Kumamoto J. Sci. (Math.)
Vol. 16, 39～47 March (1984)

ON THE AUTOMNRPHISMS OF CHEVALLEY GROUPS OVER $\mathfrak{p}$-ADIC INTEGER RINGS

Dedicated to Professor Kenzo Iizuka on his 60th birthday

Kazuo SUZUKI

(Recived November 18, 1983)

Introduction. Let G be a simple and simply connected Chevalley-Demazure group scheme assocated with a connected complex simple Lie group of type Φ. For any commutative ring R with 1, let $G(R)$ be the group of R-points of G (for the defintion see E. Abe [1]). It is well known that if R is a perfect field, then any automorphism of $G(R)$ can be expressed as the product of an inner, a diagonal, a graph and a field automorphism (see R. Steinberg [5]).

In this note, for a local ring, we shall give a sufficient condition for an automorphism of $G(R)$ to be expressed by the same product as above, and futhermore for a complete $\mathfrak{p}$-adic integer ring $\mathfrak{o}_{\mathfrak{p}}$, we shall show that any automorphism of $G(\mathfrak{o}_{\mathfrak{p}})$ is expressed by the same product as above except some cases. The main theorems are stated in Section 1 and are proved in Section 2 and 3.

1. The statement of the main theorems.

1.1. Let G be a simple and simply conneced Chevalley-Demazure group scheme and R be a local ring. Let $x_\alpha(t)$ be the unipotent element of $G(R)$ associated with a root α of Φ and $t\in R$. Assume that 2 is a unit elemet of R if G is symplectic. Then in M. Stein [4], it is known that for rank $\Phi>1$ (resp. rank $\Phi=1$), the following relations A, B, C, (resp. A, B', C) is a complete set of relations for $G(R)$.

A) $x_\alpha(s)x_\alpha(t)=x_\alpha(s+t)$ for root $\alpha\in\Phi$ and $s, t\in R$

B) $[x_\alpha(s), x_\alpha(t)]=\prod_{i\alpha+j\beta} x_{i\alpha+j\beta}(N_{\alpha\beta ij}s^it^j)$

for all $\alpha,\beta\in\Phi$ such that $\alpha+\beta\neq 0$, where the product is in some order and $N_{\alpha\beta ij}$ are certain integers depend only the root system (for elements a, b of a group, we denote by $[a, b]=aba^{-1}b^{-1}$).

B') $w_\alpha(t)x_\alpha(u)w_\alpha(-t)=x_{-\alpha}(-t^{-2}u)$ for $t\in R^*$ and $u\in R$.

where R^* is the set of unit elements of R and $w_\alpha(t)=x_\alpha(t)x_{-\alpha}(-t^{-1})x_\alpha(t)$ for

$t \in R^*$.

C) $h_\alpha(uv)=h_\alpha(u)h_\alpha(v)$ for $u, v \in R^*$,

where $h_\alpha(u)=w_\alpha(u)w_{-\alpha}(-1)$ for $u \in R^*$.

Now we descrive the three kinds of automorphisms of $G(R)$. Denote by L_0 the Z-module generated by all simple roots. Let $f_\alpha \in R^*$ for any simple root $\alpha \in \Phi$. Let f be extended to a homomorphism of L_0 into R^*. Define the map ϕ_1 of $\{x_\alpha(t)\}_{\alpha \in \Phi, t \in R}$ onto itself by

$$x_\alpha(t) \longrightarrow x_\alpha(f_\alpha t) \text{ for all } \alpha \in \Phi,\ t \in R.$$

On the other hand, let γ be a ring automorphism of R. Define the map ϕ_2 of $\{x_\alpha(t)\}_{\alpha \in \Phi, t \in R}$ onto itself by

$$x_\alpha(t) \longrightarrow x_\alpha(t^\gamma) \text{ for all } \alpha \in \Phi,\ t \in R.$$

Then the maps ϕ_1 and ϕ_2 can be extended to automorphisms $\bar{\phi}_1$ and $\bar{\phi}_2$ of $G(R)$ respectively since the relations A, B and C are preserved by them. $\bar{\phi}_1$ and $\bar{\phi}_2$ are called the diagonal and the ring automorphism of $G(R)$ respectively.

Let $p=(\alpha_0, \alpha_0)/(\beta_0, \beta_0)$ with α_0 long, β_0 short. Let σ be an angle preserving permutation of the simple root, $\sigma \neq 1$. If two length occur, assume that p is equal to 0 in R and the map: $x \longrightarrow x^p$ is a ring automorphism of R. Then there exists an automorphism ϕ_3 of $G(R)$ and signs ε_α ($\varepsilon_\alpha=1$ if α or $-\alpha$ simple) such that

$$\phi_3 x_\alpha(t) = \begin{cases} x_{\sigma\alpha}(\varepsilon_\alpha t) & \text{if } \alpha \text{ is long or all roots are of one length.} \\ x_{\sigma\alpha}(\varepsilon_\alpha t^p) & \text{if } \alpha \text{ is short.} \end{cases}$$

The automorphism of $G(R)$ of this type are called graph automorphisms (see R. Steinberg [5]).

1.2. THEOREM A. *Let G be a simple and simply connected Chevalley-Demazure group scheme, R a local ring, M the maximal ideal of R, $k=R/M$ the residue class field. Assume that $ch(k) \neq 2$, and further if G is of type A_1 then $k \neq F_3$, if G is of type G_2 then $ch(k) \neq 3$. Let σ be an automorphism of $G(R)$ such that $\sigma U(R)=U(R)$ where $U(R)$ is the subgroup of $G(R)$ generated by $x_\alpha(t)$ for all positive roots and $t \in R$. Then σ is expressed by the product of an inner, a graph and a ring automorphism.*

N. B. Under the hypotheses of the above Theorem, there exists no graph automorphism of $G(R)$ if two length occur.

1.3. Let K be an algebraic number field of finite degree over Q, $\mathfrak{o}$ its ring of integers, $\mathfrak{p}$ a prime of $\mathfrak{o}$, $K_{\mathfrak{p}}$ the completion of K at the valution with respect to $\mathfrak{p}$, $\mathfrak{o}_{\mathfrak{p}}$ the ring of integers of $K_{\mathfrak{p}}$, $\mathfrak{p}'$ the maximal ideal of $\mathfrak{o}_{\mathfrak{p}}$ and $k=\mathfrak{o}_{\mathfrak{p}}/\mathfrak{p}'$.

THEOREM B. *Let G be the same as theorem A. Assume that $ch(k)\neq 2$, and if G are of type A_3, $B_m(m\geqq 2)$, C_n, D_n $(n\geqq 3)$ and F_4 then $k\neq F_3$ and if G is of type G_2* then $ch(k)\neq 3$. *Then any automorphism of $G(\mathfrak{o}_{\mathfrak{p}})$ can be expressed as the product of an inner, a diagonal, a graph and a ring automorphism.*

2 A proof of theorem A.

2.1. Under the same notations as above, let $\mathfrak{a}$ be an ideal of a commutative ring R. Denote by $U(\mathfrak{a})$ (resp. $V(\mathfrak{a})$) the subgroup of $G(R)$ generated by $x_{\alpha}(t)$ for all positive (resp. all negarive) roots α of Φ and $t\in\mathfrak{a}$. Let T be the maximal torus of G. Set $B(R)=U(R)T(R)$, and let $N(R)$ be the subgroup of $G(R)$ generated by $w_{\alpha}(u)$ for all roots α of Φ and $u\in R^*$ where denote $w_{\alpha}(u)=x_{\alpha}(u)x_{-\alpha}(-u^{-1})x_{\alpha}(u)$ and $N(R)/T(R)\cong W(\Phi)$. Let $G(f)$: $G(R)\longrightarrow G(R/\mathfrak{a})$ be the group homomorphism induced by the natural homomorphism $R\longrightarrow R/\mathfrak{a}$. Denote by $G(R,\mathfrak{a})$ (resp. $G^*(R,\mathfrak{a})$) the kernel (resp. the inverse image of the center of $G(R/\mathfrak{a})$) of $G(f)$.

2.2. LEMMA. *Let G be a simple and simply connected Chevalley-Demazure group scheme, R a local ring, $\mathfrak{m}$ the maximal ideal of R and $k=R/\mathfrak{m}$. Assume that if G is of type A_1 then $ch(k)\neq 2$ and $k\neq F_3$, and if G is of type B_2 or G_2 then $k\neq F_2$. Then $G^*(R,\mathfrak{m})$, $G(R,\mathfrak{m})$ are characteristic subgroups of $G(R)$.*

PROOF. Let σ be any automorphism of $G(R)$. By a characterization of normal subgroups of $G(R)$ in E. Abe [1], there exists an ideal $\mathfrak{a}$ of R such that

$$G^*(R,\mathfrak{a})\supset\sigma G^*(R,\mathfrak{m})\supset G(R,\mathfrak{a}).$$

Let $E(R)$ be the subgroup of $G(R)$ generated by $x_{\alpha}(t)$ for all roots α and $t\in R$. Then by the simply connectedness of G, $G(R)=E(R)$, hence the map: $G(R)\longrightarrow G(R/\mathfrak{b})$ is surjective for any ideal $\mathfrak{b}$ of R, therefore we have $G^*(R,\mathfrak{m})\supset G^*(R,\mathfrak{a})$. By the maximality of $\sigma G^*(R,\mathfrak{m})$, we have $\sigma G^*(R,\mathfrak{m})=G^*(R,\mathfrak{m})$.

On the other hand, for the normal subgroup $\sigma G(R,\mathfrak{m})$, there exists an ideal $\mathfrak{c}$ such that

$$G^*(R, \mathfrak{c}) \supset \sigma G(R, \mathfrak{m}) \supset G(R, \mathfrak{c}).$$

Firstly assume $\mathfrak{m}=\mathfrak{c}$. Since $G^*(R, \mathfrak{m})/G(R, \mathfrak{m})$ is isomorphic to the center of $G(k)$, we have $[G^*(R, \mathfrak{m})\colon G(R, \mathfrak{m})] < \infty$ where for a subgroup B of a group A, we denote by $[A\colon B]$ the index of B in A. Then we have $\sigma G(R, \mathfrak{m}) \supset G(R, \mathfrak{m})$ and

$$[G^*(R, \mathfrak{m})\colon G(R, \mathfrak{m})] = [\sigma G^*(R, \mathfrak{m})\colon \sigma\, G(R, \mathfrak{m})] = [G^*(R, \mathfrak{m})\colon \sigma G(R, \mathfrak{m})]$$

and so $G(R, \mathfrak{m}) = \sigma G(R, \mathfrak{m})$. Secondly we will show that $\mathfrak{m}=\mathfrak{c}$. Assume $\mathfrak{m} \supsetneqq \mathfrak{c}$. In E. Abe [1] we have seen that

$G^*(R, \mathfrak{d}) = U(\mathfrak{d})T^*(\mathfrak{d})V(\mathfrak{d})$ for any proper ideal $\mathfrak{d}$ of R where $T^*(\mathfrak{d}) = T(R) \cap G(R, \mathfrak{d})$. For $m \in \mathfrak{m}$, $m \notin \mathfrak{c}$,

$[x_\alpha(m),\ x_{-\alpha}(1)] = x_\alpha(-m^2/(1-m))h_\alpha(1/(1-m))x_{-\alpha}(m/(1-m)) \notin G^*(R, \mathfrak{c})$. On the other hand,

$$[x_\alpha(m),\ x_{-\alpha}(1)] \in G(R, \mathfrak{m}).$$

Therefore, $G^*(R, \mathfrak{c}) \not\supset \sigma G[R, \mathfrak{m})$. This is a contradiction, and it must be $\mathfrak{m}=\mathfrak{c}$.

2.4. LEMMA. *Assume the hypothese of Theorem A,* Set $B(R) = U(R)T(R)$, $B^-(R) = V(R)T(R)$. *Then for an automorphism σ of $G(R)$ such that $\sigma(U(R)) = U(R)$, there exists an element u in $U(R)$ which satisfy the followings*

$u \circ \sigma(B(R)) = (R)$, $u \circ \sigma(B^-(R)) = R^-(B)$, $u \circ \sigma(V(R)) = V(R)$, $u \circ \sigma(T(R)) = T(R)$ where $u \circ \sigma$ is the product of σ and the inner automorphism induced by u.

PROOF. By 2.1. Lemma, we see that σ induces the automorphism $\bar{\sigma}$ of $G(k)$ and $\bar{\sigma}U(k) = U(k)$. There exists an element $\bar{w}_0$ of $N(k)$ such that $\bar{w}_0 U(k) \bar{w}_0{}^{-1} = U(k)$ and $\bar{w}_0 = \bar{w}_{\alpha_1}(1) \cdots\cdots \bar{w}_{\alpha_r}(1)$ where $\bar{w}_{\alpha_i}(1)$ is the image of $w_{\alpha_i}(1)$ under the map: $G(R) \longrightarrow G(k)$. We can write $\bar{\sigma}(\bar{w}_0) = \bar{u}\bar{w}\bar{u}'\bar{t}$ where $\bar{u}, \bar{u}' \in U(k)$, $\bar{w} \in N(k)$, $\bar{t} \in T((k)$, then $\bar{\sigma}(V(k)) = \bar{\sigma}(\bar{w}_0)\bar{\sigma}(U(k))\bar{\sigma}(\bar{w}_0)^{-1} = \bar{u}\bar{w}U(k)\bar{w}^{-1}\bar{u}^{-1}$. Since $U(k) \cap V(k) = 1$, we have $\bar{u}\bar{w}U(k)\bar{w}^{-1}\bar{u}^{-1} \cap U(k) = 1$, hence $\bar{w}U(k)\bar{w}^{-1} \cap U(k) = 1$ and so $\bar{w} = \bar{h}\bar{w}_0$ for some $\bar{h} \in T(k)$. Let $T(\mathfrak{m})$ be the subgroup of $T(R)$ generated by all $h(\chi)$ such that $\chi(\alpha) \equiv 1 \bmod \mathfrak{m}$ for all $\alpha \varepsilon \Phi$. In E. Abe [1] we have seen that $G(R, \mathfrak{m}) = U(\mathfrak{m})\, T(\mathfrak{m}) V(\mathfrak{m})$ Setting $w_0 = w_{\alpha_1}(1) \cdots\cdots w_{\alpha_r}(1)$ in $G(R)$, it holds $\sigma(w_0) = utvw_0u'$ where $u, u' \in U(R)$, $t \in T(R)$, $v \in V(\mathfrak{m})$. Hence we have $\sigma(V(R)) = uV(R)u^{-1}$. Setting $\sigma' = u^{-1} \circ \sigma$, we have $\sigma'(U(R)) = U(R)$, $\sigma'(V(R)) = V(R)$. Since the normalizer of $U(R)$ (resp. $V(R)$) is $B(R)$ (resp. $B^-(R)$)(see N. B. of 3.2. Lemma), it holds that $\sigma'(B(R)) = B(R)$ and

$\sigma'(B^-(R))=B^-(R)$. By $B(R)\cap B^-(R)=T(R)$, it holds $\sigma'(T(R))=T(R)$.

2.5. Let σ be an automorphism of $G(R)$ normalized in above Lemma. Assume that 2 is a unit element of R, then $N(R)$ is the normalizer of $T(R)$, hence σ fixes $N(R)$. Setting $B=B(R)V(\mathfrak{m})$, we have $B=B(R)G(R,\mathfrak{m})$ and $\sigma B=B$. On the other hand α is a simple root if and only if $B\cap Bw_\alpha(1)B$ is a subgroup of $G(R)$. Hence we have $\sigma(w_\alpha(1))=w_\beta(1)t$ for some simple root β and $t\in T(R)$.

$V(R)\cap w_\alpha(1)B(R)w_\alpha(1)^{-1}=x_{-\alpha}(R)$ for simple root α, it holds

$$\begin{aligned}\sigma(x_{-\alpha}(R))&=\sigma(V(R)\cap w_\alpha(1)B(R)w_\alpha(1)^{-1})\\&=V(R)\cap w_\beta(1)B(R)w_\beta(1)^{-1}\\&=x_{-\beta}(R).\end{aligned}$$

By the same way we have $\sigma(x_\alpha(R))=x_\beta(R)$.

Let ρ be the map of the set of simple roots into itself defined by $\alpha\longrightarrow\beta$. Since $\sigma(x_\alpha(1))=x_\beta(t)$ for some $t\in R^*$, we can define the map χ of the set of simple roots into R^* by $\alpha\longrightarrow t$. Hence by the map $x_\alpha(a)\longrightarrow x_\alpha(\chi(\alpha)^{-1}a)$ for simple roots α, we have the diagonal automorphism ψ of $G(R)$ such that $\psi\circ\sigma(x_\alpha(1))=x_{\rho\alpha}(1)$ for any simple root α.

The following three Lemmas are proved by the same way as in R. Steinberg [4], therefore we shall omit the proof of them.

2.6. LEMMA. *Under the hypotheses of Theorem A, an automorphism σ of $G(R)$ such that $\sigma(U(R))=U(R)$ is mormalized by a diagonal automorphism so that*

1) $\sigma x_\alpha(1)=x_{\rho\alpha}(1)$ *for all simple root α,*
2) $w_\alpha(1)=w_{\rho\alpha}(1)$,
3) *ρ preserves angles.*

2.7. LEMMA. *Under the hypotheses of Theorem A, an automorphism σ of $G(R)$ such that $\sigma(U(R))=U(R)$ is normalized by a diagonal and graph automorphism so that*

$\sigma x_\alpha(R)=x_\alpha(R)$ and $\sigma x_\alpha(1)=x_\alpha(1)$ for any simple roots α.

2.8. LEMMA. *Let σ be an automorphism of $G(R)$. Assume that $\sigma x_\alpha(R)=x_\alpha(R)$*

and $\sigma x_\alpha(1)=x_\alpha(1)$ *for all simple roots* α. *Then* σ *is a ring automorphism.*

Here we have proved Theorem A completely.

3. A proof of Theorem B.

Throughout this Section we shall use the same notation as 3.1. Let $\mathfrak{o}_\mathfrak{p}$ be the ring of integers of $K_\mathfrak{p}$, $\mathfrak{p}$ the maximal ideal of $\mathfrak{o}_\mathfrak{p}$.

3.1. LEMMA. *Let* G *be a simple and simply connected Chevalley-Demazure group scheme. Assume that if* G *is of type* A_1 *then* $ch(k)\neq 2$ *and* $k\neq F_3$, *and if* G *is of type* B_2 *or* G_2 *then* $k\neq F_2$. *Then for each rational integer* i, $G^*(\mathfrak{o}_\mathfrak{p},\mathfrak{p}^i)$ *and* $G(\mathfrak{o}_\mathfrak{p},\mathfrak{p}^i)$ *are characteristic subgroups of* $G(\mathfrak{o}_\mathfrak{p})$.

PROOF. Denoting $R=\mathfrak{o}_\mathfrak{p}$, we shall prove this Lemma by using the induction for i. If $i=1$, then by 1.2. Lemma, our Lemma is clear. For any automorphism σ of $G(R)$, we have that for $G^*(R,\mathfrak{p}^i)$, there exists an ideal $\mathfrak{p}^j$ such that

$$G^*(R,\mathfrak{p}^j)\supset\sigma G^*(R,\mathfrak{p}^i)\supset G(R,\mathfrak{p}^j).$$

Assume $i>j$. By the induction assumptions $G^*(R,\mathfrak{p}^j)$ and $G(R,\mathfrak{p}^j)$ are characteristic, hence we have

$$G^*(R,\mathfrak{p}^j)\supset G^*(R,\mathfrak{p}^i)\supset G(R,\mathfrak{p}^j).$$

On the other hand, for $t\in\mathfrak{p}^j$ and $t\notin\mathfrak{p}^i$, we have $x_\alpha(t)\in G(R,\mathfrak{p}^j)$ and $x_\alpha(t)\notin G^*(R,\mathfrak{p}^i)$. Therefore $G^*(R,\mathfrak{p}^i)\not\supset G(R,\mathfrak{p}^j)$, it is contradiction. Hence it must be $i\leqq j$. Using the induction assumption, it holds

$$\begin{aligned}G^*(R,\mathfrak{p}^{i-1})/G^*(R,\mathfrak{p}^i)&\cong\sigma G^*(R,\mathfrak{p}^{i-1})/\sigma G^*(R,\mathfrak{p}^i)\\&\cong G^*(R,\mathfrak{p}^{i-1})/\sigma G^*(R,\mathfrak{p}^i).\end{aligned}$$

Since the order of $G(R)/G(R,\mathfrak{p}^j)$ is finite and $G^*(R,\mathfrak{p}^{i-1})/G^*(R,\mathfrak{p}^j)$ is a homomorphic image of $G^*(R,\mathfrak{p}^{i-1})/G^*(R,\mathfrak{p}^i)$, we have

$$[G^*(R,\mathfrak{p}^{i-1})\colon G^*(R,\mathfrak{p}^i)]\geqq[G^*(R,\mathfrak{p}^{i-1})\colon G^*(R,\mathfrak{p}^j)]$$

On the other hand

$$[G^*(R,\mathfrak{p}^{i-1})\colon G^*(R,\mathfrak{p}^i)][G^*(R,\mathfrak{p}^i)\colon G^*(R,\mathfrak{p}^j)]$$
$$=[G^*(R,\mathfrak{p}^{i-1})\colon G^*(R,\mathfrak{p}^j)]$$

Hence $G^*(R,\mathfrak{p}^i)=G^*(R,\mathfrak{p}^j)$. Therefore we have

$$[G^*(R,\mathfrak{p}^{i-1})\colon G^*(R,\mathfrak{p}^i)]=[\sigma G^*(R,\mathfrak{p}^{i-1})\colon \sigma G^*(R,\mathfrak{p}^i)]$$
$$=[G^*(R,\mathfrak{p}^{i-1})\colon G^*(R,\mathfrak{p}^j)][G^*(R,\mathfrak{p}^j)\colon \sigma G^*(R,\mathfrak{p}^i,)]$$
$$=[G^*(R,\mathfrak{p}^{i-1})\colon G^*(R,\mathfrak{p}^i)][G^*(R,\mathfrak{p}^i)\colon \sigma G^*(R,\mathfrak{p}^i)].$$

Hence $[G^*(R,\mathfrak{p}^i)\colon \sigma G^*(R,\mathfrak{p}^i)]=1$ and we have $G^*(R,\mathfrak{p}^i)=\sigma G^*(R,\mathfrak{p}^i)$. By the same way, we can easily prove that $G(R,\mathfrak{p}^i)$ is characteristic.

3.2. LEMMA. *Let* $\mathfrak{a}$ *be a non-zero ideal of* $\mathfrak{o}_\mathfrak{p}$. *Then the normalizer of* $U(\mathfrak{a})$ *in* $G(\mathfrak{o}_\mathfrak{p})$ *is equal to* $B(\mathfrak{o}_\mathfrak{p})$.

PROOF. Denote $R=\mathfrak{o}_\mathfrak{p}$. It is clear that $B(R)$ normalizes $U(\mathfrak{a})$. Hence it is sufficient to show that for any element x of $G(R)$ such that $x\notin B(R)$, it holds ${}^xU(\mathfrak{a})\not\subset U(\mathfrak{a})$ where ${}^xA=xAx^{-1}$ for a subgroup A of $G(R)$. By the Bruchat decomposition of $G(R)$, we can write $x=uvwtu'$ where $u, u'\in U(R)$, $v\in V(\mathfrak{p})$, $w\in W$ where we denote a representative element of w in $N(R)$ by the same symbol w. Firstly assume $w\neq 1$. Denote by $V'(\mathfrak{a})$ the subgroup generated by $x_{w(\alpha)}(\mathfrak{a})$ for all $w(\alpha)<0$ and $\sigma\in\Phi$. Then it holds $V'(\mathfrak{a})\neq 1$ and so ${}^{uv}V'(\mathfrak{a})\subset {}^xU(\mathfrak{a})\subset G(R,\mathfrak{a})$, hence ${}^{uv}V'(\mathfrak{a})\subset U(\mathfrak{a})T(\mathfrak{a})V(\mathfrak{a})$. If ${}^{uv}V'(\mathfrak{a})\subset U(\mathfrak{a})$, then it holds ${}^vV'(\mathfrak{a})\subset U(\mathfrak{a})\cap V(\mathfrak{a})=1$, it is contradictory to ${}^vV'(\mathfrak{a})\neq 1$. Hence ${}^xU(\mathfrak{a})\not\subset U(\mathfrak{a})$. Secondly assume $w=1$. Then we can write $x=vut$ where $u\varepsilon U(R)$, $v\varepsilon V(P)$, $t\varepsilon T(R)$ and $v\neq 1$. Hence ${}^xU(\mathfrak{a})={}^vU(\mathfrak{a})$. Set $v=x_{-\beta_1}(b_1)\cdot\cdots\cdot x_{-\beta_r}(b_r)$ where each β_i are positive roots and $\beta_1<\dots<\beta_r$, $b_i\varepsilon R$ $(b_i\neq 0)$ $i=1,2,\dots,r$. If β_1 is a simple root, then for $a\in\mathfrak{a}$ $(a\neq 0)$ we have

$$[x_{\beta_1}(a),\ v]=[x_{\beta_1}(a),\ x_{-\beta_1}(b_1)]y$$
$$=x_{\beta_1}(c)h_{\beta_1}(d)y$$

where $c\in\mathfrak{a}$, $d=1/(1+ab_1)\neq 1$, $y\in V(\mathfrak{a})$, hence ${}^vx_{\beta_1}(a)\notin U(\mathfrak{a})$. If β_1 is not simple, then there exists a positive root α such that $\alpha-\beta_1$ is a negative root and $i\alpha-j\beta_k<\alpha-\beta_1$ if $i\alpha-j\beta_k$ are roots. Here let $a\in\mathfrak{a}(a\neq 0)$ and V'' be the subgroup generated by $x_{-\gamma}(t)$ for all roots $-\gamma<\alpha-\beta_1$ and $t\in\mathfrak{o}_\mathfrak{p}$. We have

$$[x_\alpha(a),\ v]x_{\alpha-\beta_1}(\varepsilon ab_1)y$$

where $\varepsilon = \pm 1, \pm 2, \pm 3$ and $y \in V''$. Hence ${}^{v}x_{\alpha}(a) \notin U(\mathfrak{a})$ and so ${}^{x}U(\mathfrak{a}) \not\subset U(\mathfrak{a})$.

N. B. Let R be local ring and assume that $ch(k) \neq 2$ and if G is of type G_2 then $ch(k) \neq 3$. Then observing the above proof of 3.2. Lemma, we can see that the normalizer of $U(R)$ is $B(R)$.

3.3. LEMMA. *Under the hypotheses of Theorem B, let U_0 be a subgroup of $U(\mathfrak{o}_{\mathfrak{p}})$ normalized by $B(\mathfrak{o}_{\mathfrak{p}})$ such that $[U(\mathfrak{o}_{\mathfrak{p}}) : U^0] < \infty$. Then there exists a non-zero ideal A of $\mathfrak{o}_{\mathfrak{p}}$ such that $U(\mathfrak{a}) \subset U_0$*

PROOF. In K. Suzuki [3], we can see that there exists ideal $\mathfrak{a}_{\alpha}$ of $\mathfrak{o}_{\mathfrak{p}}$ for each positive root α of Φ such that

$$\mathfrak{a}_{\alpha}\mathfrak{a}_{\beta} \subset \mathfrak{a}_{\alpha+\beta} \qquad \text{for positive roots } \alpha, \beta, \alpha+\beta,$$
$$U_0 = \Pi_{\alpha \in \Phi} x_{\alpha}(\mathfrak{a})$$

By $[U(\mathfrak{o}_{\mathfrak{p}}) : U_0] < \infty$ we see $\mathfrak{a}_{\alpha} \neq 0$ for each positive root α. Let $\mathfrak{a}$ be the maximal ideal of $\{\mathfrak{a}_{\alpha}\}_{\alpha \in \Phi^+}$. Then $\mathfrak{a}$ is non-zero and we have $U(\mathfrak{a}) \subset U_0$.

3.4. Here we shall prove the Theorem B. Let σ be an automorphism of $G(\mathfrak{o}_{\mathfrak{p}})$, $\bar{K}_{\mathfrak{p}}$ the algebraic closure of $K_{\mathfrak{p}}$, $\overline{\sigma B(\mathfrak{o}_{\mathfrak{p}})}$ the closure of $\sigma B(\mathfrak{o}_{\mathfrak{p}})$ in $G(\bar{K}_{\mathfrak{p}})$, $(\overline{\sigma B(\mathfrak{o}_{\mathfrak{p}})})_0$ the connected component of the unit element of $\overline{\sigma B(\mathfrak{o}_{\mathfrak{p}})}$. Then $(\overline{\sigma B(\mathfrak{o}_{\mathfrak{p}})})_0$ is solvable and defined over $K_{\mathfrak{p}}$. Thus there exists an element g of $G(K_{\mathfrak{p}})$ such that $g(\overline{\sigma B(\mathfrak{o}_{\mathfrak{p}})})_0 g^{-1} \subset B(\bar{K}_{\mathfrak{p}})$. In [2], it is shown that $g = st$ for some $s \in B(K_{\mathfrak{p}})$ and $t \in G(\mathfrak{o}_{\mathfrak{p}})$. Now let B_0 be the set of elements x of $B(\mathfrak{o}_{\mathfrak{p}})$ such that $\sigma x \in (\overline{\sigma B(\mathfrak{o}_{\mathfrak{p}})})_0$. Denoting the commutator subgroup of $B(\mathfrak{o}_{\mathfrak{p}})$ by $\mathfrak{D}B(\mathfrak{o}_{\mathfrak{p}})$, we have $\mathfrak{D}B(\mathfrak{o}_{\mathfrak{p}}) = U(\mathfrak{o}_{\mathfrak{p}})$ and $t\sigma\mathfrak{D}B_0 t^{-1} \subset U(\mathfrak{o}_{\mathfrak{p}})$. Since the order of $B(\mathfrak{o}_{\mathfrak{p}})/B_0$ is finite, by 3.3 Lemma, there exists a non-zero ideal $\mathfrak{a}$ of $\mathfrak{o}_{\mathfrak{p}}$ such that $\mathfrak{D}B_0 \supset U(\mathfrak{a})$. Thus setting $\sigma' = t \circ \sigma$, we have $\sigma' U(\mathfrak{a}) \subset U(\mathfrak{a})$ since $G(\mathfrak{o}_{\mathfrak{p}}, \mathfrak{a})$ is characteristic and $\sigma' U(\mathfrak{a}) \subset G(\mathfrak{o}_{\mathfrak{p}}, \mathfrak{a}) \cap U(\mathfrak{o}_{\mathfrak{p}}) = U(\mathfrak{a})$. For any non-zero ideal $\mathfrak{b}$ contained in $\mathfrak{a}$, $U(\mathfrak{a}/\mathfrak{b})$ is finite, and so $\sigma' U(\mathfrak{a}/\mathfrak{b}) = U(\mathfrak{a}/\mathfrak{b})$. Since $\varprojlim_{\mathfrak{b} \subset \mathfrak{a}} U(\mathfrak{a}/\mathfrak{b}) = U(\mathfrak{a})$, we have $\sigma' U(\mathfrak{a}) = U(\mathfrak{a})$. By 3.2 Lemma, it holds $\sigma' B(\mathfrak{o}_{\mathfrak{p}}) = B(\mathfrak{o}_{\mathfrak{p}})$, hence $\sigma' U(\mathfrak{o}_{\mathfrak{p}}) = U(\mathfrak{o}_{\mathfrak{p}})$. Therefore from Theorem A we have proved completely Theorem B.

Ackowledgement The anthor extends his thanks to professor E. Abe for his kind attention in the preparation of the munuscript.

References

[1] E. Abe, Chevalley groups over local rings, Tôhoku Math. Jour. **21**(1969), 474-494.
[2] A. Borel, Density and maximality of arithmetic groups, J. Reine Angew. Math. **224** (1966), 78-89.
[3] K. Suzuki, On parabolic subgroups of Chevalley groups over local rings, Tôhoku Math. Jour. **28** (1976), 57-66
[4] M. Stein, Central extentions of Chevalley groups over commutative rings, Cloumbia Univ. Ph. D., 1970 Math. .
[5] R. Steinberg, Lecture on Chevalley groups, Lecture notes, Yale Univ. 1968.

Depertment of Mathematics
Faculty of General Education
Kumamoto University