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SOME STUDIES ON p-BLOCKS WITH ABELIAN DEFECT GROUPS

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1. Introduction

Let p be a prime number and G be a finite group. The structures of p-blocks with abelian defect groups and with inertial index 1 are well known (see Lemma 2). In this paper we study p-blocks with abelian defect groups and with inertial index 2, 3, 4 and 5. Let B be a p-block of G, χ be an ordinary irreducible character in B and π be a p-element of G. Then we have

$$\chi(\pi\rho)=\sum_{b}\sum_{\phi\in IBr(b)} d(\chi, \pi, \phi)\phi(\rho)$$

for all p-regular elements ρ of the centralizer $C(\pi)$ of π in G. Here b runs over the p-blocks of $C(\pi)$ with $b^G=B$. After Brauer [1], let $\chi^{(\pi, b)}$ be the central function on G which is defined as folllows. For a p-regular element ρ of $C(\pi)$, $\chi^{(\pi, b)}(\pi\rho)=\sum_{\phi\in b} d(\chi, \pi, \phi)\phi(\rho)$ and $\chi^{(\pi, b)}$ vanishes outside of the p-section of π. Let D be a defect group of B, b be a root of B in $C(D)D$ and $T(b)$ be the inertial group of b in $N(D)$, where $C(D)$ and $N(D)$ are the centralizer of D in G and the normalizer of D in G, respectively. Here we suppose D is abelian. Then by Brauer [3, (4G) and (6C)] each subsection associated with B is conjugate to a subsection $(\pi, b^{C(\pi)})$ $(\pi\in D)$. Furthermore, for $\pi_1, \pi_2\in D$, $(\pi_1, b^{C(\pi_1)})$ and $(\pi_2, b^{C(\pi_2)})$ are conjugate if and only if π_1 and π_2 are conjugate in $T(b)$. Therefore by Broué and Puig [5, Theorem], if η is a $T(b)$-invariant character of D, then $\chi*\eta$ is a generalized character of G. Here

$$\chi*\eta=\sum_{\pi\in U}\eta(\pi)\chi^{(\pi, b^{C(\pi)})}$$

where U is a set of representatives for the $T(b)$-conjugacy classes of D. This fact plays an important rôle in our arguments. If B has inertial index 1, then the ordinary irreducible characters in B are $\chi*\lambda$'s, where λ runs over the linear characters of D (see Lemma 2). If B has inertial index 2 or 3 and if some assumptions are satisfied, then we have the following.

THEOREM 1. *Let G be a finite group and B be a p-block of G with an abelian defect group D and with inertial index e, and b be a root of B in $C(D)$. We denote $D_1=D\cap C(T(b))$ and $D_2=[T(b), D]$. Let Λ_1 be the set of non trivial linear characters of D_1 and Λ_2 be a set of representatives for the $T(b)$-conjugacy classes of non trivial linear characters of D_2. If $p\geqq 5$, $e=2$* or 3 *and if $|D_2|\leqq p^2$, then the followings hold.*

(*i*) *B contains exactly e irreducible Brauer characters and exactly $|D_1|(e+(|D_2|-1)/e)$ ordinary irreducible characters χ_i, $1\leqq i\leqq e$, χ_μ, $\mu\in\Lambda_2$, $\chi_i*\lambda$, $1\leqq i\leqq e$, $\lambda\in\Lambda_1$* and *$\chi_\mu*\lambda$, $\lambda\in\Lambda_1$, $\mu\in\Lambda_2$.*

(*ii*) *There exist $\varepsilon, \varepsilon_1, \varepsilon_2, \ldots, \varepsilon_e=\pm 1$ such that*

$$\chi_\mu^{(1,B)}=\varepsilon\sum_{i=1}^{e}\varepsilon_i\chi_i^{(1,B)} \qquad (\mu\in\Lambda_2)$$

(*iii*) *Let π be an element of D such that $C(\pi)\cap T(b)=C(D)$. Then $b^{C(\pi)}$ contains a unique irreducible Brauer character $\phi^{(\pi)}$. There exists a sign $\gamma_\pi=\pm 1$* such that

$$d(\chi_i, \pi, \phi^{(\pi)})=\varepsilon_i\gamma_\pi \qquad (1\leqq i\leqq e),$$

$$d(\chi_\mu, \pi, \phi^{(\pi)})=\varepsilon\gamma_\pi\eta_\mu(\pi) \qquad (\mu\in\Lambda_2),$$

where η_μ is the sum of characters of D which is $T(b)$-conjugate to μ.

The proof of Theorem 1 is given in § 3. In § 4 we show that if $p\geqq 19$ and G has an abelian Sylow p-subgroup P such that $|N(P):C(P)|=2$, 3, 4 or 5, then the the principal p-block of G contains exactly 2, 3, 4 or 5 irreducible Brauer characters, respectively.

Let K be the algebraic closure of the p-adic number field, $\mathfrak{o}$ be the ring of local integers in K and $\mathfrak{p}$ be the maximal ideal of $\mathfrak{o}$. For a p-block B, $\mathrm{Irr}(B)$ denotes the set of ordinary irreducible characters in B and $\mathrm{IBr}(B)$ denotes the set of irreducible Brauer characters in B. We put $k(B)=|\mathrm{Irr}(B)|$ and $l(B)=|\mathrm{IBr}(B)|$.

2. Lemmas.

In this section we fix a p-block B of G and we assume that a defect group D of B is abelian and b is a root of B in $C(D)$.

LEMMA 1. *Let χ be an ordinary irreducible character in B and μ be a linear character of D. Then any ordinary irreducible character appearing in $\chi*\eta_\mu$ belongs to* Irr(B), *where η_μ is the sum of characters of D which is $T(b)$-conjugate to μ. If χ is of height* 0 *and χ' is an ordinary irreducible character in B, then χ' appears in $\chi*\eta_\mu$ for some linear character μ of D.*

Proof. By the definition of $\chi*\eta_\mu$ and by the second main theorem on p-blocks, the first half is evident. Let Λ be a set of representatives for the $T(b)$-conjugacy classes of linear characters of D. Then we have

$$\sum_{\mu\in\Lambda}\chi*\eta_\mu=|D|\chi^{(1,B)}.$$

By [1, (5H)], if χ is of height 0, then the inner prodduct $(\chi^{(1,B)}, \chi')\neq 0$ for any $\chi'\in \mathrm{Irr}(B)$. This completes the proof.

If B has inertial index 1, then B is a nilpotent p-block. Hence we have the following by Broué and Puig [4, Theorem 1. 2].

LEMMA 2. *If B has inertial index* 1, *then the followings hold.*

(*i*) $l(B)=1$ and $k(B)=|D|$.

(*ii*) $\{(\pi, b^{C(\pi)})|\pi\in D\}$ *is a set of representatives for the conjugacy classes of subsections associated with B. Moreover $b^{C(\pi)}$ contains a unique irreducible Brauer character $\phi^{(\pi)}$.*

(*iii*) *B contains an ordinary irreducible character χ_0 such that $d(\chi_0, \pi, \phi^{(\pi)})=\pm 1$ for all $\pi\in D$. Moreover* $\mathrm{Irr}(B)=\{\chi_0*\lambda|\lambda$ *is a linear character of* $D\}$.

In the above lemma, $\chi_0*\lambda$ is an irreducible character of G, because the inner product $(\chi_0*\lambda, \chi_0*\lambda)$ is equal to 1 and $(\chi_0*\lambda)$ (1) is positive.

LEMMA 3. *Under the assumption of Lemma 2, let ϕ and ϕ_0 be the irreducible Brauer characters in B and b, respectively. Then we have*

$$|G:C(D)|\phi_0(1)/\phi(1)\equiv\pm 1 \qquad (\mathrm{mod}\ p).$$

Proof. We prove by induction on $|G|$. We may assume $G\neq C(D)$. There exists an element $\pi\in D$ such that $C(\pi)\neq G$. Since $b^{C(\pi)}$ has a defect group D and inertial index 1, by the induction hypothesis,

(1) $$|C(\pi):C(D)|\phi_0(1)/\phi^{(\pi)}(1)\equiv\pm 1 \pmod{p},$$

where $\phi^{(\pi)}$ is the irreducible Brauer character in $b^{C(\pi)}$. On the other hand, by [1, (4B)],

$$|G:C(\pi)|\chi^{(\pi, b^{C(\pi)})}(\pi)/\chi(1)\equiv|T(b):T(b)\cap C(\pi)|\equiv 1 \pmod{\mathfrak{p}},$$

for all $\chi\in \mathrm{Irr}(B)$. Lemma 2 implies that for any $\chi\in \mathrm{Irr}(B)$, $d(\chi, \pi, \phi^{(\pi)})\equiv\pm 1 \pmod{\mathfrak{p}}$ and χ is irreducible as a Brauer character. In particular $\chi(1)=\phi(1)$. So we obtain

(2) $$|G:C(\pi)|\phi^{(\pi)}(1)/\phi(1)\equiv\pm 1 \pmod{p}.$$

(1) and (2) yield $|G:C(D)|\phi_0(1)/\phi(1)\equiv\pm 1 \pmod{p}$.

LEMMA 4. *If π_1 and π_2 are elements of D such that $T(b)\cap C(\pi_1)=T(b)\cap C(\pi_2)=C(D)$ and if $\phi^{(\pi_1)}$ and $\phi^{(\pi_2)}$ are the irreducible Brauer characters in $b^{C(\pi_1)}$ and $b^{C(\pi_2)}$, respectively, then there exists a sign $\gamma=\pm 1$ with the property that*

$$d(\chi, \pi_1, \phi^{(\pi_1)})\equiv\gamma d(\chi, \pi_2, \phi^{(\pi_2)}) \pmod{\mathfrak{p}}$$

for all $\chi\in \mathrm{Irr}(B)$.

Proof. By the assumption and by [1, (4B)],

$$|G:C(\pi_i)|d(\chi, \pi_i, \phi^{(\pi_i)})\phi^{(\pi_i)}(1)/\chi(1)\equiv|T(b):C(D)| \pmod{\mathfrak{p}}$$

for all $\chi\in \mathrm{Irr}(B)$ and for $i=1, 2$. Since $b^{C(\pi_i)}$ has the same defect as B has, $\chi(1)/|G:C(\pi_i)|\phi^{(\pi_i)}(1)$ belongs to $\mathfrak{o}$. Therefore we have

$$d(\chi, \pi_1, \phi^{(\pi_1)})\equiv\{(|C(\pi_1)|\phi^{(\pi_2)}(1))/(|C(\pi_2)|\phi^{(\pi_1)}(1))\}d(\chi, \pi_2, \phi^{(\pi_2)}) \pmod{\mathfrak{p}}$$

for all $\chi\in \mathrm{Irr}(B)$. Applying Lemma 3 to $b^{C(\pi_i)}$,

$$|C(\pi_i):C(D)|\phi_0(1)/\phi^{(\pi_i)}(1)\equiv\pm 1 \pmod{p}.$$

Hence we have

$$(|C(\pi_1)|\phi^{(\pi_2)}(1))/(|C(\pi_2)|\phi^{(\pi_1)}(1))\equiv\pm 1 \pmod{p}.$$

This completes the proof.

3. Proof of Theorem 1.

Proof of Theorem 1. We prove by induction on $|G|$. We put $p^d=|D|$, $p^{d_1}=|D_1|$ and $p^{d_2}=|D_2|$. By [8, Chapter 5, Theorem 2. 3], $D=D_1\times D_2$. Let S be a set of representatives for the $T(b)$-conjugacy classes of D and put $U=S-D_1$. Then $|U|=p^{d_1}(p^{d_2}-1)/e$. For $\tau\in S$, we put $b_\tau=b^{C(\tau)}$. Then $\{(\tau, b_\tau)|\tau\in S\}$ is a set of representatives for the conjugacy classes of subsections associated with B. For χ, $\chi'\in$ Irr(B), we denote $m^{(\tau)}_{\chi,\chi'}$ the contribution of (τ, b_τ) to the inner product (χ, χ'). By [1, (5B)],

$$(3)\qquad p^d=\sum_{\tau\in S} p^d m^{(\tau)}_{\chi,\chi}\qquad \text{for all } \chi\in \mathrm{Irr}(B).$$

If $\pi\in U$, then b_π has inertial index 1 and hence b_π contains a unique irreducible Brauer character $\phi^{(\pi)}$. Moreover $p^d m^{(\tau)}_{\chi,\chi}=|d(\chi, \pi, \phi^{(\pi)})|^2$. The orthogonality relations for the decomposition numbers yield

$$(4)\qquad p^d=\sum_{\chi\in B}|d(\chi, \pi, \phi^{(\pi)})|^2\qquad \text{for all } \pi\in U.$$

Since $p\geqq 5$ and $e=2$ or 3, B contains exactly $l(B)$ p-rational ordinary irreducible characters. Let χ_1 be one of them.

Step 1. $\qquad d(\chi_1, \pi, \phi^{(\pi)})=\pm 1\qquad$ *for all* $\pi\in U$.

Proof of Step 1. For $\tau\in S$ and any $\chi\in$ Irr(B), $m^{(\tau)}_{\chi,\chi}$ does not vanish by [1, (4C)]. In particular $m^{(\tau)}_{\chi_1,\chi_1}$ is a positive rational integer. If $d(\chi_1, \pi, \phi^{(\pi)})\not\equiv\pm 1$ (mod p) for some $\pi\in U$, then by Lemma 4 and (3), $p^d\geqq p^{d_1}+4p^{d_1}(p^{d_2}-1)/e$. Since $e=2$ or 3, $p^{d_1}+4p^{d_1}(p^{d_2}-1)/e>p^d$. This is a contradiction. Hence we have $d(\chi_1, \pi, \phi^{(\pi)})\equiv\pm 1$ (mod p) for all $\pi\in U$. We assume $p^{d_1}=1$. If $d(\chi_1, \pi, \phi^{(\pi)})\neq\pm 1$ for some $\pi\in U$, then $|d(\chi_1, \pi, \phi^{(\pi)})|=|d(\chi_1, \pi^k, \phi^{(\pi^k)})|\geqq p-1$, where $(k, p)=1$. Since $p\geqq 5$ and $e=2$ or 3, π is not $T(b)$-conjugate to π^k for some k. Hence $\sum_{\tau\in S} p^d m^{(\tau)}_{\chi_1,\chi_1}\geqq 2(p-1)^2>|D|$. This contradicts (3).

Next we assume that $p^{d_1}\neq 1$. For $\tau\in S$, $\bar{b}_\tau$ be the p-block of $C(\tau)/\langle\tau\rangle$ such that Irr$(b_\tau)\supseteqq$ Irr$(\bar{b}_\tau)$ when we regard the characters of $C(\tau)/\langle\tau\rangle$ as characters of $C(\tau)$. If $\tau\in D_1-\{1\}$, then we can apply the induction hypothesis to $\bar{b}_\tau$(see Olsson [10, Theorem 1. 5]. Hence $l(b_\tau)=l(\bar{b}_\tau)=e$. Therefore we obtain $k(B)=l(B)+e(p^{d_1}-1)+p^{d_1}(p^{d_2}-1)/e$. Let χ_0 be an element of Irr(B) and $F(\chi_0)$ be the family of p-conjugate characters in B which contains χ_0. By the theorem on the arithmetical and geometrical mean,

$$\sum_{\chi \in F(\chi_0)} |d(\chi, \pi, \phi^{(\pi)})|^2 \geqq |F(\chi_0)|.$$

Suppose that $d(\chi_1, \pi, \phi^{(\pi)}) \neq \pm 1$ for some $\pi \in U$. Then $|d(\chi_1 * \lambda, \pi, \phi^{(\pi)})|^2 = |\lambda(\pi) d(\chi_1, \pi, \phi^{(\pi)})|^2 = |d(\chi_1, \pi, \phi^{(\pi)})|^2 \geqq (p-1)^2$ for all $\lambda \in \Lambda_1$. Since $\{\chi_1 * \lambda \mid \lambda \in \Lambda_1\}$ is a union of families of p-conjugate characters in B, by (4) and the assumption, we have

$$\begin{aligned} p^d &\geqq p^{d_1}(p-1)^2 + k(B) - p^{d_1} \\ &\geqq p^{d_1}(p-1)^2 + (e-1)(p^{d_1}-1) + p^{d_1}(p^{d_2}-1)/e > p^d. \end{aligned}$$

Hence $d(\chi_1, \pi, \phi^{(\pi)}) = \pm 1$ for all $\pi \in U$.

Step 2. *Let μ, μ' be elements of Λ_2. We have*

$$(\chi_1 * \eta_\mu, \chi_1 * \eta_\mu) = e^2 - e + 1, \qquad (\chi_1 * \eta_\mu, \chi_1) = e - 1.$$

If $\mu \neq \mu'$, then $(\chi_1 * \eta_\mu, \chi_1 * \eta_{\mu'}) = e^2 - e$.

Proof of Step 2. By Step 1, $m^{(\pi)}_{\chi_1, \chi_1} = 1/p^d$ for all $\pi \in U$. Hence we have $(\chi_1 * \eta_\mu, \chi_1) = e \sum_{\sigma \in D_1} m^{(\sigma)}_{\chi_1, \chi_1} + (1/p^d) \sum_{\pi \in U} \eta_\mu(\pi)$. By the way (3) yields $\sum_{\sigma \in D_1} m^{(\sigma)}_{\chi_1, \chi_1} = 1 - |U|/p^d$, so we have

$$\sum_{\sigma \in D_1} m^{(\sigma)}_{\chi_1, \chi_1} = ((e-1)p^d + p^{d_1})/ep^d. \tag{5}$$

On the other hand, by the orthogonality relations for the irreducible characters of D, $\sum_{\tau \in D} \eta_\mu(\tau) = ep^{d_1} + e \sum_{\pi \in U} \eta_\mu(\pi) = 0$. Hence

$$\sum_{\pi \in U} \eta_\mu(\pi) = -p^{d_1}. \tag{6}$$

By (5) and (6), we obtain $(\chi_1 * \eta_\mu, \chi_1) = e - 1$. This yields

$$\begin{aligned} (\chi_1 * \eta_\mu, \chi_1 * \eta_\mu) &= (\chi_1, \chi_1 * (\eta_\mu \eta_{\mu^{-1}})) = e^2 - e + 1, \\ (\chi_1 * \eta_\mu, \chi_1 * \eta_{\mu'}) &= (\chi_1, \chi_1 * (\eta_{\mu'} \eta_{\mu^{-1}})) = e^2 - e. \end{aligned}$$

Step 3. *$l(B) = e$ and $k(B) = p^{d_1}(e + (p^{d_2}-1)/e)$.*

Proof of Step 3. By Step 2 and by the assumption $e = 2$ or 3, there exist $e - 1 + (p^{d_2}-1)/e$ distinct ordinary irreducible characters χ_i, $2 \leqq i \leqq e$, χ_μ, $\mu \in \Lambda_2$, in B and signs ε', $\varepsilon_i' = \pm 1$, $2 \leqq i \leqq e$, such that

(7) $$\chi_1 * \eta_\mu = (e-1)\chi_1 + \sum_{i=2}^{e} \varepsilon_i' \chi_i + \varepsilon' \chi_\mu \quad \text{for all } \mu \in \Lambda_2$$

except for the case $e=2$ and $p^{d_2}=7$. If $e=2$ and $p^{d_2}=7$, then (7) or the following (7′) holds. Let $\Lambda_2 = \{\mu_1, \mu_2, \mu_3\}$.

(7′) $$\begin{cases} \chi_1 * \eta_{\mu_1} = \chi_1 + \delta_2 \chi_2 + \delta_3 \chi_3, \\ \chi_1 * \eta_{\mu_2} = \chi_1 + \delta_2 \chi_2 + \delta_4 \chi_4, \\ \chi_1 * \eta_{\mu_3} = \chi_1 + \delta_3 \chi_3 + \delta_4 \chi_4, \end{cases}$$

where $\delta_2, \delta_3, \delta_4 = \pm 1$ and $\chi_2, \chi_3, \chi_4 \in \mathrm{Irr}(B)$. We assume that $p^{d_1}=1$. Then the trivial character 1_D of D and Λ_2 form a set of representatives for the $T(b)$-conjugacy classes of linear characters of D. If $e \neq 2$ or $p^{d_2} \neq 7$, then by (7) and Lemma 1, $k(B) = e + (p^{d_2}-1)/e$ and hence $l(B) = e$. If $e=2$ and $p^{d_2}=7$, then the statement follows from Dade's theorem on p-blocks with cyclic defect groups (Dade [6]), and hence (7′) does not hold.

We assume that $p^{d_1} \neq 1$. $\{1_D\} \cup \Lambda_1 \cup \Lambda_2 \cup \{\lambda\mu \mid \lambda \in \Lambda_1, \mu \in \Lambda_2\}$ is a set of representatives for the $T(b)$-conjugacy classes of linear characters of D. Let $\lambda \in \Lambda_1$. (7) and (7′) yield

(8) $$\chi_1 * \eta_{\lambda\mu} = (e-1)(\chi_1 * \lambda) + \sum_{i=2}^{e} \varepsilon_i'(\chi_i * \lambda) + \varepsilon'(\chi_\mu * \lambda) \quad \text{for all } \mu \in \Lambda_2,$$

(8′) $$\begin{cases} \chi_1 * \eta_{\mu_1\lambda} = \chi_1 * \lambda + \delta_2(\chi_2 * \lambda) + \delta_3(\chi_3 * \lambda), \\ \chi_1 * \eta_{\mu_2\lambda} = \chi_1 * \lambda + \delta_2(\chi_2 * \lambda) + \delta_4(\chi_4 * \lambda), \\ \chi_1 * \eta_{\mu_3\lambda} = \chi_1 * \lambda + \delta_3(\chi_3 * \lambda) + \delta_4(\chi_4 * \lambda), \end{cases}$$

respectively. If (7) holds, then by Lemma 1 the set $\{\chi_i \mid 1 \leqq i \leqq e\} \cup \{\chi_\mu \mid \mu \in \Lambda_2\} \cup \{\chi_i * \lambda \mid 1 \leqq i \leqq e, \lambda \in \Lambda_1\} \cup \{\chi_\mu * \lambda \mid \lambda \in \Lambda_1, \mu \in \Lambda_2\}$ coincides with Irr(B). If (7′) holds, then $\{\chi_1, \chi_2, \chi_3, \chi_4\} \cup \{\chi_i * \lambda \mid 1 \leqq i \leqq 4, \lambda \in \Lambda_1\}$ coincides with Irr(B). Suppose that (7) holds and put $I = \{\chi_i \mid 1 \leqq i \leqq e\} \cup \{\chi_\mu \mid \mu \in \Lambda_2\} \cup \{\chi_i * \lambda \mid 1 \leqq i \leqq e, \lambda \in \Lambda_1\} \cup \{\chi_\mu * \lambda \mid \lambda \in \Lambda_1, \mu \in \Lambda_2\}$. If $\chi = \chi'$ for some $\chi, \chi' \in I$, then $\chi * \lambda = \chi' * \lambda$ for all $\lambda \in \Lambda_1$, and the cardinal number $|I|$ does not exceed $(e-1)p^{d_1} + p^{d_1}(p^{d_2}-1)/e$. On the other hand, $k(B) \geqq 1 + e(p^{d_1}-1) + p^{d_1}(p^{d_2}-1)/e$ by the induction hypothesis. Since $p \geqq 5$ and $e=2$ or 3, this is a contradiction. Hence the characters in I are distinct, and so $k(B) = ep^{d_1} + p^{d_1}(p^{d_2}-1)/e$ and $l(B) = e$. If (7′) holds, then we have $k(B) \leqq 4p^{d_1}$. This contradicts $k(B) \geqq 1 + 2(p^{d_1}-1) + 3p^{d_1} = 5p^{d_1} - 1$. Hence (7′) does not holds.

Step 4. *Conclusion*. By (7), we can show that χ_i $(1 \leqq i \leqq e)$ is a p-rational character. By Step 1 and Lemma 4, there exist signs ε_i, $\gamma_\pi = \pm 1$ $(1 \leqq i \leqq e,\ \pi \in U)$ such that $d(\chi_i, \pi, \phi^{(\pi)}) = \varepsilon_i \gamma_\pi$. Then we can show $(\chi_1 * \eta_\mu, \chi_i) = -\varepsilon_1 \varepsilon_i$ for all $i \geqq 2$. In fact

$$(\chi_1 * \eta_\mu,\ \chi_i) = e \sum_{\sigma \in D_1} m_{\chi_1, \chi_i}^{(\sigma)} + \varepsilon_1 \varepsilon_i \sum_{\pi \in U} \eta_\mu(\pi)/p^d.$$

Since $\sum_{\sigma \in D_1} m_{\chi_1, \chi_i}^{(\sigma)} + \varepsilon_1 \varepsilon_i |U|/p^d = (\chi_1, \chi_i) = 0$, $\sum_{\sigma \in D_1} m_{\chi_1, \chi_i}^{(\sigma)} = -\varepsilon_1 \varepsilon_i p^{d_1}(p^{d_2} - 1)/ep^d$. This and (6) yield $(\chi_1 * \eta_\mu, \chi_i) = -\varepsilon_1 \varepsilon_i$. So (7) can be written as

$$\chi_1 * \eta_\mu = (e-1)\chi_1 - \sum_{i=2}^{e} \varepsilon_1 \varepsilon_i \chi_i + \varepsilon \varepsilon_1 \chi_\mu,$$

where $\varepsilon = \varepsilon' \varepsilon_1$. This implies

$$\eta_\mu(\sigma) \chi_1^{(\sigma, b_\sigma)} = (e-1) \chi_1^{(\sigma, b_\sigma)} - \sum_{i=2}^{e} \varepsilon_1 \varepsilon_i \chi_i^{(\sigma, b_\sigma)} + \varepsilon \varepsilon_1 \chi_\mu^{(\sigma, b_\sigma)} \tag{9}$$

for all $\sigma \in S$. If $\sigma = 1$, then by (9)

$$e \chi_1^{(1, B)} = (e-1) \chi_1^{(1, B)} - \sum_{i=2}^{e} \varepsilon_1 \varepsilon_i \chi_i^{(1, B)} + \varepsilon \varepsilon_1 \chi_\mu^{(1, B)}.$$

Hence we have (*ii*). If $\pi \in U$, then by (9)

$$\eta_\mu(\pi) \varepsilon_1 \gamma_\pi = \varepsilon \varepsilon_1 d(\chi_\mu, \pi, \phi^{(\pi)}).$$

This complets the proof of Theorem 1·

REMARK 1. Theorem 1 holds for $p = 2$ and 3.

REMARK 2. If G is a p-solvable group, then the conclusion in Theorem 1 holds without the assumption that $p \geqq 5$ and $|D_2| \leqq p^2$ (see Watanabe [11]).

4. Principal blocks

Let G be a finite group with an abelian Sylow p-subgroup P and B_0 be the principal p-block of G throghout this section. A root b of B_0 in $C(P)$ is the principal p-block of $C(P)$ and hence $T(b) = N(P)$. For an element $\pi \in P$, $b^{C(\pi)}$ is also the principal p-block of $C(\pi)$. We put $b_\pi = b^{C(\pi)}$. For a linear character μ of P, η_μ denotes the sum of the characters of P which is $N(P)$-conjugate to μ.

THEOREM 2. *Suppose that B is the principal p-block B_0 in Theorem* 1 *and suppose that $p\geqq 3$ and $e=2$* or 3. *Then the conclusion in Theorem* 1 *holds.*

Proof. Steps 1 and 2 in the proof of Theorem 1 hold for the trivial character 1_G of G. Hence by Step 3, B contains exactly e irreducible Brauer characters and exactly $|D_1|(e+(|D_2|-1)/e)$ distinct ordinary irreducible characters $\chi_1=1_G$, χ_i, $2\leqq i\leqq e$, χ_μ, $\mu\in\Lambda_2$, $\chi_i*\lambda$, $1\leqq i\leqq e$, $\lambda\in\Lambda_1$, $\chi_\mu*\lambda$, $\lambda\in\Lambda_1$, $\mu\in\Lambda_2$. Moreover we have

$$1_G*\eta_\mu=(e-1)1_G+\sum_{i=2}^{e}\varepsilon_i'\chi_i+\varepsilon'\chi_\mu \tag{10}$$

for all $\mu\in\Lambda_2$, where ε_i', $\varepsilon'=\pm 1$. By (10) we can show that χ_i is p-rational. Let π be an element of $D-D_1$. b_π has a unique irreducible Brauer character $1_{C(\pi)}$. Set $d(\chi_i,\pi,1_{C(\pi)})=a_i$, $2\leqq i\leqq e$, and set $c=e-1+\sum_{i=2}^{e}\varepsilon_i'a_i$. Then a_i is a rational integer and $a_i\neq 0$, because b_π has the same defect as B has. By the argument of Step 4, if $a_i=\pm 1$, then Theorem 2 is proved. By (10), it follows that

$$d(\chi_\mu,\pi,1_{C(\pi)})=\varepsilon'(\eta_\mu(\pi)-c),$$
$$d(\chi_i*\lambda,\pi,1_{C(\pi)})=\lambda(\pi)a_i,$$
$$d(\chi_\mu*\lambda,\pi,1_{C(\pi)})=\varepsilon'\lambda(\pi)(\eta_\mu(\pi)-c)$$

for all i $(1\leqq i\leqq e)$, λ $(\lambda\in\Lambda_1)$ and μ $(\mu\in\Lambda_2)$. Since $\sum_{\chi\in B}|d(\chi,\pi,1_{C(\pi)})|^2=|D|$, we have

$$|D|=|D_1|(1+\sum_{i=2}^{e}a_i^2+\sum_{\mu\in\Lambda_2}|\eta_\mu(\pi)-c|^2)$$
$$=|D_1|(1+\sum_{i=2}^{e}a_i^2+\sum_{\mu\in\Lambda_2}|\eta_\mu(\pi)|^2-c\sum_{\mu\in\Lambda_2}\eta_\mu(\pi)-c\sum_{\mu\in\Lambda_2}\eta_\mu(\pi^{-1}))+c^2(|D_2|-1)/e).$$

By the orthogonality relations for the irreduicble characters of D_2, $\sum_{\mu\in\Lambda_2}|\eta_\mu(\pi)|^2=|D_2|-e$ and $\sum_{\mu\in\Lambda_2}\eta_\mu(\pi)=-1$. Hence it follows that

$$|D_2|=1+\sum_{i=2}^{e}a_i^2+|D_2|-e+2c+c^2(|D_2|-1)/e.$$

This yields

$$1+\sum_{i=2}^{e}a_i^2-e+2c+c^2(|D_2|-1)/e=0.$$

Since $1+\sum_{i=2}^{e}a_i^2-e\geqq 0$ and $2c+c^2(|D_2|-1)/e\geqq 0$, $a_i=\pm 1$ for all i. This completes the proof.

We add the following to Theorem 2,

PROPOSITION. *Suppose that in Theorem* 1 *B is the principal p-block B_0 and suppose that $p\geqq 3$ and $e=2$* or 3. *Then we have*

$$(1_G^{(1,B)}, 1_G^{(1,B)})=\{(e-1)|D_2|+1\}/e|D|.$$

Proof. We prove by induction on $|G|$. We use the notation in the proof of Theorem 1. Let $\sigma\in D_1-\{1\}$. Then applying the induction hypothesis to $\bar{b}_\sigma$, we obtain $(1_{C(\sigma)}^{(1,b_\sigma)}, 1_{C(\sigma)}^{(1,b_\sigma)})=\{(e-1)p^{d_2}+1\}/ep^d$. It is evident that $(1_G^{(\sigma,b_\sigma)}, 1_G^{(\sigma,b_{\sigma})})=(1_{C(\sigma)}^{(1,b_\sigma)}, 1_{C(\sigma)}^{(1,b_\sigma)})$. If $\pi\in U$, then $(1_G^{(\pi,b_\pi)}, 1_G^{(\pi,b_\pi)})=1/p^d$. By (3), we can show $(1_G^{(1,B)}, 1_G^{(1,B)})=\{(e-1)p^{d_2}+1\}/ep^d$.

THEOREM 3. *If $p\geqq 19$ and $N(P)/C(P)$ is a cyclic group of order* 4, *then* $l(B_0)=4$.

Proof. We prove by induction on $|G|$. Let t be a generator of $N(P)/C(P)$. $\langle t\rangle$ acts on P and on the set of linear characters of P. We call the orbits $\langle t\rangle$-conjugacy classes. If $P_1=C_P(t)$, $P_2=[t, C_P(t^2)]$ and $P_3=[t^2, P]$, where $C_P(t)$ is the set of elements of P which is fixed by t and $C_P(t^2)$ is the set of elements of P which is fixed by t^2, then by [8, chapter 5, Theorem 2.3],

$$P=C_P(t^2)\times[t^2, P]=C_P(t)\times[t, C_P(t^2)]\times P_3=P_1\times P_2\times P_3.$$

We put $p^a=|P|$ and $p^{a_i}=|P_i|$ for $i=1, 2, 3$. Let U be a set of representatives for the $\langle t\rangle$-conjugacy classes of $P_1\times P_2-P_1$ and T be a set of representatives for the $\langle t\rangle$-conjugacy classes of $P-P_1\times P_2$. If $\pi\in U$, the stabilizer of π in $\langle t\rangle$ is $\langle t^2\rangle$. If $\tau\in T$, then the stabilizer of τ in $\langle t\rangle$ is the identity group. Hence $|U|=(p^{a_1}p^{a_2}-p^{a_1})/2$ and $|T|=(p^a-p^{a_1}p^{a_2})/4$. Moreover $P_1\cup U\cup T$ is a set of representatives for the $\langle t\rangle$-conjugacy classes of P. For any element $\sigma\in P$, we put $m^{(\sigma)}=(1_G^{(\sigma,b_\sigma)}, 1_G^{(\sigma,b_\sigma)})$. If $\sigma\in P_1-\{1\}$, then b_σ has inertial index 4 and hence $l(b_\sigma)=4$ by the induction hypothesis. If $\pi\in U$, then b_π has inertial index 2 and $l(b_\pi)=2$ by Theorem 2. Moreover by Proposition,

$$m^{(\pi)}=(p^{a_3}+1)/2p^a. \tag{11}$$

If $\tau\in T$, then b_τ has inertial index 1 and hence $m^{(\tau)}=1/p^a$. Since $\sum_{\tau\in P_1} m^{(\sigma)}+$

$$\sum_{\pi \in U} m^{(\pi)} + \sum_{\tau \in T} m^{(\tau)} = (1_G, 1_G) = 1,$$

(12) $$\sum_{\sigma \in P_1} m^{(\sigma)} = (2p^a + p^{a_1}p^{a_3} + p^{a_1})/4p^a.$$

Let Λ_i be a set of representatives for the $\langle t \rangle$-conjugacy classes of non trivial linear characters of P_i, $i=1, 2, 3$. If $\mu \in \Lambda_2$, then the stabilizer of μ in $\langle t \rangle$ is $\langle t^2 \rangle$. If $\nu \in \Lambda_3$, then the stabilzer of ν in $\langle t \rangle$ is the identity group. Moreover $\{1_D\} \cup \Lambda_1 \cup \Lambda_2 \cup \Lambda_3 \cup \{\mu\nu \mid \mu \in \Lambda_2, \ \nu \in \Lambda_3\} \cup \{\mu^{-1}\nu \mid \mu \in \Lambda_2, \ \nu \in \Lambda_3\} \cup \{\lambda\mu \mid \lambda \in \Lambda_1, \ \mu \in \Lambda_2\} \cup \{\lambda\nu \mid \lambda \in \Lambda_1, \ \nu \in \Lambda_3\} \cup \{\lambda\mu\nu \mid \lambda \in \Lambda_1, \mu \in \Lambda_2, \nu \in \Lambda_3\} \cup \{\lambda\mu^{-1}\nu \mid \lambda \in \Lambda_1, \mu \in \Lambda_2, \nu \in \Lambda_3\}$ is a set of representatives for the $\langle t \rangle$-conjugacy classes of linear characters of P.

By the orthogonality relations for the characters of P_3 we obtain

(13) $$\sum_{\tau \in T} \eta_\nu(\tau) = -p^{a_1}p^{a_2} \qquad (\nu \in \Lambda_3).$$

Combining this with (11) and (12),

(14) $$(1_G * \eta_\nu, 1_G) = 3 \qquad (\nu \in \Lambda_3).$$

This yields the following.

(15) $$(1_G * \eta_\nu, 1_G * \eta_\nu) = (1_G, 1_G * (\eta_{\nu^{-1}} \eta_\nu)) = 13 \qquad (\nu \in \Lambda_3).$$

(16) $$(1_G * \eta_\nu, 1_G * \eta_{\nu'}) = (1_G, 1_G * (\eta_{\nu^{-1}} \eta_{\nu'})) = 12 \qquad (\nu, \nu' \in \Lambda_3, \nu \neq \nu').$$

By (15) and (16) and the assumption $p \geqq 19$, there exist $4 + (p^{a_3} - 1)/4$ distinct ordinary irreducible characters $\chi_1 = 1_G$, χ_2, χ_3, χ_4, χ_ν, $\nu \in \Lambda_3$, in B such that

(17) $$1_G * \eta_\nu = 3\, 1_G + \varepsilon_2\chi_2 + \varepsilon_3\chi_3 + \varepsilon_4\chi_4 + \varepsilon\chi_\nu$$

for all $\nu \in \Lambda_3$, where ε_2, ε_3, ε_4, $\varepsilon = \pm 1$. Therefore if $p^{a_1}p^{a_2} = 1$, then by Lemma 1 $k(B_0) = 4 + (p^{a_3} - 1)/4$. Since $k(B_0) = 1(B_0) + (p^{a_3} - 1)/4$, $l(B_0) = 4$.

Next we suppose that $p^{a_1} \neq 1$ and $p^{a_2} = 1$. By Lemma 1, $I = \{\chi_i \mid 1 \leqq i \leqq 4\} \cup \{\chi_\nu \mid \nu \in \Lambda_3\} \cup \{\chi_i * \lambda \mid 1 \leqq i \leqq 4, \ \ \lambda \in \Lambda_1\} \cup \{\chi_\nu * \lambda \mid \lambda \in \Lambda_1, \ \ \nu \in \Lambda_3\}$ is the set of ordinary irreducible characters in B. Since the cardinal number $|I|$ does not exceed $4p^{a_1} + (p^{a_1}p^{a_3} - p^{a_1})/4$ and since $k(B_0) \geqq 1 + 4(p^{a_1} - 1) + (p^{a_1}p^{a_3} - p^{a_1})4$ by the induction hypothesis, we can see that the characters in I are distinct. Hence we obtain $k(B_0) = 4p^{a_1} + (p^{a_1}p^{a_3} - p^{a_1})/4$ and $l(B_0) = 4$.

We assume that $p^{a_2} \neq 1$ in the rest of the proof. By the orthogonality relations for the linear characters of P_2 and P_3, the following hold.

(18) $$\sum_{\pi \in U} \eta_\mu(\pi) = -p^{a_1} \text{ and } \sum_{\tau \in T} \eta_\mu(\tau) = 0 \qquad (\mu \in \Lambda_2),$$

(19) $$\sum_{\pi \in U} \eta_{\mu\nu}(\pi) = -2p^{a_1} \text{ and } \sum_{\tau \in T} \eta_{\mu\nu}(\tau) = 0 \qquad (\mu \in \Lambda_2,\ \nu \in \Lambda_3).$$

By (11), (12) and (18), we have

(20) $$(1_G{*}\eta_\mu,\ 1_G) = 1,\ \ (1_G{*}\eta_\mu,\ 1_G{*}\eta_\mu) = 3 \qquad (\mu \in \Lambda_2),$$

(21) $$(1_G{*}\eta_\mu,\ 1_G{*}\eta_{\mu'}) = 2 \qquad (\mu,\ \mu' \in \Lambda_2,\ \mu \neq \mu').$$

By (11), (12) and (19), we have

(22) $$(1_G{*}\eta_{\mu\nu},\ 1_G) = 2 \qquad (\mu \in \Lambda_2,\ \nu \in \Lambda_3).$$

By (20) and (21), there exist $1 + (p^{a_2} - 1)/2$ distinct ordinary irreducible characters χ_2', χ_μ, $\mu \in \Lambda_2$, in B_0 such that

(23) $$1_G{*}\eta_\mu = 1_G + \varepsilon'\chi_2' - \varepsilon'\chi_\mu \qquad (\mu \in \Lambda_2),$$

where $\varepsilon' = \pm 1$. Since $(1_G{*}\eta_\nu,\ 1_G{*}\eta_\mu) = (1_G{*}(\eta_{\mu_{\nu}-1} + \eta_{\mu\nu}),\ 1_G) = 4$ by (22), we may assume $\chi_2' = \chi_2$ and $\varepsilon' = \varepsilon_2$. Moreover χ_μ is different from χ_3, χ_4 and χ_ν $(\nu \in \Lambda_3)$. Hence (23) can be written as

(24) $$1_G{*}\eta_\mu = 1_G + \varepsilon_2\chi_2 - \varepsilon_2\chi_\mu \qquad (\mu \in \Lambda_2).$$

By (14), (20) and (22), we can show the following. Let μ, $\mu' \in \Lambda_2$ and ν, $\nu' \in \Lambda_3$ and $\mu \neq \mu'$ and $\nu \neq \nu'$.

(25) $$(1_G{*}\eta_{\mu\nu},\ 1_G{*}\eta_{\mu\nu}) = 11.$$

(26) $$(1_G{*}\eta_{\mu\nu},\ 1_G{*}\eta_{\mu\nu'}) = 10.$$

(27) $$(1_G{*}\eta_{\mu\nu},\ 1_G{*}\eta_{\mu^{-1}\nu}) = (1_G{*}\eta_{\mu\nu},\ 1_G{*}\eta_{\mu^{-1}\nu'}) = 10.$$

(28) $$(1_G{*}\eta_\mu,\ 1_G{*}\eta_{\mu\nu}) = 5.$$

(29) $$(1_G{*}\eta_{\mu'},\ 1_G{*}\eta_{\mu\nu}) = 4.$$

(30) $$(1_G{*}\eta_{\mu\nu},\ 1_G{*}\eta_{\mu'\nu}) = (1_G{*}\eta_{\mu\nu},\ 1_G{*}\eta_{\mu'\nu'}) = 8.$$

(31) $$(1_G{*}\eta_\nu,\ 1_G{*}\eta_{\mu\nu}) = (1_G{*}\eta_{\nu'},\ 1_G{*}\eta_{\mu\nu'}) = 8.$$

By (22), (24), (25), (28), (29) and the assumption $p \geqq 19$, $(1_G{*}\eta_{\mu\nu},\ \chi_2) = 2\varepsilon_2$ and $(1_G{*}\eta_{\mu\nu},\ \chi_\mu) = -\varepsilon_2$. Hence by (22), (26) and (27), B contains ordinary irreducible characters χ_μ' $(\mu \in \Lambda_2)$, $\chi_{\mu\nu}$ and $\chi_{\mu^{-1}\nu}$ $(\mu \in \Lambda_2$ and $\nu \in \Lambda_3)$ with the property that

(32) $$\begin{cases} 1_G{*}\eta_{\mu\nu} = 2\,1_G + 2\varepsilon_2\chi_2 - \varepsilon_2\chi_\mu + \varepsilon_\mu'\chi_\mu' + \varepsilon_\mu\chi_{\mu\nu}, \\ 1_G{*}\eta_{\mu^{-1}\nu} = 2\,1_G + 2\varepsilon_2\chi_2 - \varepsilon_2\chi_\mu + \varepsilon_\mu'\chi_\mu' + \varepsilon_\mu\chi_{\mu^{-1}\nu} \end{cases}$$

for all $\mu \in \Lambda_2$ and $\nu \in \Lambda_3$, where ε_μ', $\varepsilon_\mu = \pm 1$. Then for $\mu \in \Lambda_2$, 1_G, χ_2, χ_μ, χ_μ', $\chi_{\mu\nu}$ and $\chi_{\mu^{-1}\nu}$ $(\nu \in \Lambda_3)$ are distinct. From (29), if $\mu' \neq \mu$, then $\chi_{\mu'}$ is different from χ_μ', $\chi_{\mu\nu}$ and $\chi_{\mu^{-1}\nu}$ $(\nu \in \Lambda_3)$. Combining this with (30), if $\mu \neq \mu'$ $(\mu, \mu' \in \Lambda_2)$, then $\chi_{\mu'}' \neq \chi_\mu'$, $\chi_{\mu\nu}$, $\chi_{\mu^{-1}\nu}$ $(\nu \in \Lambda_3)$, $\chi_{\mu\nu} \neq \chi_{\mu\nu'}$, $\chi_{\mu^{-1}\nu'}$ $(\nu, \nu' \in \Lambda_3)$ and $\chi_{\mu'^{-1}\nu'} \neq \chi_{\mu^{-1}\nu}$ $(\nu, \nu' \in \Lambda_3)$. This and (31) yield that χ_3, χ_4 and χ_ν $(\nu \in \Lambda_3)$ are different from χ_μ', $\chi_{\mu\nu}$ and $\chi_{\mu^{-1}\nu}$ $(\mu \in \Lambda_2$ and $\nu \in \Lambda_3)$.

If $p^{a_1}=1$, then by Lemma 1, (17), (24) and (32), $k(B_0)=4+2|\Lambda_2|+|\Lambda_3|+2|\Lambda_2||\Lambda_3|=4+p^{a_2}-1+p^{a_2}(p^{a_3}-1)/4$. By the induction hypothesis and Theorem 2, $k(B_0)=l(B_0)+2|U|+|T|=l(B_0)+p^{a_2}-1+(p^{a_2}p^{a_3}-p^{a_2})/4$, so $l(B_0)=4$. If $p^{a_1}\neq 1$, then by the same argument as in the case $p^{a_1}\neq 1$ and $p^{a_2}=1$, we can show $l(B_0)=4$.

THEOREM 4. *If $p \geqq 5$ and $N(P)/C(P)$ is an elementary abelian group of order 4, then $l(B_0)=4$.*

Proof. We prove by induction on $|G|$. Put $X=N(P)/C(P)$ and $X=\{\bar{1}, t_1, t_2, t_3\}$, where $\bar{1}$ is the identity element of X. By [8, chapter 5, Theorem 2.3] we have

$$P=P_0\times P_1\times P_2\times P_3,$$

where $P_0=C_P(X)$, $P_0\times P_i=C_P(t_i)$ and P_i is X-invariant for all i. For $\sigma_i \in P_i-\{1\}(i=1, 2, 3)$, if $j\neq i$, then $\sigma_i^{t_j}=\sigma_i^{-1}$. Let U_i be a set of representatives for the X-conjugacy classes of $P_i-\{1\}$ for $i=1$, 2 and 3. Then $P_0 \cup \{\sigma\sigma_i | \sigma \in P_0, \sigma_i \in U_i, i=1, 2, 3\} \cup \{\sigma\sigma_1\sigma_2 | \sigma \in P_0, \sigma_1 \in U_1, \sigma_2 \in U_2\} \cup \{\sigma\sigma_1\sigma_3 | \sigma \in P_0, \sigma_1 \in U_1, \sigma_3 \in U_3\} \cup \{\sigma\sigma_2\sigma_3 | \sigma \in P_0, \sigma_2 \in U_2, \sigma_3 \in U_3\} \cup \{\sigma\sigma_1\sigma_2\sigma_3 | \sigma \in P_0, \sigma_i \in U_i\} \cup \{\sigma\sigma_1^{-1}\sigma_2\sigma_3 | \sigma \in P_0, \sigma_i \in U_i\}$ is a set of representatives for the X-conjugacy classes of P. We put $p^a=|P|$ and $p^{a_i}=|P_i|$ $(i=0, 1, 2, 3)$. Since the stabilizer of $\sigma\sigma_i$ in X is $\langle t_i \rangle$ and the stabilizers of $\sigma\sigma_i\sigma_j$ and $\sigma\sigma_1\sigma_2\sigma_3$ in X are $\langle \bar{1} \rangle$, by the induction hypothesis and Theorem 2 we can show

$$\begin{aligned}(33)\qquad k(B_0)&=l(B_0)+4(p^{a_0}-1)+2p^{a_0}\sum_{i=1}^{3}(p^{a_i}-1)/2\\&\quad+p^{a_0}(p^{a_1}-1)(p^{a_2}-1)/4+p^{a_0}(p^{a_1}-1)(p^{a_3}-1)/4\\&\quad+p^{a_0}(p^{a_2}-1)(p^{a_3}-1)/4+p^{a_0}(p^{a_1}-1)(p^{a_2}-1)(p^{a_3}-1)/4.\end{aligned}$$

We put $S_i=P_0\times P_i-P_0$ $(i=1, 2, 3)$ and $S=P-(P_0\cup S_1\cup S_2\cup S_3)$. If $\pi_i \in S_i$, then by Proposition

(34) $$m^{(\pi i)}=1/2p^{a}+1/2p^{a_0}p^{a_i}.$$

If $\pi\in S$, then

(35) $$m^{(\pi)}=1/p^{a}.$$

Since $\sum_{\sigma\in P_0}m^{(\sigma)}=1-\frac{1}{2}\sum_{i=1}^{3}\sum_{\pi i\in S_i}m^{(\pi i)}-\frac{1}{4}\sum_{\pi\in S}m^{(\pi)}$, we have

(36) $$\sum_{\sigma\in P_0}m^{(\sigma)}=1-\left\{\frac{1}{2}\sum_{i=1}^{3}(1/2p^{a}+1/2p^{a_0}p^{a_i})|S_i|+\frac{1}{4}(p^{a}-\sum_{i=1}^{3}p^{a_0}p^{a_i}+2p^{a_0})/p^{a}\right\}.$$

If μ_i is a non trivial linear character of P_i ($i=1, 2, 3$), then by the orthogonality relations for the characters of P_i,

(37) $$\sum_{\pi\in S_i}\eta_{\mu_i}(\pi)=-2p^{a_0}\text{ and }\sum_{\pi\in S}\eta_{\mu_i}(\pi)=4p^{a_0}-2\sum_{j\neq i,0}p^{a_0}p^{a_j}.$$

If μ is a linear character of P whose stabilizer in X is $\langle\bar{1}\rangle$, then

(38) $$\sum_{\pi\in S_i}\eta_{\mu}(\pi)=\begin{cases}-4p^{a_0}, & \mu\neq 1_{P_i}\text{ on }P_i,\\ 4p^{a_0}p^{a_i}-4p^{a_0}, & \mu=1_{P_i}\text{ on }P_i.\end{cases}$$

(39) $$\sum_{\pi\in S}\eta_{\mu}(\pi)=-(4p^{a_0}+\sum_{i=1}^{3}\sum_{\pi\in S_i}\eta_{\mu}(\pi)).$$

By (34), (35), (36) and (37), if μ_i is a non trivial linear character of P_i ($i=1, 2, 3$), then

(40) $$(1_G*\eta_{\mu_i},\ 1_G)=1.$$

By (34), (35), (36), (38) and (39), if μ_i is a non trivial linear character of P_i ($i=1, 2, 3$), then

(41) $$(1_G*\eta_{\mu_i\mu_j},\ 1_G)=1 \qquad (i\neq j),$$

(42) $$(1_G*\eta_{\mu_1\mu_2\mu_3},\ 1_G)=0.$$

Let Λ_i be a set of representatives for the X-conjugacy classes of non trivial linear charactrs of P_i, and let μ_i and μ_i' be distinct characters in Λ_i, $i=1, 2, 3$. By (40), (41) and (42) we can show the following.

(43) $$(1_G*\eta_{\mu_i},\ 1_G*\eta_{\mu_i})=3, \qquad (1_G*\eta_{\mu_i},\ 1_G*\eta_{\mu_i'})=2.$$

(44) $$(1_G*\eta_{\mu_i},\ 1_G*\eta_{\mu_j})=1 \qquad (i\neq j).$$

(45) $$(1_G*\eta_{\mu_i},\ 1_G*\eta_{\mu_i\mu_j})=3 \qquad (i\neq j).$$

(46) $$(1_G*\eta_{\mu_i},\ 1_G*\eta_{\mu_{i'}\mu_j})=2 \qquad (i\neq j).$$

(47) $$(1_G*\eta_{\mu_i\mu_j},\ 1_G*\eta_{\mu_i\mu_j})=9 \qquad (i\neq j).$$

(48) $$(1_G*\eta_{\mu_i\mu_j},\ 1_G*\eta_{\mu_i\mu_{j'}})=6 \qquad (i\neq j).$$

(49) $$(1_G*\eta_{\mu_i\mu_j},\ 1_G*\eta_{\mu_{i'}\mu_{j'}})=4 \qquad (i\neq j).$$

If i, j and k are distinct, then

(50) $$(1_G*\eta_{\mu_i},\ 1_G*\eta_{\mu_j\mu_k})=0.$$

If i, j and k are distinct, then

(51) $$(1_G*\eta_{\mu_i\mu_j},\ 1_G*\eta_{\mu_j\mu_k})=2,\ (1_G*\eta_{\mu_i\mu_{j'}},\ 1_G*\eta_{\mu_j\mu_k})=0$$

(52) $$(1_G*\eta_{\mu_1\mu_2\mu_3},\ 1_G*\eta_{\mu_1\mu_2\mu_3})=7.$$

(53) $$(1_G*\eta_{\mu_i},\ 1_G*\eta_{\mu_1\mu_2\mu_3})=1.$$

(54) $$(1_G*\eta_{\mu_{i'}},\ 1_G*\eta_{\mu_1\mu_2\mu_3})=0$$

(55) $$(1_G*\eta_{\mu_i\mu_j},\ 1_G*\eta_{\mu_1\mu_2\mu_3})=4 \qquad (i\neq j)$$

(56) $$(1_G*\eta_{\mu_i\mu_{j'}},\ 1_G*\eta_{\mu_1\mu_2\mu_3})=2 \qquad (i\neq j).$$

(57) $$(1_G*\eta_{\mu_{i'}\mu_{j'}},\ 1_G*\eta_{\mu_1\mu_2\mu_3})=0 \qquad (i\neq j).$$

(58) $$(1_G*\eta_{\mu_{1'}\mu_{2'}\mu_{3'}},\ 1_G*\eta_{\mu_1\mu_2\mu_3})=0.$$

(59) $$(1_G*\eta_{\mu_{1'}\mu_{2'}\mu_3},\ 1_G*\eta_{\mu_1\mu_2\mu_3})=2.$$

(60) $$(1_G*\eta_{\mu_{1'}\mu_2\mu_3},\ 1_G*\eta_{\mu_1\mu_2\mu_3})=4.$$

We may assume that p^{a_1}, $p^{a_2}\neq 1$. By (40) and (43), B contains ordinary irreducible characters 1_G, χ_1, χ_2, χ_{μ_1} $(\mu_1\in\Lambda_1)$, χ_{μ_2} $(\mu_2\in\Lambda_2)$ such that

(61) $$1_G*\eta_{\mu_i}=1_G+\varepsilon_i\chi_i-\varepsilon_i\chi_{\mu_i}$$

for all $\mu_i\in\Lambda_i$ and $i=1$, 2, where $\varepsilon_i=\pm 1$. If $p^{a_3}\neq 1$, then B contains ordinary irreduicble characters χ_3, χ_{μ_3} $(\mu_3\in\Lambda_3)$ such that

(62) $$1_G*\eta_{\mu_3}=1_G+\varepsilon_3\chi_3-\varepsilon_3\chi_{\mu_3}$$

for all $\mu_3\in\Lambda_3$, where $\varepsilon_3=\pm 1$. By (44), 1_G, χ_1,χ_2, χ_3, χ_{μ_1} $(\mu_1\in\Lambda_1)$, χ_{μ_2} $(\mu_2\in\Lambda_2)$ and χ_{μ_3} $(\mu_3\in\Lambda_3)$ are distinct.

Let $\mu_1\in\Lambda_1$ and $\mu_2\in\Lambda_2$. By (41), (45), (46), (47), (49) and the assumption $p\geqq 5$, we obtain the following.

$$(1_G*\eta_{\mu_1\mu_2},\ \chi_i)=\varepsilon_i \text{ and } (1_G*\eta_{\mu_1\mu_2},\ \chi_{\mu_i})=-\varepsilon_i.$$

for $i=1, 2$. Moreover $(1_G*\eta_{\mu_1\mu_2},\ \chi)=0$, 1 or -1 for any $\chi\in\mathrm{Irr}(B_0)$. By the way,

$$(63)\qquad 1_G*\eta_{\mu_1\mu_2}=(1_G*\eta_{\mu_1})*\eta_{\mu_2}=1_G+\varepsilon_2\chi_2-\varepsilon_2\chi_{\mu_2}+\varepsilon_1\chi_1*\eta_{\mu_2}-\varepsilon_1\chi_{\mu_1}*\eta_{\mu_2}.$$

Hence $(\chi_1*\eta_{\mu_2},\ \chi_1*\eta_{\mu_2})=3$ and $(\chi_{\mu_1}*\eta_{\mu_2},\ \chi_{\mu_1}*\eta_{\mu_2})=3$, because $(\chi_1*\eta_{\mu_2},\ \chi_1*\eta_{\mu_2})\leqq 3$ and $(\chi_{\mu_1}*\eta_{\mu_2},\ \chi_{\mu_1}*\eta_{\mu_2})\leqq 3$. In particular we have $(\chi_1,\ \chi_1*\eta_{\mu_2}^2)=1$ and $(\chi_{\mu_1}, \chi_{\mu_1}*\eta_{\mu_2}^2)=1$, and hence $(\chi_1,\ \chi_1*\eta_{\mu_2})=1$ and $(\chi_{\mu_1},\ \chi_{\mu_1}*\eta_{\mu_2})=1$ for all $\mu_2\in\Lambda_2$. These yield

$$(64)\qquad \chi_1*\eta_{\mu_2}=\chi_1+\varepsilon'\chi_3'+\delta'\chi_{\mu_2}',\qquad \chi_{\mu_1}*\eta_{\mu_2}=\chi_{\mu_1}+\varepsilon_{\mu_1}'\chi_{\mu_1}'+\delta_{\mu_1}'\chi_{\mu_1\mu_2}$$

for all $\mu_2\in\Lambda_2$, where χ_3', χ_{μ_1}', χ_{μ_2}', $\chi_{\mu_1\mu_2}\in\mathrm{Irr}(B_0)$ and ε', δ', ε_{μ_1}', $\delta_{\mu_1}'=\pm1$. Here we note that χ_3', χ_{μ_1}, χ_{μ_1}', χ_{μ_2}', $\mu_2\in\Lambda_2$, are distinct and that $\chi_{\mu_1\mu_2}$, $\mu_2\in\Lambda_2$, are distinct. On substituting (64) in (63), we have

$$1_G*\eta_{\mu_1\mu_2}=1_G+\varepsilon_1\chi_1-\varepsilon_1\chi_{\mu_1}+\varepsilon_2\chi_2-\varepsilon_2\chi_{\mu_2}+\varepsilon\chi_3'+\delta_2\chi_{\mu_2}'+\varepsilon_{\mu_1}\chi_{\mu_1}'+\delta_{\mu_1}\chi_{\mu_1\mu_2}$$

for all $\mu_1\in\Lambda_1$ and $\mu_2\in\Lambda_2$, where ε, δ_2, ε_{μ_1}, $\delta_{\mu_1}=\pm1$. If μ_1, $\mu_1'\in\Lambda_1$ and $\mu_1\neq\mu_1'$, then $(1_G*\eta_{\mu_1\mu_2},\ 1_G*\eta_{\mu_1'\mu_2})=6$ for any $\mu_2\in\Lambda_2$. From this we see $\chi_{\mu_1}'\neq\chi_{\mu_1'}'$. On the other hand for μ_2, $\mu_2'\in\Lambda_2$, $(\chi_1*\eta_{\mu_2'},\ \chi_{\mu_1}*\eta_{\mu_2})=(\chi_1*\eta_{\mu_2'\mu_2-1},\ \chi_{\mu_1})=0$ by (64). Hence $\chi_{\mu_2}'\neq\chi_{\mu_1\mu_2'}$. It also holds that

$$(63')\qquad 1_G*\eta_{\mu_1\mu_2}=(1_G*\eta_{\mu_2})*\eta_{\mu_1}=1_G+\varepsilon_1\chi_1-\varepsilon_1\chi_{\mu_1}+\varepsilon_2\chi_2*\eta_{\mu_1}-\varepsilon_2\chi_{\mu_2}*\eta_{\mu_1}.$$

By the above argument we have

$$(64')\qquad \chi_2*\eta_{\mu_1}=\chi_2\pm\tilde{\chi}_3\pm\tilde{\chi}_{\mu_1},\quad \chi_{\mu_2}*\eta_{\mu_1}=\chi_{\mu_2}\pm\tilde{\chi}_{\mu_2}\pm\tilde{\chi}_{\mu_1\mu_2}.$$

for all $\mu_1\in\Lambda_1$, where $\tilde{\chi}_3$, $\tilde{\chi}_{\mu_1}$, $\tilde{\chi}_{\mu_2}$, $\tilde{\chi}_{\mu_1\mu_2}\in\mathrm{Irr}(B_0)$. Then $\tilde{\chi}_3$, $\tilde{\chi}_{\mu_2}$, χ_{μ_2}, $\tilde{\chi}_{\mu_1}$, $\mu_1\in\Lambda_1$ are distinct and $\tilde{\chi}_{\mu_1\mu_2}$, $\mu_1\in\Lambda_1$ are distinct. If μ_2, $\mu_2'\in\Lambda_2$ and $\mu_2\neq\mu_2'$, then $\tilde{\chi}_{\mu_2}\neq\tilde{\chi}_{\mu_2'}$. Moreover if μ_1, $\mu_1'\in\Lambda_1$, then $\tilde{\chi}_{\mu_1}\neq\tilde{\chi}_{\mu_1'\mu_2}$. On substituting (64′) in (63′),

$$1_G*\eta_{\mu_1\mu_2}=1_G+\varepsilon_1\chi_1-\varepsilon_1\chi_{\mu_1}+\varepsilon_2\chi_2-\varepsilon_2\chi_{\mu_2}+\tilde{\varepsilon}\tilde{\chi}_3+\delta_1\tilde{\chi}_{\mu_1}+\tilde{\varepsilon}_{\mu_2}\tilde{\chi}_{\mu_2}+\tilde{\delta}_{\mu_2}\tilde{\chi}_{\mu_1\mu_2}$$

for all $\mu_1\in\Lambda_1$ and $\mu_2\in\Lambda_2$, where $\tilde{\varepsilon}$, δ_1, $\tilde{\varepsilon}_{\mu_2}$, $\tilde{\delta}_{\mu_2}=\pm1$. Therefore we can see $\chi_3'=\tilde{\chi}_3$, $\varepsilon_{\mu_1}=\delta_1$, $\chi_{\mu_1}'=\tilde{\chi}_{\mu_1}$, $\delta_2=\tilde{\varepsilon}_{\mu_2}$, $\chi_{\mu_2}'=\tilde{\chi}_{\mu_2}$, $\delta_{\mu_1}=\tilde{\delta}_{\mu_2}$ and $\chi_{\mu_1\mu_2}=\tilde{\chi}_{\mu_1\mu_2}$. We put $\delta_{12}=\delta_{\mu_1}=\tilde{\delta}_{\mu_2}$. Hence we have

$$(65)\qquad 1_G*\eta_{\mu_1\mu_2}=1_G+\varepsilon_1\chi_1-\varepsilon_1\chi_{\mu_1}+\varepsilon_2\chi_2-\varepsilon_2\chi_{\mu_2}+\varepsilon\chi_3'+\delta_2\chi_{\mu_2}'+\delta_1\chi_{\mu_1}'+\delta_{12}\chi_{\mu_1\mu_2}.$$

Furthermore if μ_1, $\mu_1'\in\Lambda_1$, then $\chi_{\mu_1}'\neq\chi_{\mu_1\mu_2}$. Hence by (48) we can see that if $\mu_1\neq\mu_1'$, then $\chi_{\mu_1'}\neq\chi_{\mu_1}'$, $\chi_{\mu_1\mu_2}$. On the other hand if μ_2, $\mu_2'\in\Lambda_2$ and $\mu_2\neq\mu_2'$, by (48)

$\chi_{\mu_2'} \neq \chi_{\mu_2}'$, $\chi_{\mu_1\mu_2}$. By (49) we can show if $\mu_1 \neq \mu_1'$ and $\mu_2 \neq \mu_2'$, then $\chi_{\mu_1\mu_2} \neq \chi_{\mu_1'\mu_2'}$. Hence 1_G, χ_1, χ_2, χ_3', $\chi_{\mu_1}(\mu_1 \in \Lambda_1)$, $\chi_{\mu_1}'(\mu_1 \in \Lambda_1)$, $\chi_{\mu_2}(\mu_2 \in \Lambda_2)$, $\chi_{\mu_2}'(\mu_2 \in \Lambda_2)$, $\chi_{\mu_1\mu_2}(\mu_1 \in \Lambda_1, \mu_2 \in \Lambda_2)$ are distinct.

Suppose that $p^{a_3}=1$ and $p^{a_0}=1$. Then $\{1_P\} \cup \Lambda_1 \cup \Lambda_2 \cup \{\mu_1\mu_2 \mid \mu_1 \in \Lambda_1, \mu_2 \in \Lambda_2\}$ is a set of representatives for the X-conjugacy classes of linear characters of P. Hence by Lemma 1, (61) and (65), $k(B_0)=4+2|\Lambda_1|+2|\Lambda_2|+|\Lambda_1||\Lambda_2|$. Since $|\Lambda_i|=(p^{a_i}-1)/2$, we have $l(B_0)=4$ by (33). Suppose that $p^{a_3}=1$ and $p^{a_0}\neq 1$. Then $\Lambda_0 \cup \{\lambda\mu_1 \mid \lambda \in \Lambda_0, \mu_1 \in \Lambda_1\} \cup \{\lambda\mu_2 \mid \lambda \in \Lambda_0, \mu_2 \in \Lambda_2\} \cup \{\lambda\mu_1\mu_2 \mid \lambda \in \Lambda_0, \mu_1 \in \Lambda_1, \mu_2 \in \Lambda_2\}$ is a set of representatives for the X-conjugacy classes of linear characters of P, where Λ_0 is the set of linear characters of P_0. By Lemma 1, (61) and (65) we can show $k(B_0) \leqq 4p^{a_0}+2p^{a_0}(p^{a_1}-1)/2+2p^{a_0}(p^{a_2}-1)/2+p^{a_0}(p^{a_1}-1)(p^{a_2}-1)/4$. On the other hand by (33) $k(B_0)=l(B_0)-4+4p^{a_0}+2p^{a_0}(p^{a_1}-1)/2+2p^{a_0}(p^{a_2}-1)/2+p^{a_0}(p^{a_1}-1)(p^{a_2}-1)/4$. Since $p \geqq 5$, by the same argument as in Step 3 in Theorem 1, we have $l(B_0)=4$.

We suppose that $p^{a_3}\neq 1$ in the rest of the proof. By (50) and the assumption $p \geqq 5$, $\varepsilon=-\varepsilon_3$ and $\chi_3'=\chi_3$. By (51) and (65) we can show the following.

$$(66)\quad \begin{cases} 1_G*\eta_{\mu_1\mu_3}=1_G+\varepsilon_1\chi_1-\varepsilon_1\chi_{\mu_1}-\varepsilon_2\chi_2+\varepsilon_3\chi_3-\varepsilon_3\chi_{\mu_3}+\delta_1\chi_{\mu_1}'+\delta_3\chi_{\mu_3}'+\delta_{13}\chi_{\mu_1\mu_3} \\ 1_G*\eta_{\mu_2\mu_3}=1_G-\varepsilon_1\chi_1+\varepsilon_2\chi_2-\varepsilon_2\chi_{\mu_2}+\varepsilon_3\chi_3-\varepsilon_3\chi_{\mu_3}+\delta_2\chi_{\mu_2}'+\delta_3\chi_{\mu_3}'+\delta_{23}\chi_{\mu_2\mu_3} \end{cases}$$

for all $\mu_i \in \Lambda_i$, $i=1, 2, 3$, where χ_{μ_3}', $\chi_{\mu_1\mu_3}$, $\chi_{\mu_2\mu_3} \in \mathrm{Irr}(B_0)$ and $\delta_3, \delta_{13}, \delta_{23}=\pm 1$.

Let $\mu_i \in \Lambda_i$, $i=1, 2, 3$. By (42), (52), (53), (54), (55), (56) and by the assumption $p \geqq 5$, $(1_G*\eta_{\mu_1\mu_2\mu_3}, \chi_i)=0$, $(1_G*\eta_{\mu_1\mu_2\mu_3}, \chi_{\mu_i})=-\varepsilon_i$ and $(1_G*\eta_{\mu_1\mu_2\mu_3}, \chi_{\mu_i}')=\delta_i$ for $i=1, 2, 3$. Therefore we have

$$(67)\quad \begin{cases} 1_G*\eta_{\mu_1\mu_2\mu_3}=-\sum_{i=1}^{3}\varepsilon_i\chi_{\mu_i}+\sum_{i=1}^{3}\delta_i\chi_{\mu_i}'+\delta\chi_{\mu_1\mu_2\mu_3}, \\ 1_G*\eta_{\mu_1^{-1}\mu_2\mu_3}=-\sum_{i=1}^{3}\varepsilon_i\chi_{\mu_i}+\sum_{i=1}^{3}\delta_i\chi_{\mu_i}'+\delta\chi_{\mu_1^{-1}\mu_2\mu_3}, \end{cases}$$

where $\chi_{\mu_1\mu_2\mu_3}$, $\chi_{\mu_1^{-1}\mu_2\mu_3} \in \mathrm{Irr}(B_0)$ and $\delta=\pm 1$. By (51), (54), (56), (57), (58), (59) and (60), 1_G, χ_1, χ_2, χ_3, χ_{μ_i} $(\mu_i \in \Lambda_i, i=1, 2, 3)$, χ_{μ_i}' $(\mu_i \in \Lambda_i, i=1, 2, 3)$, $\chi_{\mu_1\mu_2}$ $(\mu_1 \in \Lambda_1, \mu_2 \in \Lambda_2)$, $\chi_{\mu_1\mu_3}$ $(\mu_1 \in \Lambda_1, \mu_3 \in \Lambda_3)$, $\chi_{\mu_2\mu_3}$ $(\mu_2 \in \Lambda_2, \mu_3 \in \Lambda_3)$, $\chi_{\mu_1\mu_2\mu_3}$ $(\mu_i \in \Lambda_i, i=1, 2, 3)$, $\chi_{\mu_1^{-1}\mu_2\mu_3}$ $(\mu_i \in \Lambda_i, i=1, 2, 3)$ are distinct. $\Lambda_0 \cup \{\lambda\mu_i \mid \lambda \in \Lambda_0, \mu_i \in \Lambda_i, i=1, 2, 3\} \cup \{\lambda\mu_1\mu_2 \mid \lambda \in \Lambda_0, \mu_1 \in \Lambda_1, \mu_2 \in \Lambda_2\} \cup \{\lambda\mu_1\mu_3 \mid \lambda \in \Lambda_0, \mu_1 \in \Lambda_1, \mu_3 \in \Lambda_3\} \cup \{\lambda\mu_2\mu_3 \mid \lambda \in \Lambda_0, \mu_2 \in \Lambda_2, \mu_3 \in \Lambda_3\} \cup \{\lambda\mu_1\mu_2\mu_3 \mid \lambda \in \Lambda_0, \mu_i \in \Lambda_i\} \cup \{\lambda\mu_1^{-1}\mu_2\mu_3 \mid \lambda \in \Lambda_0, \mu_i \in \Lambda_i\}$ is a set of representatives for the X-conjugacy classes of linear characters of P. If $p^{a_0}=1$, then by (61), (62), (65), (66) and (67) we obtain $k(B_0)=4+2|\Lambda_1|+2|\Lambda_2|+2|\Lambda_3|+|\Lambda_1||\Lambda_2|+|\Lambda_1||\Lambda_3|+|\Lambda_2||\Lambda_3|+2|\Lambda_1||\Lambda_2||\Lambda_3|$. Hence by (33), $l(B_0)=4$. If $p^{a_0}\neq 1$, then we can show

$l(B_0)=4$ by the same argument as in the case $p^{a_3}=1$ and $p^{a_0}\neq 1$. Thus the theorem is proved.

THEOREM 5. *If* $p\geqq 7$ *and* $|N(P)/C(P)|=5$, *then* $l(B_0)=5$.

Proof. We prove by induction on $|G|$. We may assume that the maximal normal p'-subgroup $O_{p'}(G)$ of G is the identity group. We put $P_1=C(N(P))\cap P$ and $P_2=[N(P), P]$. By [8, chapter 5, Theorem 2. 3], $P=P_1\times P_2$. Let Λ be a set of the representatives for the $N(P)$-conjugacy classes of non trivial linear characters of P_2. For μ, $\mu'\in\Lambda$ with $\mu\neq\mu'$, we have

$$(1_G*\eta_\mu,\ 1_G)=4,\quad (1_G*\eta_\mu,\ 1_G*\eta_\mu)=21, \tag{68}$$

$$(1_G*\eta_\mu,\ 1_G*\eta_{\mu'})=20$$

by Step 2 in Theorem 1. We assume that $1_G*\eta_{\mu_0}=41_G+2\varepsilon\chi_1-\varepsilon\chi_{\mu_0}$ for some $\mu_0\in\Lambda$, where $\varepsilon=\pm 1$ and χ_1, $\chi_{\mu_0}\in\mathrm{Irr}(B_0)$. Then by (68),

$$1_G*\eta_\mu=41_G+2\varepsilon\chi_1-\varepsilon\chi_\mu \tag{69}$$

for all $\mu\in\Lambda$, where $\chi_\mu\in\mathrm{Irr}(B_0)$. For any linear character λ of P_1,

$$1_G*\eta_{\lambda\mu}=4(1_G*\lambda)+2\varepsilon(\chi_1*\lambda)-\varepsilon(\chi_\mu*\lambda).$$

Hency by Lemma 1, $k(B_0)\leqq|P_1|(2+|\Lambda|)$. Since $k(B_0)=l(B_0)+5(|P_1|-1)+|P_1||\Lambda|$ by the induction hypothesis, we have $|P_1|=1$ and $l(B_0)=2$. If G is p-solvable, then $l(B_0)=5$ by Fong [7, Theorem (3C)] and Okuyama and Wajima [9, Theorem]. Hence G is not solvable, and as is well known G has even order. Let π be an element of $P-\{1\}$. We put $d(\chi_1, \pi, 1_{C(\pi)})=d$ and $c=2d+4\varepsilon$. We note that χ_1 is p-rational by (69) and hence d is a rational integer. (69) implies $d(\chi_\mu, \pi, 1_{C(\pi)})=c-\varepsilon\eta_\mu(\pi)$. On substituting these in $|P|=\sum_{\chi\in B_0}|d(\chi, \pi, 1_{C(\pi)})|^2$, we have $d^2-4+2c\varepsilon+(|P_2|-1)c^2/5=0$. Then we see $c=0$, and hence

$$d(\chi_1, \pi, 1_{C(\pi)})=-2\varepsilon \text{ and } d(\chi_\mu, \pi, 1_{C(\pi)})=-\varepsilon\eta_\mu(\pi). \tag{70}$$

Let t be an involution of G. Then $\chi_\mu(t)=2\chi_1(t)-\varepsilon$ and $\chi_\mu(1)=2\chi_1(1)-\varepsilon$. Since π and π^{-1} are not conjugate by the assumption, by Brauer [2, II, Proposition 4 and Corollary 1],

$$\sum_{\chi\in B_0}d(\chi, \pi, 1_{C(\pi)})\chi(t)^2/\chi(1)=0.$$

From this we obtain $\chi_1(t)=\chi_1(1)$ and $\chi_\mu(t)=\chi_\mu(1)$ for all $\mu\in\Lambda$. Hence $t\in O_{p'}(G)$

by Brauer [2, I Theorem 1]. This contradicts $O_{p'}(G)=\{1\}$. Therefore (69) does not hold and hence the following holds by the assumption $p\geqq 7$ and (68).

(71) $$1_G{}^*\eta_\mu = 4\,1_G+\varepsilon_2\chi_2+\varepsilon_3\chi_3+\varepsilon_4\chi_4+\varepsilon_5\chi_5+\delta\chi_\mu$$

for all $\mu\in\Lambda$, where ε_i, $\delta=\pm 1$ and χ_i, $\chi_\mu\in \mathrm{Irr}(B_0)$. Furthermore 1_G, χ_i, $2\leqq i\leqq 5$, and χ_μ, $\mu\in\Lambda$ are distinct. Hence if $|P_1|=1$, then by Lemma 1 and (71) we have $l(B_0)=5$. If $|P_1|\neq 1$, by the same argument as in Step 3 in Theorem 1, $k(B_0)=|P_1|(5+|\Lambda|)$ and hence $l(B_0)=5$.

REMARK 3. Theorems 2, 3, 4 and 5 hold for all prime numbers.

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