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ISOMETRY GROUP OF A COMPACT RIEMANNIAN MANIFOLD WITH SIMPLE SPECTRUM

Akira IKEDA

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Introduction.

Let M be a compact connected Riemannian manifold. We denote Δ the Laplace-Beltrami operator acting on the smooth functions on M. It is well known that Δ has a discrete eigenvalues with finite multiplicities. We say the spectrum of M is *simple* when each eigenvalues has multiplicity one. When the spectrum of M is simple, it seems that the group of all isometries of M, $I(M)$, is very small. In this short note we give an answer to it. We obtain the following Theorem.

THEOREM. *Let M be a compact connected Riemannian manifold with simple spectrum. Then the isometry group $I(M)$ is of Z_2-power, i. e., it is isomorphic to Z_2^k for some* $k \geqq 0$.

REMARK. The finiteness of the isometry group $I(M)$ was obtained in [2] in a different manner.

1. Preliminaries.

Here we recall some results about the spectrum of Δ for proving our Theorem. For details, see [1].

Let $M=(M, g_{ij})$ be a compact connected Riemannian manifold and $C^\infty(M)$ the space of *real* valued smooth functions on M. The Laplace-Beltrami operator acting on $C^\infty(M)$ is defined by

$$\Delta = -\frac{1}{\sqrt{g}} \sum \frac{\partial}{\partial x^i}\left(\sqrt{g}\, g^{ij} \frac{\partial}{\partial x^j}\right), \tag{1.1}$$

where $g^{ij}=(g_{ij})^{-1}$ and $g=\det(g_{ij})$.

Then Δ has a discrete eigenvalues with finite multiplicities. Let E_λ be the space of all eigenfunctions with eigenvalue λ. Then we see

(1.2) *the space* $\sum_{\lambda} E_{\lambda}$ *is dense in* $C^{\infty}(M)$ *in the sense of the uniform convergence topology.*

On the space $C^{\infty}(M)$ we introduce an inner product (,) by

$$(\phi, \psi)=\int_M \phi(x)\psi(x)\sqrt{g}\, dx \qquad \text{for } \phi, \psi \in C^{\infty}(M). \tag{1.3}$$

We denote $I(M)$ the group of all isometries of M and $I_0(M)$ its connected component containing the unit. Then the group $I(M)$ act on $C^{\infty}(M)$ as

$$(h\phi)(x)=\phi(h^{-1}x) \qquad \text{for } h \in I(M) \text{ and } \phi \in C^{\infty}(M). \tag{1.4}$$

This action induces an orthogonal transformation of $C^{\infty}(M)$ equipped with the inner product and this represantation is denoted by ρ.

Then each eigenspace E_{λ} is preserved by ρ. The restriction to E_{λ} of ρ is denoted by ρ_{λ}

$$(\rho_{\lambda}, E_{\lambda})=(\rho_{C^{\infty}(M)}, E_{\lambda}) \tag{1.5}$$

Each representation $(\rho_{\lambda}, E_{\lambda})$ is a finite real orthogonal representation. If the dimension of E_{λ} is equal to one, we have for $g \in I(M)$,

$$\rho_{\lambda}(g)=\pm 1 \qquad \text{on } E_{\lambda} \tag{1.6}$$

and

$$\rho_{\lambda}(g^2)=1 \qquad \text{on } E_{\lambda}, \tag{1.7}$$

and for $g \in I_0(M)$ we have

$$\rho_{\lambda}(g)=1 \qquad \text{on } E_{\lambda}. \tag{1.8}$$

2. Proof of Theorem.

Our proof of Theorem consists of the following successive Lemmas.

LEMMA 1. *For different points p and q on M, we can take smooth function f* *satisfying* $f(p)\neq f(q)$.

PROOF. Easy.

LEMMA 2. *Let* $g\in I(M)$. *Assune that for any eigenvalue* λ *of* Δ, $\rho_\lambda(g)=1$ *on* E_λ. *Then g is the unit element.*

PROOF. Assume that g is *not* the unit. Then there is a point p with $q=g(p)\neq p$. From our assumption any eigenfunction takes the same value at p and q. By (1.2), we see that any smooth function takes the same value at p and q. This contradicts to Lemma 1. Q. E. D,

From now on we assume that the spectrum of M is *simple*.

LEMMA 3. $I(M)$ *is a finite group.*

PROOF. Together (1.8) and Lemma 2, we see that $I_0(M)$ consists of only the unit element. Since $I(M)$ is compact, $I(M)$ is a finite group. Q. E. D.

LEMMA 4. *Any non-unite element of* $I(M)$ *has order* 2.

PROOF. This follows from (1.7) and Lemma 2. Q. E. D.

LEMMA 5. *Let G be a finite group. Assume that each non-unit elememt has order* 2. *Then G is abelian and of* Z_2*-power.*

PROOF. Let e denote the unit element of G. For elements a and b of H, we have $a=a^{-1}$ and $b=b^{-1}$. Then $aba^{-1}b^{-1}=abab=(ab)^2=e$. Hence G is abelian. From the fundamental Theorem of abelian group, we can see easily that G is of Z_2-*power*. Q. E. D.

Thus our proof of Theorem is completed.

References

[1] Berger, M, Gaudachon, P and E. Mazet, Le spectre d'une variétè riemannienne, Lecture Notes in Math., 194 Springer-Verlag, Berlin-Heidelberg New York, 1971.

[2] Bando, S and H. Urakawa, Generic properties of the eigenvalue of the Laplacian for compact Riemannian manifolds, Tohoku Math. Journ. **35**(1983), 155-172.

Department of Mathematics
Faculty of General Education
Kumamoto University