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# SOME COMMENTS ON THE INVARIANT SEQUENTIAL DECISION PROBLEM

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## 1. Introduction

For a sequential decision problem Bahadur [1] shows that the set of rules based on a sufficient and transitive sequence forms an essentially complete class. In this paper we consider a sequential decision problem which is invariant under a certain group of transformations. The purpose of this paper is to show that an analogous result holds even if the rules are confined to invariant ones.

In Section 2 we define the invariant sequential decision problem. The concept of transitivity introduced by Bahadur [1] is stated in Section 3. In Section 4 we show that the set of invariant rules based on a sufficient and transitive sequence is essentially complete in the class of all invariant sequential decision rules. In Section 5 we consider the essential completeness of the set of invariant rules with nonrandomized terminal decision rules, and the application of the result to some examples is given.

## 2. Invariant sequential decision problem

In this section we define a sequential decision problem which is invariant under a certain group of transformations. For the details see Chapter 7 in Ferguson [2].

Let $X=(X_1, X_2, \ldots)$ be a sequence of random variables. We denote the sample space of $X$ by $\mathfrak{X}$ and the $\sigma$-field on $\mathfrak{X}$ by $\mathfrak{A}$. For each $n\geq 1$ let $\mathfrak{A}_n$ be the subfield of $\mathfrak{A}$ generated by $X_1, \ldots, X_n$. The distribution of $X$ depends on the unknown parameter $\theta\varepsilon\Theta$ where $\Theta$ is a parameter space.

Let $D$ be a decision space and $\mathfrak{C}$ a $\sigma$-field on $D$, Let $L(\theta, d)$ be a real-valued function defined on $\Theta\times D$, which represents the loss when $\theta$ is the true parameter and $d$ is the decision. For each $n\geq 1$ let $c_n(\theta; x)$ be a real-valued function defined on $\Theta\times\mathfrak{X}$ and we assume that for fixed $\theta\varepsilon\Theta$ $c_n(\theta; .)$ is $\mathfrak{A}_n$-measurable. The number $c_n(\theta; x)$ represents the cost when we terminate the sampling at the *n-th* observation and $X=x$.

Suppose that there exists a group $G$ of transformations on $\mathfrak{X}$ such that for each $g \varepsilon G$

$$g\mathfrak{A}_n=\mathfrak{A}_n,\ n=1, 2,\dots$$

and the distribution of $gX$, given $\theta$, is equal to that of $X$, given $\bar{g}\theta$, where $\bar{g}$ is a transformation on $\Theta$ corresponding to $g$. It is also supposed that for each $g \varepsilon G$

$$L(\bar{g}\theta,\ \tilde{g}d)=L(\theta,\ d),\ \theta\varepsilon\Theta,\ d\varepsilon D$$

where $\tilde{g}$ is a transformation on $D$ corresponding to $g$, and for each $n$ and each $g \varepsilon G$

$$c_n(\bar{g}\theta;\ gx)=c_n(\theta;\ x),\ \theta\varepsilon\Theta,\ x\varepsilon\mathfrak{X}.$$

We say that the problem is invariant under $G$ if these conditions are satisfied.

A sequential decision rule consists of a stopping rule and a terminal decision rule. A stopping rule is a sequence of functions $\{\phi_n(x);\ n\geq 1\}$ such that for each $n$ $\phi_n$ is $\mathfrak{A}_n$-measurable and $0\leq\phi_n\leq 1$. Given $X=x$, $\phi_n(x)$ represents the conditional probability that we terminate sampling at the $n$-$th$ observation. A terminal decision rule is a sequence of functions $\{\delta_n(C|x);\ n\geq 1\}$ such that for each $n$ $\delta_n(C|.)$ is $\mathfrak{A}_n$-measurable for every $C\varepsilon\mathfrak{C}$ and $\delta_n(.|x)$ is a probability measure on $\mathfrak{C}$ for every $x\varepsilon\mathfrak{X}$. Then the risk function of a sequential decision rule $\lambda=(\{\phi_n\},\ \{\delta_n\})$ is given by

$$R(\theta,\ \lambda)=\sum_{n=1}^{\infty}E_\theta\{\psi_n(X)[\int L(\theta,\ s)\delta_n(ds|X)+c_n(\theta;X)]\} \tag{2.1}$$

where

$$\psi_n(x)=(1-\phi_1(x))(1-\phi_2(x))\dots(1-\phi_{n-1}(x))\phi_n(x),\ n=1, 2,\dots \tag{2.2}$$

and $\psi_n(x)$ represents the conditional probability of not stopping before the $n$-$th$ observation and then stopping after the $n$-$th$ observation, given $X=x$.

For an invariant sequential decision problem, it is natural to confine our attentions to invariant sequential decision rules. A stopping rule $\{\phi_n\}$ is said to be invariant under $G$ if for each $n$

(2.3) $$\phi_n(gx)=\phi_n(x),\ g\varepsilon G,\ x\varepsilon\mathfrak{X}$$

and a terminal decision rule $\{\delta_n\}$ is said to be invariant under $G$ if for each $n$

(2.4) $$\delta_n(\bar{g}C|gx)=\delta_n(C|x),\ g\varepsilon G,\ C\varepsilon\mathfrak{C},\ x\varepsilon\mathfrak{X}.$$

Then we say that a sequential decision rule $(\{\phi_n\},\ \{\delta_n\})$ is invariant under $G$ if both $\{\phi_n\}$ and $\{\delta_n\}$ are invariant under $G$.

Let $\{\mathfrak{B}_n;\ n\geq 1\}$ be a sequence of subfields of $\mathfrak{A}$ and $\{f_n(x);\ n\geq 1\}$ is a sequence of functions defined on $\mathfrak{X}$. Then we say that $\{f_n(x)\}$ is $\{\mathfrak{B}_n\}$-measurable if for each $n$ $f_n(x)$ is $\mathfrak{B}_n$-measurable. The terminal decision rule $\{\delta_n\}$ is said to be $\{\mathfrak{B}_n\}$-measurable if for each $C\varepsilon\mathfrak{C}$ $\{\delta_n(C|.)\}$ is $\{\mathfrak{B}_n\}$-measurable.

For each $n\geq 1$ let $\mathfrak{A}_{In}=\{A\varepsilon\mathfrak{A}_n;\ gA=A$ for all $g\varepsilon G\}$. It is easy to see that $\mathfrak{A}_{In}$ is a subfield of $\mathfrak{A}_n$ and that a stopping rule $\{\phi_n\}$ is invariant under $G$ if and only if $\{\phi_n\}$ is $\{\mathfrak{A}_{In}\}$-measurale.

## 3. Transitivity

Let $\{\mathfrak{B}_n;\ n\geq 1\}$ and $\{\mathfrak{B}_{0n};\ n\geq 1\}$ be two sequences of subfields of $\mathfrak{A}$ such that $\mathfrak{A}_n\supset\mathfrak{B}_n\supset\mathfrak{B}_{0n}$ for each $n$. We say that $\{\mathfrak{B}_{0n}\}$ is a sufficient sequence for $\{\mathfrak{B}_n\}$ if for each $n$ $\mathfrak{B}_{0n}$ is sufficient for $\mathfrak{B}_n$. Bahadur [1] introduced the concept of transitivity in sequential problem. The sequence $\{\mathfrak{B}_{0n}\}$ is said to be transitive for $\{\mathfrak{B}_n\}$ if for every $n$, every $\mathfrak{B}_{0(n+1)}$-measurable function $f$ and every $\theta\varepsilon\Theta$,

$$E_\theta(f(X)|\mathfrak{B}_n)=E_\theta(f(X)|\mathfrak{B}_{0n})\text{ a.e.}$$

The following lemma, which is obtained by Hall, et al. (see Theorem 4.3 in [3]) is useful to show that a sequence is transitive.

LEMMA 1. *Suppose that* $\{\mathfrak{C}_n;\ n\geq 1\}$ *is a sequence of independent subfields of* $\mathfrak{A}$ *and for each* $n$ $\mathfrak{B}_n=\mathfrak{C}_1\vee\ldots\vee\mathfrak{C}_n$ *(the smallest subfield which contains* $\mathfrak{C}_1,\ldots.\mathfrak{C}_n$*). If for each* $n$ $\mathfrak{B}_{0(n+1)}\subset\mathfrak{B}_{0n}\vee\mathfrak{C}_{n+1}$, *then* $\{\mathfrak{B}_{0n}\}$ *is a transitive sequence for* $\{\mathfrak{B}_n\}$.

The important property of a sufficient and transitive sequence in sequential problem is the following lemma, the proof of which is obtained by Bahadur (see Theorem 11.4 in [1]).

LEMMA 2. *If* $\{\mathfrak{B}_{0n}\}$ *is a sufficient and transitive sequence for* $\{\mathfrak{B}_n\}$, *for each* $\{\mathfrak{B}_n\}$*-measurable stopping rule* $\{\phi_n\}$ *there exists a* $\{\mathfrak{B}_{0n}\}$*-measurable stopping rule* $\{\phi_n^*\}$ *such that for each* $n$ *and every* $\theta\varepsilon\Theta$

$$E_\theta(\psi_n(X)\,|\,\mathfrak{B}_{0n})=E_\theta(\psi_n^*(X)\,|\,\mathfrak{B}_{0n}) \quad \text{a.e.}$$

*where* $\psi_n$ *and* $\psi_n^*$ *are constructed by* (2.2).

## 4. Essential completeness

In the sequel we suppose that $\{\mathfrak{A}_{Sn}\}$ is a sufficient and transitive sequence for $\{\mathfrak{A}_n\}$. Bahadur [1] showed that the set of rules $(\{\phi_n\}, \{\delta_n\})$ in which both $\{\phi_n\}$ and $\{\delta_n\}$ are $\{\mathfrak{A}_{Sn}\}$-measurable is essentially complete among all sequential decision rules. In this section we show that an analogous result holds in the invariant sequential decision problem under some assumptions.

ASSUMPTION 1. There exists a $\{\mathfrak{A}_{Sn}\}$-measurable terminal decision rule which is invariant under $G$.

For each $n$ let $\mathfrak{A}_{SIn}=\mathfrak{A}_{Sn}\cap\mathfrak{A}_{In}$.

ASSUMPTION 2. For each $n$ there exists a set $A_n\varepsilon\mathfrak{A}_{SIn}$ of probability measure 1 for every $\theta\varepsilon\Theta$ and a real-valued function $Q_n(A|x)$ defined on $\mathfrak{A}_n\times\mathfrak{X}$ with $Q_n(A|x)=0$ for $A\varepsilon\mathfrak{A}_n$ and $x\notin A_n$, such that

(i) for any $x\varepsilon A_n$, $Q_n(.\,|x)$ is a probability measure on $\mathfrak{A}_n$;

(ii) for any $A\varepsilon\mathfrak{A}_n$, $Q_n(A|.)$ is a version of the conditional probability of $A$, given $\mathfrak{A}_{Sn}$;

(iii) for any $x\varepsilon\mathfrak{X}$, $A\varepsilon\mathfrak{A}_n$ and $g\varepsilon G$,

$$Q_n(gA\,|\,gx)=Q_n(A\,|\,x).$$

LEMMA 3. *If Assumptions* 1 *and* 2 *hold, then the set of invariant rules with* $\{\mathfrak{A}_{Sn}\}$*-measurable terminal decision rules is essentially complete in the class of all invariant sequential decision rules.*

PROOF. Let $\lambda=(\{\phi_n\}, \{\delta_n\})$ be any invariant rules. For each $n$ let

$$\xi_n(C|x)=\int\psi_n(y)\delta_n(C|y)Q_n(dy|x), \qquad C\varepsilon\mathfrak{C},\ x\varepsilon\mathfrak{X},$$

where $\psi_n$ is (2.2). Then Assumption 2 implies that for any $\theta \varepsilon \Theta$

$$\xi_n(C|x) = E_\theta\{\psi_n(X)\delta_n(C|x)|\mathfrak{A}_{Sn}\} \quad \text{a. e.}$$

and by (2.3) and (2.4)

$$\text{(4.1)} \qquad \xi_n(\bar{g}C|gx) = \xi_n(C|x), \qquad g\varepsilon G,\ C\varepsilon\mathfrak{C},\ x\varepsilon\mathfrak{X}.$$

For each $n$, $C\varepsilon\mathfrak{C}$ and $x\varepsilon\mathfrak{X}$, let

$$\begin{aligned} \delta_n^*(C|x) &= \xi_n(C|x)/\xi_n(D|x) && \text{if } \xi_n(D|x)>0, \\ &= \delta_n^0(C|x) && \text{otherwise,} \end{aligned}$$

where $\{\delta_n^0\}$ is the invariant terminal decision rule in Assumption 1. Cleary $\{\delta_n^*\}$ is a $\{\mathfrak{A}_{Sn}\}$-measurable terminal decision rule and invariant under $G$ by (4.1), so that $\lambda^* = (\{\phi_n\}, \{\delta_n^*\})$ is an invariant sequential decision rule with $\{\mathfrak{A}_{Sn}\}$-measurable terminal decision rule.

It follows from Theorem 10.1 in Bahadur [1] that the risk function of $\lambda^*$ is equal to that of $\lambda$, which completes the proof.

The following lemmas are proved by Hall, et al. (see Theorem 3.2, Corollary 4.1 and Theorem 6.1 in [3]).

LEMMA 4. *If Assumption 2 holds, then* $\{\mathfrak{A}_{SIn}\}$ *is a sufficient and transitive sequence for* $\{\mathfrak{A}_{In}\}$.

LEMMA 5. *If Assumption 2 holds, then for each* $n$ $\mathfrak{A}_{Sn}$ *and* $\mathfrak{A}_{In}$ *are conditionally independent, given* $\mathfrak{A}_{SIn}$, *that is, for any* $\mathfrak{A}_{Sn}$*-measurable function* $f_1$ *and* $\mathfrak{A}_{In}$*-measurable function* $f_2$,

$$E_\theta(f_1(X)f_2(X)|\mathfrak{A}_{SIn}) = E_\theta(f_1(X)|\mathfrak{A}_{SIn})E_\theta(f_2(X)|\mathfrak{A}_{SIn}) \quad a.\ e..$$

ASSUMPTION 3. For any $\theta\varepsilon\Theta$, $\{c_n(\theta;.)\}$ is $\{\mathfrak{A}_{Sn}\}$-measurable.

Now we have the following result which corresponds to Theorem 10.2 in Bahadur [1].

THEOREM 1. *If Assumptions 1 through 3 hold, then the set of invariant rules* *with* $\{\mathfrak{A}_{SIn}\}$*-measurable stopping rule and* $\{\mathfrak{A}_{Sn}\}$*-measurable terminal decision rule is essentially complete in the class of all invariant rules.*

PROOF. Let $\lambda=(\{\phi_n\}, \{\delta_n\})$ be any invariant sequential decision rule. By Lemma 3 we can suppose that $\{\delta_n\}$ is $\{\mathfrak{A}_{Sn}\}$-measurable. Since $\{\phi_n\}$ is $\{\mathfrak{A}_{In}\}$-measurable, it follows from Lemmas 2 and 4 that there exists a $\{\mathfrak{A}_{SIn}\}$-measurable stopping rule $\{\phi_n^*\}$ such that for each $n$ and every $\theta\varepsilon\Theta$

$$E_\theta\{\psi_n^*(X)|\mathfrak{A}_{SIn}\}=E_\theta\{\psi_n(X)|\mathfrak{A}_{SIn}\} \quad \text{a. e.} \tag{4.2}$$

where $\psi_n^*$ and $\psi_n$ are defined by (2.2). Letting $\lambda^*=(\{\phi_n^*\}, \{\delta_n\})$, $\lambda^*$ is invariant under $G$ and its stopping rule is $\{\mathfrak{A}_{SIn}\}$-measurable.

Now we show that the risk function of $\lambda^*$ is equal to that of $\lambda$. Since $\{\delta_n\}$ is $\{\mathfrak{A}_{Sn}\}$-measurable and by Assumption 3 $\{c_n(\theta;.)\}$ is $\{\mathfrak{A}_{Sn}\}$-measurable for every $\theta\varepsilon\Theta$, it follows from Lemma 5 and (4.2) that

$$\begin{aligned}R(\theta, \lambda)&=\sum_{n=1}^{\infty}E_\theta\{\psi_n(X)[\textstyle\int L(\theta, s)\delta_n(ds|X)+c_n(\theta;X)]\}\\&=\sum_{n=1}^{\infty}E_\theta\{\psi_n(X)|\mathfrak{A}_{SIn}\}E_\theta\{\textstyle\int L(\theta, s)\delta_n(ds|X)+c_n(\theta;X)|\mathfrak{A}_{SIn}\}\\&=\sum_{n=1}^{\infty}E_\theta\{\psi_n^*(X)[\textstyle\int L(\theta, s)\delta_n(ds|X)+c_n(\theta;X)]\}\\&=R(\theta, \lambda^*),\end{aligned}$$

which completes the proof.

REMARK 1. In sequential testing problems, Assumptions 1 and 2 can be replaced by the assumption that $\{\mathfrak{A}_{SIn}\}$ is a sufficient and transitive sequence for $\{\mathfrak{A}_{In}\}$ and the proof of Theorem 1 becomes more easy. But in sequential estimation problems, we need Assumptions 1 and 2 (cf. Theorem 5.4.5 in Nabeya [4], p. 192).

## 5. Nonrandomized terminal decision rule

In this section we consider the essential completeness of the set of rules with nonrandomized terminal decision rules (cf. Theorem 4 in Ferguson [2], p. 329).

ASSUMPTION 4. D is a convex subset of p-dimensional Euclidean space, $L(\theta, d)$ is a convex function of $d$ for each $\theta \varepsilon \Theta$ and $L(\theta, d) \to \infty$ as $||d|| \to \infty$ where $||d||^2 = d'd$.

ASSUMPTION 5. The trasformation $\tilde{g}$ on $D$ corresponding to $g \varepsilon G$ is a linear transformation such that $\tilde{g}d = Bd + c$ where $B$ is a $p \times p$ nonsingular matrix and $c$ is a p-dimensional vector.

We replace Assumption 1 by the following assumption.

ASSUMPTION 6. There exists a nonrandomized $\{\mathfrak{A}_{Sn}\}$-measurable terminal decision rule which is invariant under $G$.

THEOREM 2. *If Assumptions 2 through 6 hold, then the set of invariant rules with $\{\mathfrak{A}_{SIn}\}$-measurable stopping rule and nonrandomized $\{\mathfrak{A}_{Sn}\}$-measurable terminal decision rule is essentially complete in the class of all invariant sequential decision rules.*

ROOF. Let $\lambda = (\{\phi_n\}, \{\delta_n\})$ be any invariant rule. By Theorem 1 we can suppose that $\{\phi_n\}$ is $\{\mathfrak{A}_{SIn}\}$-measurable and $\{\delta_n\}$ is $\{\mathfrak{A}_{Sn}\}$-measurable. Define a nonrandomized terminal decision rule $\{\delta_n^*\}$ by

$$(5.1) \qquad \begin{aligned} \delta_n^*(x) &= \int s\delta_n(ds|x) && \text{if } \int ||s|| \delta_n(ds|x) < \infty, \\ &= \delta_n^0(x) && \text{otherwise,} \end{aligned}$$

where $\{\delta_n^0\}$ is the nonrandomized $\{\mathfrak{A}_{Sn}\}$-measurable terminal decision rule which is invariant under $G$ by Assumption 6. Clearly $\{\delta_n^*\}$ is $\{\mathfrak{A}_{Sn}\}$-measurable. Since $\{\delta_n\}$ is invariant under $G$, Assumption 5 implies that for any $g \varepsilon G$

$$(5.2) \qquad \begin{aligned} \int s\delta_n(ds|gx) &= \int \tilde{g}s\delta_n(\tilde{g}ds|gx) \\ &= \tilde{g}\int s\delta_n(ds|x), \end{aligned}$$

so that

$$\int ||s|| \delta_n(ds|gx) < \infty \text{ if and only if } \int ||s|| \delta_n(ds|x) < \infty.$$

Hence it follows from (5.1) and (5.2) that for every $x \varepsilon \mathfrak{X}$ and every $g \varepsilon G$

$$\delta_n^*(gx)=\tilde{g}\delta_n^*(x).$$

Therefore $\lambda^*=(\{\phi_n\},\ \{\delta_n^*\})$ is invariant under $G$.

Now we show that the risk function of $\lambda^*$ is not greater than that of $\lambda$. We suppose $R(\theta,\ \lambda)<\infty$. Then it follows from (2.1) that for each $n$

$$E_\theta\{\psi_n(X)\int L(\theta,\ s)\delta_n(ds|X)\}<\infty. \tag{5.3}$$

Assumption 4 implies that if $\int L(\theta,\ s)\delta_n(ds|x)<\infty$, then $\int||s||\delta_n(ds|x)<\infty$ (see Remark in Ferguson [2], p. 78), so that by Jensen's inequality

$$L(\theta,\ \delta_n^*(x))\leq\int L(\theta,\ s)\delta_n(ds|x).$$

Therefore from (5.3) we have that for each $n$

$$E_\theta\{\psi_n(X)L(\theta,\ \delta_*^n(X))\}\leq E_\theta\{\psi_n(X)\int L(\theta,\ s)\delta_n(ds|X)\}.$$

Hence it follows from (2.1) that

$$R(\theta,\ \lambda^*)\leq R(\theta,\ \lambda),$$

which completes the proof.

EXAMPLE 1. Let $X_1,\ X_2,\dots$ be a sample from the normal distribution with mean $\mu$ and variance $\sigma^2$, where $\theta=(\mu,\ \sigma^2)$ is unknown. We want to estimate $\mu$ sequentially under the loss function $L(\theta,\ d)=(d-\mu)^2$ and constant cost $c$ per observation.

Clearly this problem is invariant under the group $G$ of translations,

$$g(x_1,\ x_2,\dots)=(x_1+a,\ x_2+a,\dots),\ g\in G.$$

For each $n$ let

$$\bar{X}_n=\frac{1}{n}\sum_{i=1}^n X_i, \qquad S_n=\sum_{i=1}^n(X_i-\bar{X}_n)^2$$

and $\mathfrak{A}_{Sn}$ be a subfield generated by $\bar{X}_n$ and $S_n$. Then it follows from Lemma 1 that $\{\mathfrak{A}_{Sn}\}$ is a sufficient and transitive sequence for $\{\mathfrak{A}_n\}$. In this case $\mathfrak{A}_{SIn}$ becomes the subfield generated by $S_n$. It is easy to see that assumptions in Theorem 2 are satisfied (for Assumption 2, use Theorem 7.1 in Hall, et al. [4])

and the nonrandomized estimator of $\mu$ based on $\bar{X}_n$ and $S_n$ is given by

$$\delta_n = \bar{X}_n + h_n(S_n) \tag{5.4}$$

with some function $h_n$.

Let $\lambda = (\{\phi_n\}, \{\delta_n\})$ be any invariant sequential decision rule with $\{\mathfrak{A}_{SIn}\}$-measurable stopping rule and $\{\mathfrak{A}_{Sn}\}$-measurable terminal decision rule. By (5.4) we have that

$$\begin{aligned} R(\theta, \lambda) &= \sum_{n=1}^{\infty} E_\theta\{\psi_n(X)[(\delta_n(X)-\mu)^2+cn]\} \\ &= \sum_{n=1}^{\infty} E_\theta\left\{\psi_n(X)\left[\frac{\sigma^2}{n}+h_n^2(S_n)+cn\right]\right\} \\ &\geq R(\theta, \lambda^*), \end{aligned}$$

where $\lambda^* = (\{\phi_n\}, \{\bar{X}_n\})$, because $\bar{X}_n$ is independent of $(S_1, \ldots, S_n)$ (see Lemma 10.9.3 in Zacks [6]). Hence for this problem the set of rules, in which the stopping rule is determined by $S_n$ at the n-th observation and the terminal decision rule is the sample mean, is essentially complete in the class of all invariant sequential decision rules.

EXAMPLE 2. We consider a life test with replacement on $M$ machines. For failure times we assume an exponential distribution with the probability density function

$$f(x) = \theta^{-1} exp\,(-x/\theta),\; x>0,\; \theta>0.$$

Let $X_n$ be the time at which the n-th failure would occur if the life test were allowed to operate indefinitely. We want to estimate $\theta$ sequentially under the loss function $L(\theta, d) = (d-\theta)^2/\theta^2$ and the cost function proportional to the test time, i. e. $cx_n/\theta$ if we stop the life test at the n-th failure and $x_n$ is the failure time ($c$ is some positive constant).

It is easy to see that the joint density function of $X_1, \ldots, X_n$ is given by

$$f(x_1, \ldots, x_n) = (M/\theta)^n exp(-Mx_n/\theta),\; 0<x_1<\ldots<x_n,$$

so that this problem is invariant under the group $G$ of scale transformations,

$$g(x_1, x_2, \ldots) = (ax_1, ax_2, \ldots),\; g\varepsilon G,\; a>0.$$

and $X_n$ is a sufficient statistic for $X_1, \dots, X_n$. For each $n$ let $\mathfrak{A}_{Sn}$ be the subfield generated by $X_n$. Since for each $n$ $X_1, X_2-X_1, \dots, X_n-X_{n-1}$ are mutually independent, it follows from Lemma 1 that $\{\mathfrak{A}_{Sn}\}$ is a sufficient and transitive sequence for $\{\mathfrak{A}_n\}$. In this case every $\{\mathfrak{A}_{SIn}\}$-measurable stopping rules do not depend on failure times. It is easy to see that assumptions in Theorem 2 are satisfied (for Assumption 2, use Theorem 7.1 in Hall, et al. [4]) and the nonrandomized estimator of $\theta$ based on $X_n$ is given by

$$\delta_n = b_n X_n \tag{5.5}$$

where $b_n$ is constant.

Let $\lambda=(\{\phi_n\}, \{\delta_n\})$ be any invariant sequential decision tule with $\{\mathfrak{A}_{SIn}\}$-measurable stopping rule and $\{\mathfrak{A}_{Sn}\}$-measurable terminal decision rule. Then it follows from (5.5) that

$$\begin{aligned} R(\theta, \lambda) &= \sum_{n=1}^{\infty} E_\theta\{\psi_n(X)[(\delta_n(X)-\theta)^2/\theta^2 + cX_n/\theta]\} \\ &= \sum_{n=1}^{\infty} \psi_n \left[\frac{n(n+1)}{M^2}\left(b_n - \frac{M}{n+1}\right)^2 + \frac{1}{n+1} + \frac{cn}{M}\right] \end{aligned}$$

because $\psi_n$ is constant and $2MX_n/\theta$ has a chi-square distribution with $2n$ degrees of freedom. Hence letting $\lambda^* = \left(\{\phi_n\}, \left\{\frac{M}{n+1}X_n\right\}\right)$, we have that

$$\begin{aligned} R(\theta, \lambda) &\geq R(\theta, \lambda^*) \\ &= \sum_{n=1}^{\infty} \psi_n \left[\frac{1}{n+1} + \frac{cn}{M}\right]. \end{aligned}$$

Define

$$n^* = \text{Min } \{n \geq 1 \text{ such that } c \geq M(n+1)^{-1}(n+2)^{-1}\}.$$

Then it follows easily that the non-sequential rule which terminates the life test at the $n^*$-*th* failure time and estimates $\theta$ by $\frac{M}{n^*+1} X_{n^*}$ has the minimum risk among all invariant sequential decision rules.

REMARK 2. We [5] showed that this non-sequential rule is minimax among all sequential decision rules.

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