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ON A COMPACT ABELIAN GROUP EXTENSION OF A W*-DYNAMICS

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In this note we discuss an analogue of a compact abelian group extension of a classical dynamics to a W*-dynamics. We give a necessary and sufficient condition for a compact abelian group extension of an ergodic W*-dynamics to be ergodic and its application.

1. Introduction

Let M be a von Neumann algebra, φ be a faihtful normal state on M and $L^2(M, \varphi)$ be a Hilbert space which is the completion of M by an inner product $(x|y)_\varphi = \varphi(y^*x)$. If an automorphism α of M preserves the state φ, that is, $\varphi\circ\alpha=\varphi$, we say that the triple (M, φ, α) is *an invariant W*-dynamics*. An invariant W*-dynamics (N, ψ, β) is *conjugate* to (M, φ, α) if there is an isomorphism Φ of M onto N such that $\varphi=\psi\circ\Phi$ and $\alpha=\Phi^{-1}\circ\beta\circ\Phi$ on M. Let U be a unitary operator on $L^2(M, \varphi)$ defined by $Ux=\alpha(x)$ for x in M. The von Neumann algebra M is naturally identified with a von Neumann algebra acting on the Hilbert space $L^2(M, \varphi)$. (M, φ, α) is *ergodic* if $\{\xi\in L^2(M, \varphi);\ U\xi=\xi\}=C1_M$, where 1_M is the identity in M. Let σ be a continuous action of a compact abelian group G on M such that $\varphi\circ\sigma_g=\varphi$ for g in G. If (M, φ, α) is an invariant W*-dynamics such that $\sigma_g\circ\alpha=\alpha\circ\sigma_{\kappa(g)}$ for g in G and for some automorphism κ of G, then α induces an automorphism $\alpha|_{M^\sigma}$ of the fixed point subalgebra M^σ of M under the action σ, which preserves the state $\varphi|_{M^\sigma}$. If an invariant W*-dynamics (N, ψ, β) is conjugate to $(M^\sigma, \varphi|_{M^\sigma}, \alpha|_{M^\sigma})$, we say that (M, φ, α) is *a (G, σ)-extension of (N, ψ, β) under κ*. Let Γ be the dual group of a compact abelian group G. An element γ of Γ is called *n-periodic* with respect to an automorphism κ of G if $\gamma\kappa\neq\gamma, \dots,$ $\gamma\kappa^{n-1}\neq\gamma$ and $\gamma\kappa^n=\gamma$ $(n\geqslant 1)$.

In Section 2 we prove the following theorem:

THEOREM 1. *Let (M, φ, α) be a (G, σ)-extension of an ergodic invariant W*-dynamics under κ. Then (M, φ, α) is not ergodic if and only if there exist a positive* *integer* n *and a* γ *in* Γ, *n-periodic with respect to* κ *and not equal to* 1, *and a* ξ_γ *in* $L^2(M, \varphi)$, $\xi_\gamma \neq 0$, *such that* $U^n\xi_\gamma = \xi_\gamma$ *and* $U_g\xi_\gamma = \langle g, \gamma \rangle \xi_\gamma$ *for g in G, where U and* U_g *are unitary operators on* $L^2(M, \varphi)$ *defined by* $Ux = \alpha(x)$ *and* $U_g x = \sigma_g(x)$ *for x in M, respectively.*

In Section 3 we construct an example of a compact abelian group extension $(\tilde{M}, \tilde{\varphi}, \tilde{\alpha})$ of a W*-dynamics (M, φ, α) and as an application of Theorem 1 to these compact abelian group extensions, we prove the following theorem:

THEOREM 3. *For an ergodic invariant W*-dynamics* (M, φ, α), $(\tilde{M}, \tilde{\varphi}, \tilde{\alpha})$ *is not ergodic if and only if there exist a positive integer* n *and a* γ *in* Γ, *n-periodic with respect to* κ *and not equal to* 1, *and a* ξ *in* $L^2(M, \varphi)$, $\xi \neq 0$, *such that*

$$U^n \xi = \prod_{i=0}^{n-1} \alpha^{n-i-1}(u_{\gamma\kappa^i})\xi,$$

where U is a unitary operator on $L^2(M, \varphi)$ *defined by* $Ux = \alpha(x)$ *for x in M and* $\prod_{i=0}^{n-1} a_i = a_0 a_1 a_2 \ldots a_{n-1}$.

NOTE. We have recently been informed by Professor Y. Nakagami that the usual ergodicity of an invariant W*-dynamics in the sense that the fixed point subalgebra is trivial implies our ergodicity in the L^2-sense and hence both ergodicities are equivalent (Also cf. [7]). Consequently we have a spectral condition for the ergodicity in the usual sense of the compact abelian group extension of ergodic W*-dynamics.

2. Ergodicity of a compact abelian group extension of a W*-dynamics

In this section we prove the following theorem:

THEOREM 1. *Let* (M, φ, α) *be a* (G, σ)*-extension of an ergodic invariant W*-dynamics under an automorphism* κ *of G. Then* (M, φ, α) *is not ergodic if and only if there exist a positive integer* n *and a* γ *in* Γ, *n-periodic with respect to* κ *and not equal to* 1, *and a* ξ_γ *in* $L^2(M, \varphi)$, $\xi_\gamma \neq 0$, *such that* $U^n\xi_\gamma = \xi_\gamma$ *and* $U_g\xi_\gamma = \langle g, \gamma \rangle \xi_\gamma$ *for g in G, where U and* U_g *are unitary operators on* $L^2(M, \varphi)$ *defined by* $Ux = \alpha(x)$ *and* $U_g x = \sigma_g(x)$ *for x in M, respectively.*

To do this, we need the following lemma:

LEMMA 1. *Let* (M, φ, α) *be an invariant* W^**-dynamics and* σ *a continuous action of a compact abelian group* G *on* M *such that* $\varphi\circ\sigma_g=\varphi$ *and* $\sigma_g\circ\alpha=\alpha\circ\sigma_{\kappa(g)}$ *for all* g *in* G *and some automorphism* κ *of* G. *Let* $\mathscr{V}_\gamma$ $(\gamma\in\Gamma)$ *be the set of all* ξ *in* $L^2(M, \varphi)$ *such that* $U_g\xi=\langle g, \gamma\rangle\xi$ *for* g *in* G. *Then*

$$L^2(M, \varphi)=\sum_{\gamma\in\Gamma}\oplus\mathscr{V}_\gamma \text{ (orthogonal direct sum)} \tag{1}$$

and

(2) *if* ξ *is in* $\mathscr{V}_\gamma$, *then* $U\xi$ *is in* $\mathscr{V}_{\gamma\kappa}$, *where* U_g *and* U *are unitary operators on* $L^2(M, \varphi)$ *defined by* $U_gx=\sigma_g(x)$ *and* $Ux=\alpha(x)$ *for* x *in* M, *respectively.*

PROOF. (1) For a ξ in $\mathscr{V}_\gamma$ and a ξ' in $\mathscr{V}_{\gamma'}$, we have

$$\begin{aligned}(\xi|\xi')_\varphi&=(U_g\xi|U_g\xi')_\varphi\\&=\langle g, \gamma\rangle\overline{\langle g, \gamma'\rangle}(\xi|\xi')_\varphi,\end{aligned}$$

for all g in G. If $\gamma\neq\gamma'$, $\langle g, \gamma\rangle\overline{\langle g, \gamma'\rangle}\neq 1$ for some g in G, and so ξ is orthogonal to ξ'. Suppose that a ξ in $L^2(M, \varphi)$ is orthogonal to any vector in $\bigcup_{\gamma\in\Gamma}\mathscr{V}_\gamma$. Put $\xi_\gamma=\int_G\overline{\langle g, \gamma\rangle}\,U_g\xi dg$ for γ in Γ. Then ξ_γ belongs to $\mathscr{V}_\gamma$, and we have

$$\begin{aligned}(\xi_\gamma|\xi_\gamma)_\varphi&=\int_G\int_G\overline{\langle gh^{-1}, \gamma\rangle}(U_{gh^{-1}}\xi|\xi)\,dgdh\\&=\int_G\overline{\langle g, \gamma\rangle}(U_g\xi|\xi)\,dg\\&=(\xi_\gamma|\xi)_\varphi=0.\end{aligned}$$

Hence $\xi_\gamma=0$ for γ in Γ. If we put $f(g)=(U_g\xi|\xi)_\varphi$ for g in G, the function f on G is integrable and positive definite. Hence we have

$$\begin{aligned}(U_g\xi|\xi)_\varphi&=\int_\Gamma\langle g, \gamma\rangle\hat{f}(\gamma)d\gamma\\&=\int_\Gamma\langle g, \gamma\rangle\int_G\overline{\langle h, \gamma\rangle}(U_h\xi|\xi)_\varphi\,dhd\gamma\end{aligned}$$

$$=\int_{\Gamma}<g, \gamma>(\xi_\gamma|\xi)_\varphi d\gamma$$

$$=0,$$

for all g in G. Thus $(\xi|\xi)_\varphi=0$, and so $\xi=0$. This implies the assertion (1).
(2) If ξ is in $\mathscr{V}_\gamma$, we have

$$U_g(U\xi)=U(U_{\kappa(g)}\xi)$$

$$=<\kappa(g), \gamma>U\xi$$

$$=<g, \gamma\kappa>U\xi.$$

This implies the assertion (2). q. e. d.

PROOF OF THEOREM 1. Let ξ_γ be a vector which satisfies the conditions of Theorem 1. Put $\xi=\sum_{i=0}^{n-1}U^i\xi_\gamma$. Then ξ belongs to $\sum_{i=0}^{n-1}\oplus\mathscr{V}_{\gamma\kappa^i}$ and $U\xi=\xi$. Since ξ is not a constant (a scalar multiple of 1_M), (M, φ, α) is not ergodic. Conversely, let ξ be a vector in $L^2(M, \varphi)$ which is not a constant and $U\xi=\xi$, and let $\xi=\sum_{\gamma\in\Gamma}\xi_\gamma$ with ξ_γ in $\mathscr{V}_\gamma$ be the direct sum decomposition of ξ. Then $U\xi=\sum_{\gamma\in\Gamma}U\xi_\gamma$ and $U\xi_\gamma$ is in $\mathscr{V}_{\gamma\kappa}$. From $U\xi=\xi$, we have $U\xi_\gamma=\xi_{\gamma\kappa}$ and $\|\xi_\gamma\|_\varphi=\|\xi_{\gamma\kappa}\|_\varphi$ for γ in Γ. From the orthogonality of ξ_γ' s we have $\xi_\gamma=0$ if γ is not periodic with respect to κ. Now we note that for η in $L^2(M, \varphi)$, η is U-invariant and $\{U_g;\ g\in G\}$-invariant if and only if it is U-invariant and belongs to $L^2(M^\sigma, \varphi)$. If $\xi_\gamma=0$ for all $\gamma\neq1$, then $\xi=\xi_1$, and so ξ is $\{U_g;\ g\in G\}$-invariant and U-invariant. Hence ξ belongs to $L^2(M^\sigma, \varphi)$ and U-invariant, and thus ξ is a constant from the assumption of ergodicity. This contradicts the assumption of ξ. Therefore there exists a γ in Γ, $\gamma\neq1$ such that $\xi_\gamma\neq0$. From the above, this γ is n-periodic with respect to κ for some positive integer n, and then we have $U^n\xi_\gamma=\xi_{\gamma\kappa^n}=\xi_\gamma$. q. e. d.

3. Example

In this section we construct an example of a compact abelian group extension of W*-dynamics and apply Theorem 1 to discuss the ergodicity of these compact abelian group extension given as above example.

Let (M, φ, α) be an invariant W*-dynamics, $\mathscr{H}_\varphi$ be the Hilbert space $L^2(M, \varphi)$ and U be a unitary operator on $\mathscr{H}_\varphi$ defined by $Ux=\alpha(x)$ for x in M. M is naturally identified with a von Neumann algebra acting on $\mathscr{H}_\varphi$. Let G be a compact abelian group with dual group Γ, κ be an automorphism of G, θ be an action of Γ on M and u be a mapping of Γ into M^u satisfying the following condition:

$$\begin{cases} u_{\gamma\gamma'}=u_\gamma\theta_{\gamma\kappa}(u_{\gamma'}), \\ \alpha\circ\theta_\gamma=Adu_\gamma\circ\theta_{\gamma\kappa}\circ\alpha, \end{cases} \qquad (\gamma,\gamma'\in\Gamma)$$

where M^u is the unitaries in M. For a ξ in $l^2(\mathscr{H}_\varphi, \Gamma)$, we define a unitary operator $\tilde{U}$ on $l^2(\mathscr{H}_\varphi, \Gamma)$ by

$$(\tilde{U}\xi)(\gamma)=\theta_{\gamma^{-1}}(u_{\gamma\kappa^{-1}})U\xi(\gamma\kappa^{-1}),\ (\gamma\in\Gamma).$$

Then it is straightforward to check the following lemmas:

LEMMA 2. *If we put $\tilde{\alpha}=Ad\tilde{U}$, then $\tilde{\alpha}$ is an automorphism of $M\times_\theta\Gamma$, where $M\times_\theta\Gamma$ is the crossed product von Neumann algebra of M by Γ under θ.*

PROOF. $\tilde{\alpha}(\pi_\theta(x))=\pi_\theta(\alpha(x))$, $\tilde{\alpha}(\lambda(\gamma))=\pi_\theta(u_\gamma)\lambda(\gamma\kappa)$.

LEMMA 3. *Let σ be the dual action on G of θ. Then it holds that $\sigma_g\circ\tilde{\alpha}=\tilde{\alpha}\circ\sigma_{\kappa(g)}$ for g in G.*

LEMMA 4. *Let $\tilde{M}=M\times_\theta\Gamma$, ε be the conditional expectation of $\tilde{M}$ onto M defined by $\varepsilon(x)=\pi_\theta^{-1}(\int_G\sigma_g(x)dg)$ and $\tilde{\varphi}=\varphi\circ\varepsilon$. Then $\tilde{\varphi}$ is a faithful normal state on $\tilde{M}$ which is σ-invariant and $\tilde{\alpha}$-invariant.*

Thus we have

THEOREM 2. *$(\tilde{M}, \tilde{\varphi}, \tilde{\alpha})$ is a (G, σ)-extension of (M, φ, α) under κ.*

Now as an application of Theorem 1, we have the following theorem:

THEOREM 3. *Let (M, φ, α) be an ergodic invariant W^*-dynamics and $(\tilde{M}, \tilde{\varphi}, \tilde{\alpha})$ be the (G, σ)-extension of (M, φ, α) under κ as above. Then $(\tilde{M}, \tilde{\varphi}, \tilde{\alpha})$ is not ergodic if and only if there exist a positive integer n and a γ in Γ, n-periodic with respect* *to* κ *and not equal to* 1, *and a* ξ *in* $L^2(M, \varphi)$, $\xi \neq 0$, *such that*

$$U^n \xi = \prod_{i=0}^{n-1} \alpha^{n-i-1}(u_{\gamma\kappa^i})\xi,$$

where $\prod_{i=0}^{n-1} a_i = a_0 a_1 a_2 \dots a_{n-1}$.

PROOF. It follows from Theorem 1 and the following lemma.

LEMMA 5. *Let* $\tilde{U}$ *and* $\tilde{U}_g$ *be unitary operators on* $L^2(\tilde{M}, \tilde{\varphi})$ *defined by* $\tilde{U}\tilde{x} = \tilde{\alpha}(\tilde{x})$ *and* $\tilde{U}_g \tilde{x} = \sigma_g(\tilde{x})$ *for* $\tilde{x}$ *in* $\tilde{M}$, *respectively. If* γ *is an element in* Γ *which is n-periodic with respect to* κ *for some positive integer* n *and not equal to* 1, *then the following two conditions are equivalent*:

(*i*) *There exists a* ξ *in* $L^2(M, \varphi)$, $\xi \neq 0$, *such that*

$$U^n \xi = \prod_{i=0}^{n-1} \alpha^{n-i-1}(u_{\gamma\kappa^i})\xi.$$

(*ii*) *There exists a* ξ_γ *in* $L^2(\tilde{M}, \tilde{\varphi})$, $\xi_\gamma \neq 0$, *such that*

$$\tilde{U}^n \xi_\gamma = \xi_\gamma, \ \tilde{U}_g \xi_\gamma = \langle g, \gamma \rangle \xi_\gamma, \ (g \in G).$$

PROOF. Let $V_{\pi\theta}$ be an isometry of $L^2(M, \varphi)$ into $L^2(\tilde{M}, \tilde{\varphi})$ defined by $V_{\pi\theta} x = \pi_\theta(x)$ for x in M and Λ_γ be an unitary operator on $L^2(\tilde{M}, \tilde{\varphi})$ defined by $\Lambda_\gamma \tilde{x} = \lambda(\gamma)\tilde{x}$ for $\tilde{x}$ in $\tilde{M}$. Then from $\xi_\gamma = \Lambda_\gamma^{-1} V_{\pi\theta} \xi$ or $\xi = V_{\pi\theta}^{-1} \Lambda_\gamma \xi_\gamma$ follows the equivalence of (i) and (ii). q. e. d.

The following results are immediate consequences of the above theorem.

COROLLARY 1. *If* κ *is the identity automorphism, then* $(\tilde{M}, \tilde{\varphi}, \tilde{\alpha})$ *is ergodic if and only if for* γ *in* Γ *and* ξ *in* $L^2(M, \varphi)$, $U\xi = u_\gamma \xi$ *implies* $\gamma = 1$ *or* $\xi = 0$.

COROLLARY 2. *If the action* θ *is implemented by a unitary representation* W *of* Γ *on* $L^2(M, \varphi)$ *such that* $u_\gamma = AdU(W_\gamma) W_{\gamma\kappa}$ *belong to* M *for* γ *in* Γ, *then* $(\tilde{M}, \tilde{\varphi}, \tilde{\alpha})$ *is ergodic if and only if for all* γ *in* Γ, $\gamma \neq 1$, γ *is aperiodic with respect to* κ.

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