formula in §1 the areas are given by

$$A(M)=\int_D \{(\det f_{ij})^{1/n+2}+(\det g_{ij})^{1/n+2}\}\, dx^1\dots dx^n,$$

$$A(M')=2\int_D (\det h_{ij})^{1/n+2}\, dx^1\dots dx^n.$$

Hence we have only to show the inequality

$$2(\det h_{ij})^{1/n+2}\geqq(\det f_{ij})^{1/n+2}+(\det g_{ij})^{1/n+2}.$$

This follows from the next lemma. If the equality holds, then Lemma also shows $f_{ij}=g_{ij}$, which implies $f-g$ is linear. Since $-f=g$ on ∂D, C becomes planar.

LEMMA. *Let A, B be $n\times n$ positive definite symmetric matrices. Set*

$$Q(t)=\det(A+t(B-A)).$$

Then $Q(t)^{1/n+2}$ is convex for $0\leqq t\leqq 1$, and strongly convex unless $A=\lambda B$ for some positive constant λ.

PROOF OF LEMMA. It is enough to see the convexity of the function $f(t)=Q(t)^{1/n}$. Moreover it is enough to see $f''(0)\leqq 0$ replacing A and B if necessary. A direct calculation shows

$$f''=\frac{1}{n}Q^{(1-2n)/n}(QQ''-\frac{n-1}{n}Q'^2).$$

Denote by P the complete polarized form of the determinant. Then

$$Q(0)=P(A,\dots,A)$$
$$Q'(0)=nP(A,\dots,A,B-A)$$
$$Q''(0)=n(n-1)\,P(A,\dots,A,B-A,B-A).$$

Hence the value of $QQ''-(n-1)/nQ'^2$ at $t=0$ is equal to

$$n(n-1)\{P(A\ \ldots, A)\ P(A, \ldots, A, B, B)-P(A, \ldots, A, B)^2\}.$$

And it is known that this is nonpositive and negative unless $A=\lambda B$ for some positive constant λ.

We next fix an euclidean structure of A^{n+1} and choose an orthonormal frame $\{e_i\}$ with respect to the euclidean inner product. Then $h=(h_{ij})$ defined in (1.6) is the second fundamental form tensor of M. $H=det\ h_{ij}$ is the Gaussian curvature. On the other hand the affine area element dA is given by

$$dA=H^{1/n+2}\ \omega^1\wedge\ldots\wedge\omega^n.$$

Since $dA_E:=\omega^1\wedge\ldots\wedge\omega^n$ is the euclidean area element of M, we have

(2.6) $$dA=H^{1/n+2}\ dA_E.$$

Hence by the Hölder inequality we have

$$A(M)^{n+2}=(\int dA)^{n+2}\leqq(\int_M H\ dA_E)A_E(M)^{n+1},$$

where $A_E(M)$ is the euclidean surface area. But the integral $\int HdA_E$ is known to be equal to the area of the unit n-sphere ω_n by Chern-Lashof [8], since M bounds a convex body. Therefore we have proved

(2.7) $$A(M)^{n+2}\leqq\omega_n A_E(M)^{n+1}.$$

Note that the equality holds when M is a sphere.

PROPOSITION 2. *Assume M is contained in the interior of a closed convex hyperquadric Q. Then $A(M)\leqq A(Q)$.*

PROOF. By thy invariance of the affine area, we may assume Q is an euclidean sphere. Then the twice use of (2.7) imlies

$$A(M)^{n+2}\leqq\omega_n A_E(M)^{n+1}\leqq\omega_n A_E(Q)^{n+1}=A(Q)^{n+2},$$

which proves the proposition.

PROOF OF THE INEQUALITY PART A OF THEOREM 1. Fix a number $\varepsilon>0$. Then we know the following fact due to Blaschke [2] and W. Gross: There exists a euclidean sphere $S$and a closed hypersurface M', which is obtained by certain number of succesive operations of a Steiner symmetrization on M, such that S contains M' and

$$V(S)\leqq(1+\varepsilon)V(M').$$

Then from (2.4), (2.5) and Prorosition 2, we get

$$A(M)^{n+2}\leqq c_n V(M)^n(1+\varepsilon)^n$$

where $c_n=A(S)^{n+2}/V(S)^n=(n+1)^n\omega_n^2$. Letting here $\varepsilon\longrightarrow 0$, we complete the proof.

We will give some remarks on the equality part. As the proof shows, the equality arises only when $A(M)=A(M')$ for any Steiner symmetrization M' of M. In this case by Proposion 1 the surface C is planar. Let Π be the hyperplane containing C. The hypersurface M is symmetric with respect to Π. Next picking a vector v in Π we do the process of a Steiner symmetrization along v. Then the midpoint of the line segment cutted out by C along v moves on a plane of dimension $n-1$. Does this property assure C is a quadric? When $n=2$, the answer is yes, [2]. But the author does not know the answer in general. Even if all sections C are quadrics, there remains another question whether this implies that M is itself a hyperquadric. So in the next section we will investigate the meaning of the equality in terms of the affine mean curvature and prove the latter part of Theorem 1.

§ 3. The equality case —— Hypersurface with constant affine mean curvature

The problem we treat in this section is to find convex hypersurfaces which have critical values of the surface area A with the fixed volume V. Let M be a given hypersurface which is closed and strongly convex. The deformations of M which we consider shall be also closed and strongly convex. Let us denote deformed hypersurfaces by M_t with parameter t; $M_0=M$. Then the problem is to study the property

$$\frac{d}{dt} A(M_t)|_{t=0}=0, \tag{3.1}$$

under the condition $V(M_t)$=const.. This is equivalent to find a constant λ, which is the Lagrange multiplier, satisfying

$$\frac{d}{dt} A(M_t)|_{t=0}+ \lambda \frac{d}{dt} V(M_t)|_{t=0} = 0 \tag{3.2}$$

for all deformations of M. The calculation of the first variation of $A(M)$ is done already in [4], which we follow first in the moving frame description. Second, we calculate the first variation of $V(M)$.

We choose an affine frame $(e_1, \dots, e_{n+1})$ so that $H=1$ and e_{n+1} is the affine normal vector. The point of the deformed surface M_t is written as

$$x_t=x+a^i(t,x)e_i(x)+\nu(t,x)e_{n+1}(x) \tag{3.3}$$

giving the correspondence $M \longrightarrow M_t$ by $x \longrightarrow x_t$. Then a suitable choice may allow us to pose the condition that the $e_{n+1}(x)$-component of dx_t is trivial (see Proposition 4.1 in [4]). Since an easy calculation using formulas in §1 show

$$dx_t=dx+(da^j+a^i\omega_i^j+\nu\omega_{n+1}^j)e_j+(a^i\omega_i^{n+1}+d\nu)e_{n+1},$$

this condition is equivalent to

$$d\nu+a^i\omega_i^{n+1}=0. \tag{3.4}$$

If we set $d\nu=\nu_i\omega^i$, then

$$\nu_i=-a^jh_{ji}. \tag{3.5}$$

From now on we assume this condition is satisfied. Then

$$dx_t=\omega_t^ie_i\ ;\ \omega_t^i=\omega^i+da^i+a^j\omega_j^i+\nu\omega_{n+1}^i. \tag{3.6}$$

Now letting $a^i{}_{,j}$ denote the covariant derivation of a^i, we have by (1.15)

$$a^i{}_{,j}\omega^j=da^i+a^j\omega_j^i+\frac{1}{2}h^{ik}h_{jkm}\omega^m.$$

Hence, making use of (3.5) and (1.17), we see

$$\omega_t^i = C_k^i \omega^k \text{ where } C_k^i = \delta_k^i - \nu l_k^i - h^{im}\nu_{m,k} - \frac{1}{2} a^i h_{ik}^j. \tag{3.7}$$

Put $C = \det(C_k^i)$. We have

PROPOSITION 3. *Let dA_t be the surface area element of M_t; $dA_0 = dA$. Then $dA_t = C^{(n+1)/(n+2)} dA$.*

PROOF. Let (D_i^k) be the inverse of (C_k^i). We use $(e_1, \ldots, e_{n+1})$ as an affine frame at x_t. By definition (1.6) the second fundamental tensor h_{tij} of M_t is given by

$$h_{tij} = h_{ik} D_j^k,$$

Since we have chosen $H=1$, $\det(h_{tij}) = C^{-1}$. Hence the affiine metric of M_t is

$$II_t = C^{1/n+2} h_{tij} \omega_t^i \omega_t^j = C^{1/2+2} h_{ij} C_k^j \; \omega^i \omega^k.$$

From this we get $dA_t = C^{n+1/n+2} \; \omega^1 \wedge \ldots \wedge \omega^n$ and completes the proof from the fact $dA = \omega^1 \wedge \ldots \wedge \omega^n$.

The above proposition implies

$$A(M_t) = \int_M C^{n+1/n+2} \, dA.$$

Let us compute $\delta A = dA(M_t)/dt|_{t=0}$. (In the following we denote the infinitesimal variation at $t=0$ by δ) From the formula (3.7) we get

$$\begin{aligned} \Sigma_i \, \delta C_i^i &= -\delta \nu l_i^i - h^{im}(\delta\nu)_{m,i} - \frac{1}{2} \delta a^j h_{ji}^i \\ &= -nL\delta\nu - \Delta(\delta\nu) \qquad \text{(by (1.14) and (1.23))}. \end{aligned}$$

Therefore we have

$$\delta A = -\frac{n+2}{n+1} \int (nL\delta\nu + \Delta(\delta\nu)) \, dA = -\frac{n+2}{n+1} \int nL\delta\nu \, dA. \tag{3.8}$$

Note that the above calculations are valid even if M is not closed but when the deformations are compactly supported. Namely

PROPOSITION 4 ([7], [4]). *The first variation δA is zero for all compactly supported deformations of a locally strongly convex hypersurface if and only if the mean curvature L vanishes.*

Let us next recall the integral (1.28) giving the volume of the convex body bounded by M. The volume element dV is given by

$$dV=-\frac{1}{n+1}(e_1, \ldots, e_n, x)\, \omega^1\wedge\ldots\wedge\omega^n = \frac{1}{n+1}\, p\, dA, \tag{3.9}$$

where p is the affine support function. For the surface M_t, we have

$$dV_t=-\frac{1}{n+1}\{(e_1, \ldots, e_n, x)+\nu\}\omega_t^1\wedge\ldots\wedge\omega_t^n.$$

Then by (3.7) we get

$$dV_t=\frac{1}{n+1}(p-\nu)\, C\, dA. \tag{3.10}$$

Hence we have

$$\begin{aligned} V&=\frac{1}{n+1}\int (p\delta C-\delta\nu)\, dA \\ &=-\frac{1}{n+1}\int (\delta\nu+p(nL\delta\nu+\Delta(\delta\nu)))dA \\ &=-\frac{1}{n+1}\int (1+nLp+\Delta p)\delta\nu\, dA. \end{aligned} \tag{3.11}$$

However we have the following identity for p.

PROPOSITION 5 ([10]). $\Delta p=n(1-Lp)$.

PROOF. Differentiating $p=-(e_1, \ldots, e_n, x)$ and using identities in § 1 we can see

$$p_i=-(e_1, \ldots, \underset{i}{\underbrace{e_{n+1}}}, \ldots, e_n, x). \tag{3.12}$$

Let p_{ij} denote the covariant differentiation of p_i: $p_{ij}\omega^j = dp_i - p_j\bar{\omega}_i^j$. To simplify the calculation we choose the affine frame satisfying $h_{ij}=\delta_{ij}$. Then

$$\omega_i^j + \omega_j^i = -h_{ijk}\omega^k. \tag{3.13}$$

We have

$$-dp_i = \Sigma_j\,(e_1,\ \dots,\ de_j,\ \dots,\ \overset{i}{\smile}{e_{n+1}},\ \dots,\ e_n,\ x) + (e_1,\ \dots,\ de_{n+1},\ \dots,\ e_n,\ x) + (e_1,\ \dots,\ e_n,\ dx).$$

Making use of (1.1)−(1.6), we get

$$-dp_i = \Sigma_j p_j \omega_j^i - \omega^i - p_{n+1}^i\ .$$

Then using (3.13)

$$p_{ij}\omega^j = \omega^i + \frac{1}{2}\, h_{ijk} p_j \omega^k - p l_j^i \omega^j.$$

Hence by the apolarity condition (1.14),

$$\Delta p = \Sigma\ p_{ii} = n(1 - Lp).$$

From this proposition, the variation of the volume is given by

$$\delta V = -\int \delta\nu\ dA\ . \tag{3.14}$$

Combining this with (3.8) we have proved

PROPOSITION 6. $\delta A + \lambda\delta V = -\int (\lambda + n(n+1)L/(n+2))\ \delta\nu dA.$

THEOREM 2. *Assume the hypersurface M has a critical value of the area function with the fixed volume. Then the affine mean curvature must be a positive constant.*

PROOF. $L>0$ is seen from

$$0 = \frac{1}{n}\int \Delta p\ dA = \int dA - L\int p\ dA = A(M) - (n+1)LV(M).$$

With this theorem 2 the proof of the latter part of Theorem 1 is complete.

The remaining implication is provided by the next

THEOREM B. *Let M be a closed strongly convex hypersurface with a constant mean curvature. Then M is a hyperquadric.*

REMARK. Under the assumption of Theorem B, Hsiung and Shahin [10] proved that M is an elliptic affine hypersphere. But Theorem A in § 1, then implies M is a hyperquadric. Hence we have this theorem. For the sake of completeness we will give a sketch of proof due to [10] modifying a little. They proved first the equality

$$\frac{1}{n!} d(x, \overbrace{dx\wedge \dots \wedge dx}^{n-2} \wedge de_{n+1}, e_{n+1}) = (L_2 p - L)\, dA. \tag{3.15}$$

This is seen by the formulas (1.17), (1.23) and (1.27). Then by integration and by the Stokes theorem

$$\int L\, dA = \int L_2 p\, dA. \tag{3.16}$$

Next note that an elementary inequality

$$(L)^2 \geqq L_2. \tag{3.17}$$

The equality is attained only if all k_i are equal. Then, taking an origin in the interior so that $p > 0$ and assuming L is constant, we have

$$L\int dA = \int L\, dA = \int L_2 p\, dA \leqq \int L^2 p\, dA = L\int Lp\, dA = L\int dA.$$

Hence $\int (L_2 - L^2) p\, dA = 0$ and the equality holds in (3.17). Therefore we must have $k_1 = \dots = k_n$, which implies $III = LII$, namely M is an elliptic affine hypersphere.

Generalizations of (3.15) and interesting integral formulas like (3.16) have been given in [10].

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Department of Mathematics
Faculty of Science
Kumamoto University