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EIGENFUNCTIONS OF THE 3-DIMENSIONAL LAPLACIAN DISPLAYED BY COMPUTER GRAPHICS

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1. Introduction

Using a supercomputer, we calculated eigenfunctions corresponding to the smallest 50 eigenvalues of the 3-dimensional Laplacian in several types of domains. The perspective views of the level surfaces of the eigenfunctions are shown in this paper. Especially nodal surfaces of the eigenfunctions are among them.

Eigenvalue problems play important roles in many fields of mathematics, physics, etc. (e. g., [1]). From mathematical point of view, relations between the geometric characteristics of a region and the asymptotic behavior of the eigenvalues of the Laplacian in the region attract mathematicians for many years (e. g., [6]), and there still remain unsolved problems. With the development of large and high-speed computers, we can now study graphs of 2- and 3-dimensional eigenfunctions directly with the aid of 3-dimensional graphics (see also [9]).

Calculations of the eigenfunctions were done at the large computer center of Tokyo University using the S810 model 20 vector processor, which has the maximum speed of 680 MFLOPS (Mega Floating Operations Per Second) and the maximum 18 MB (Mega Bytes) user's main memory area.

To examine the results we need a 3-dimensional graphic display, which can move surfaces on the screen smoothly. To this aim, a 3-dimensional random-scan type graphic display at the large computer center of Kyoto University was used, and its pictures were recorded by a video tape recorder. The best way to show our results is to play the video tape on a TV screen, but we should be satisfied with some perspective pictures in this paper. Pictures in this paper were printed out by the laser printers at large computer centers of Kyoto and Kyushu University.

Thanks are due to the large computer center of Tokyo University, under whose financial support most of this work was done.

2. Method of the calculation of the eigenfunctions

The eigenvalue problem for the following 3-dimensional Laplacian is considered in a 3-space region D with the boundary condition $u=0$ on the boundary of D.

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}+\lambda u=0$$

By the 5 point finite difference approximation, this eigenvalue problem is discretized. We assume that the region D is defined by a function φ in such a way that a cube is contained in D if and only if φ is negative at an edge of the cube. Then the corresponding matrix becomes symmetric and positive definite.

For example, if D is a cube and is divided into $30\times 30\times 30$ small cubes, then the order of the matrix equation becomes 24,389. Hence we cannot apply the usual tridiagonalization technique, and we should use an iterative method for a large sparse matrix. We used two steps to calculate eigenvectors in our case.

The first step is to divide the region D very roughly, for example, in $12\times 12\times 12$ for a cubic D, so that the resultant matrix has size of less than 1900. Then we can apply the usual tridiagonalization, the bisection method, and the inverse iteration method (e. g., [2], [8]). The obtained eigenvalues and the corresponding eigenvectors are used to calculate initial values for the iteration process of the next step.

The second step is to apply the Jennings's method [4] to the original matrix of the order of about 20,000. The initial vectors for the iterations are calculated by the interpolation of the vectors obtained by the first step. An iteration for the smallest 50 eigenvalues and the corresponding eigenvectors requires the solutions of 50 sparse matrix equations of the order of about 20,000. To do this we use the MICCG method ([5, 3]) with the modification due to Y. Ushiro [7] such that the algorism is vectorizable.

Calculations were done for the following three types of regions D.

(Type 1) The region D is the cube $-1<x, y, z<1$, and is equally divided into $28\times 28\times 28$. Then the number of the interior points of D is 19,683. By 90 iterations the maximum error of the eigenvectors is 3.69×10^{-8}. The used time for the iteration is 1,259 seconds.

(Type 2) The region $-1<x, y, z<1$ is equally divided into $40\times 36\times 36$ small parallelepiped, and D is the set of all small parallelepiped such that at some edge point (x, y, z) the value $2x^2+y^2+z^2-1$ is negative. Then the number of the interior points is 19,167. By 16 iterations the maximum error of the eigenvectors is 2.94×10^{-5}. The used time for the iteration is 320 seconds.

(Type 3) The region $-1<x, y, z<1$ is equally divided into $34\times34\times34$ small cubes, and D is the set of all small cubes such that at some edge point (x, y, z) the value $(x^2+y^2+z^2-1)(x^2+y^2+z^2-0.1)$ is negative. Then the number of the interior points is 19,838. By 24 iterations the maximum error of the eigenvectors is 9.12×10^{-5}. The used time for the iteration is 450 seconds.

3. Three dimensional graphic techniques to analyse data

The size of the output data for the eigenfunctions is about 5 to 10 MB in each case, and some graphic tool to analyse them is necessary. The level surface of an eigenfunction $u(x, y, z)$ is defined by $u=$constant f. To display such a surface, we used two classes of curves. The first class is $u(x, y, z)=f$ for a fixed x, which is on a plane parallel to the $y-z$ plane. The second class is $u(x, y, z)=f$ for a fixed z, which is on a plane parallel to the $x-y$ plane. To calculate these curves we applied the usual searching method of contour curves with a slight modification. Especially if f is zero, i. e. the case of nodal surfaces, some problems arise.

Since $f=0$ on the boundary and outside it, we have to cut them. In the following pictures the algorism of this process is not good enough. The second problem is to search the singular part of the surface, which is not given a priori. The third problem is the hidden line algorism near the singular part. We should invent an algorism to solve these problems.

We can use the value of $u(x, y, z)$ at any point (x, y, z) calculated by interpolation to the hidden line process. The only thing to check is whether the sign of $u-f$ changes on the segment from (x, y, z) to the eye. This algorism is very fast. In fact the used time to draw the following pictures is about 2 to 10 seconds by FACOM M382, a standard 20 MIPS large computer.

In the following figures, three perspectives of the level surfaces corresponding to $f=-0.005$, 0, 0.005 are shown in this order for every eigenvalue, which is written below each picture as $\lambda=\ldots$.

Figures 1.1−1.24 is for type 1. Figures 2.1−2.24 is for type 2. Figures 3.1−3.48 is for type 3.

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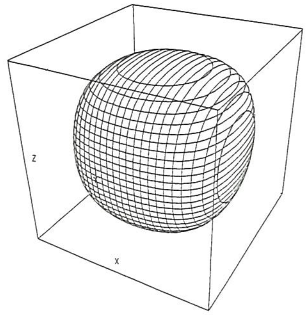


Figure 1.1. $\lambda = 7.39$

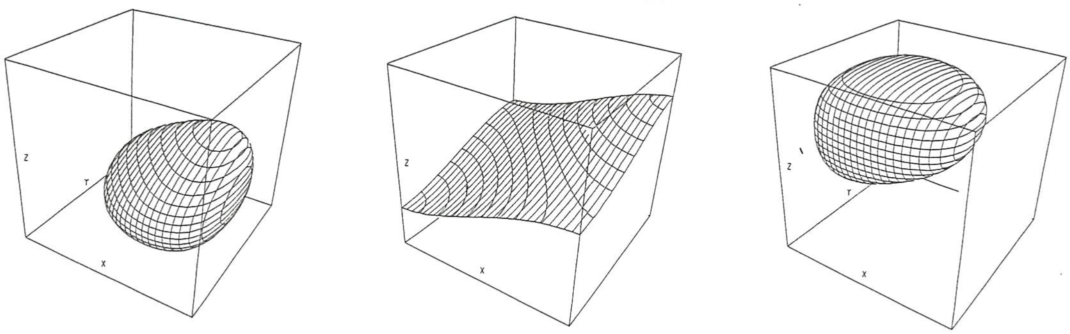
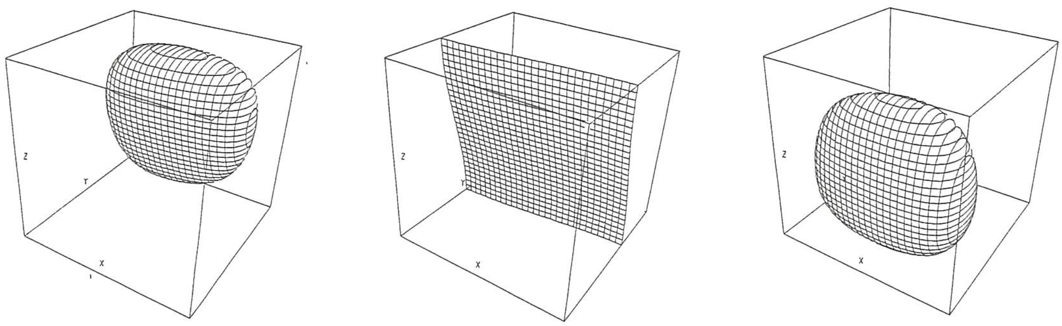


Figure 1.2. $\lambda = 14.7$



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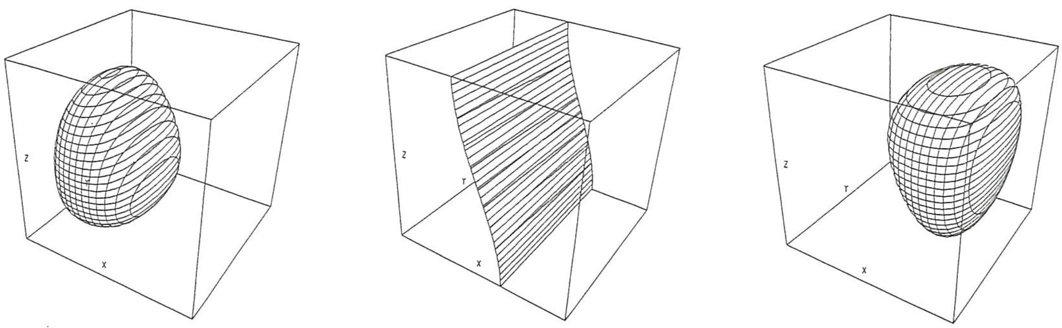

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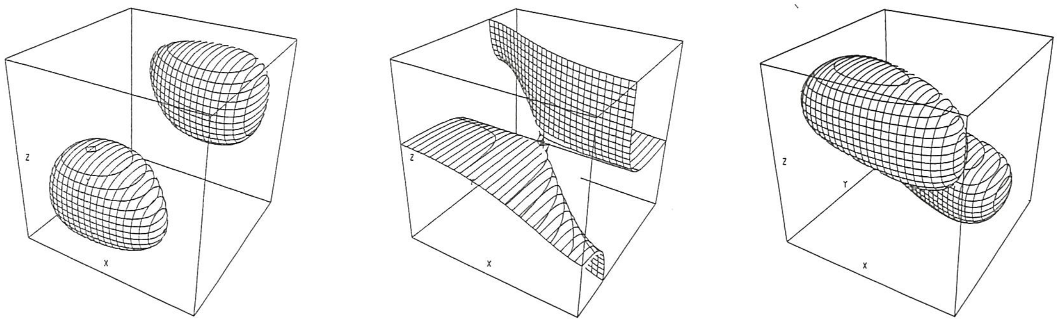


Figure 1.5. $\lambda = 22.1$

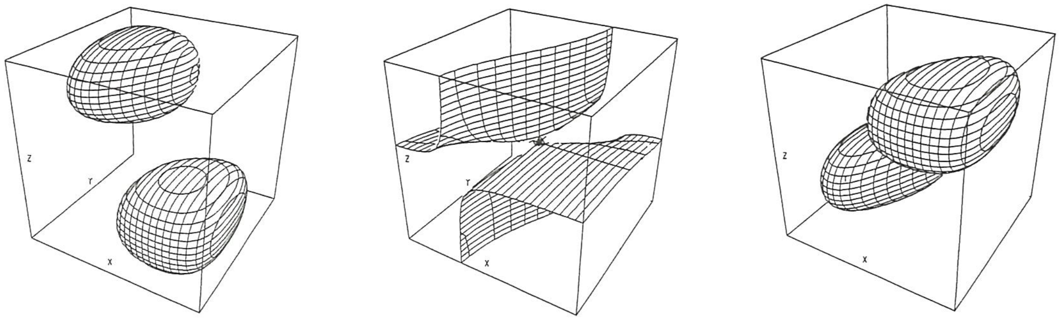
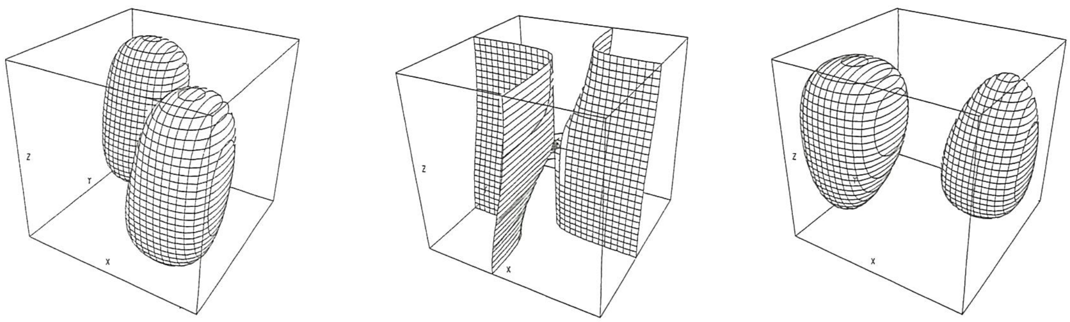


Figure 1.6. $\lambda = 22.1$



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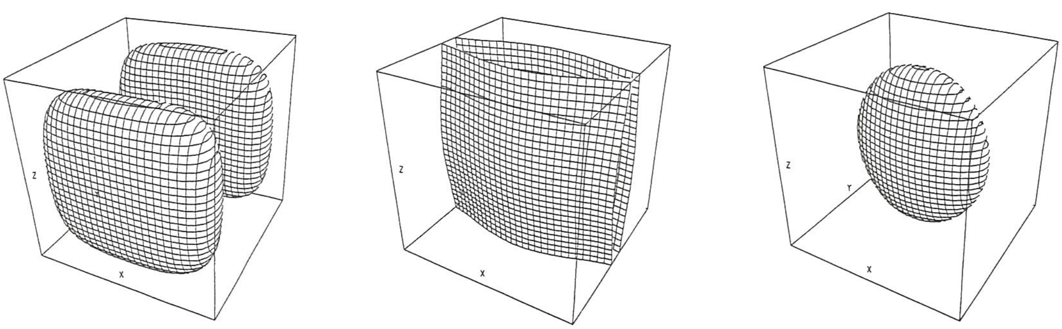

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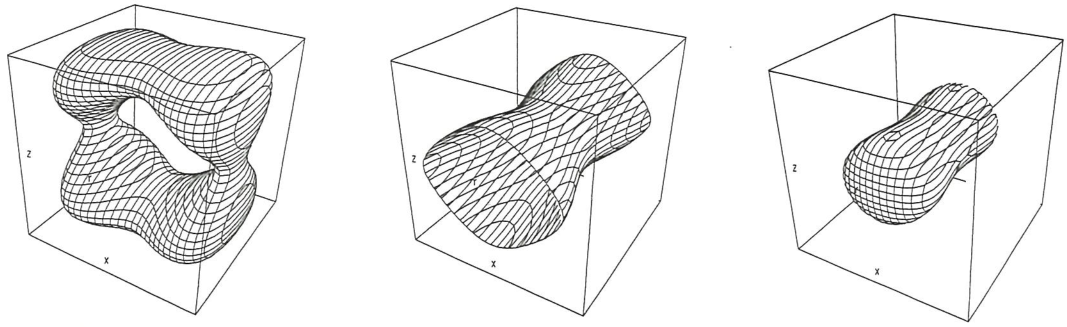

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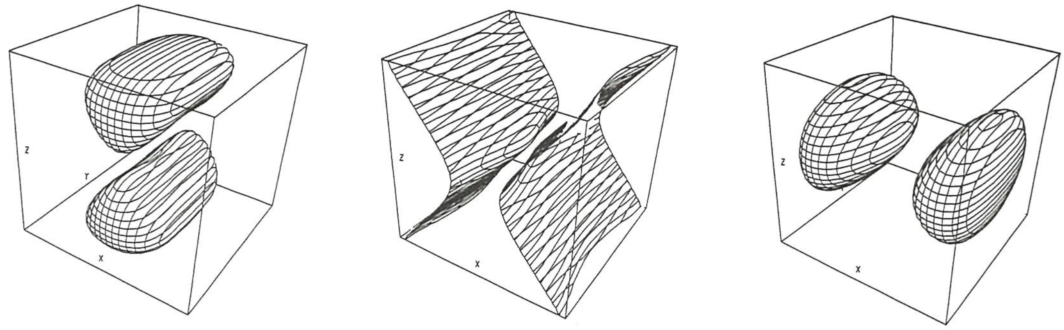

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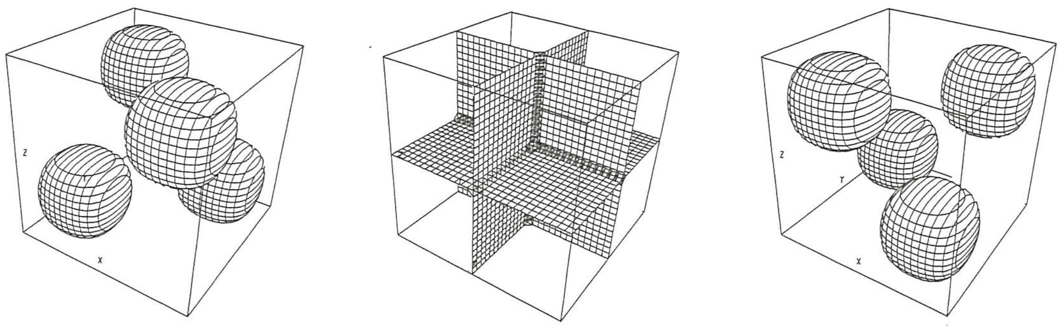

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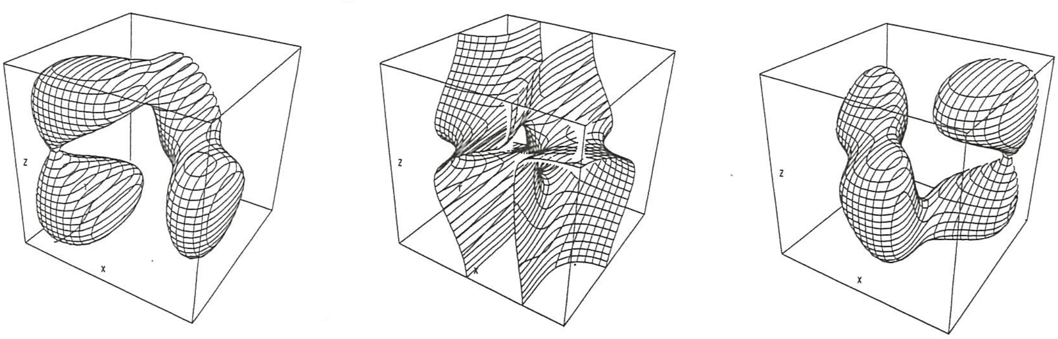

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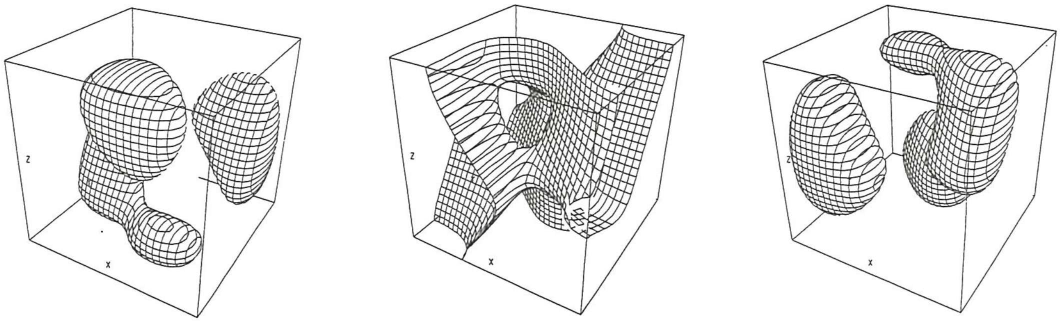

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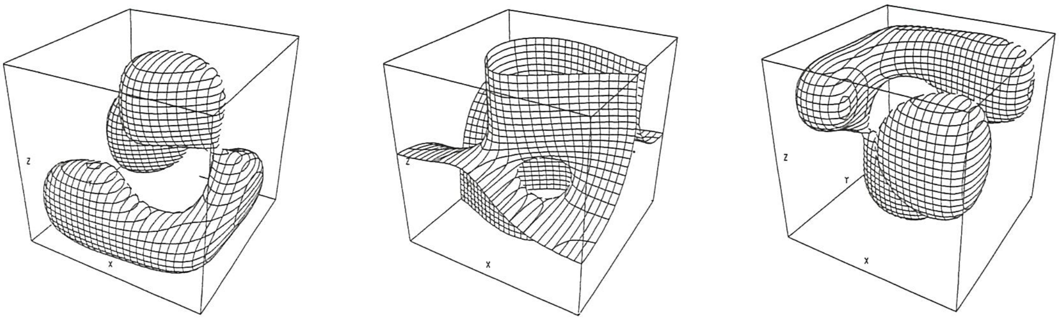

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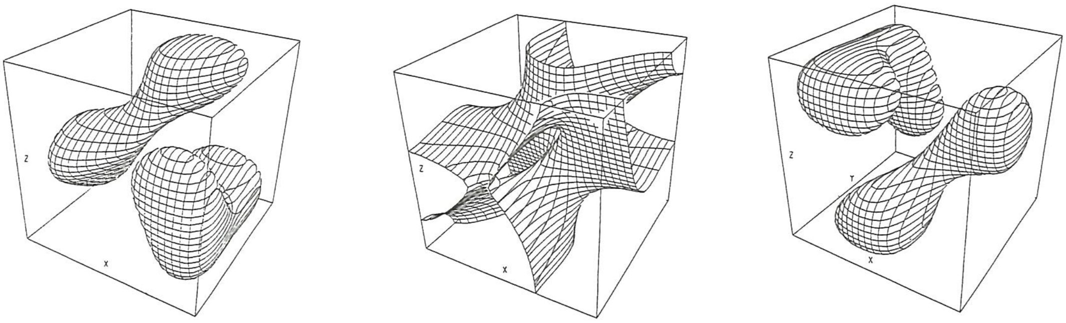

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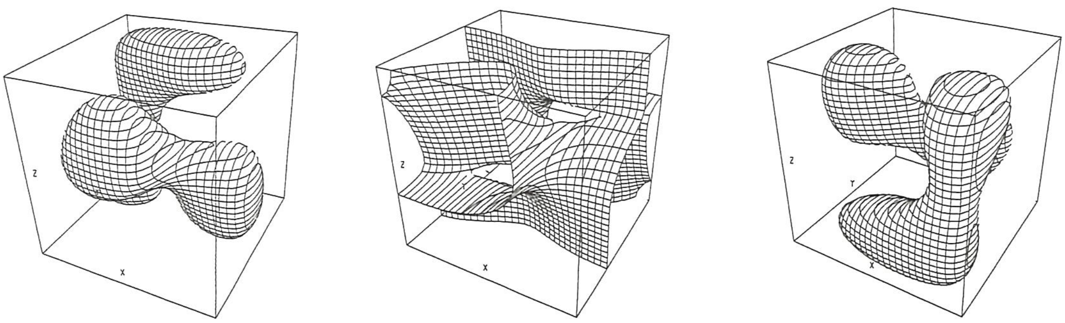

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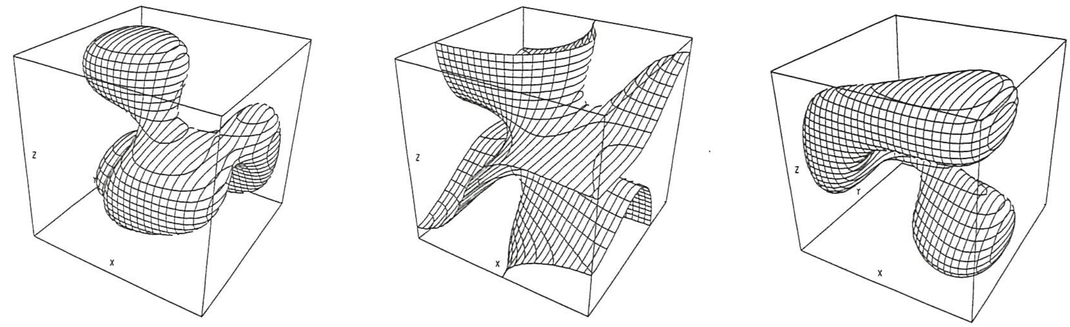

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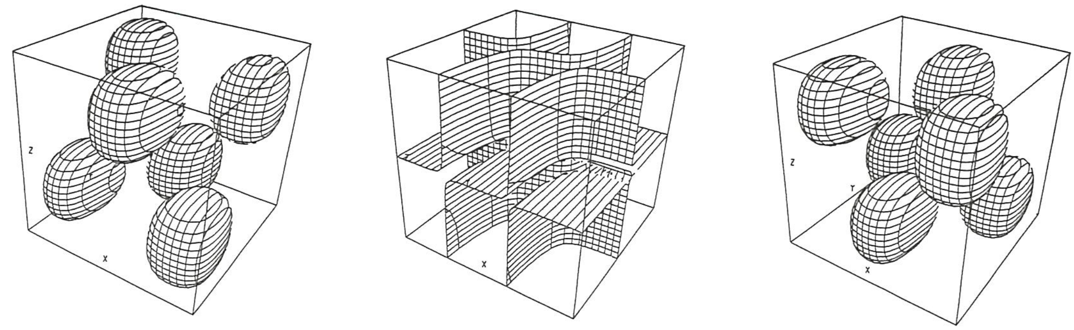

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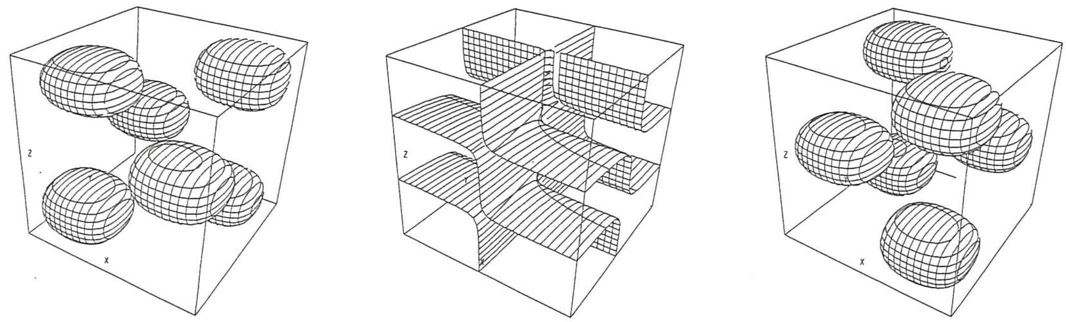

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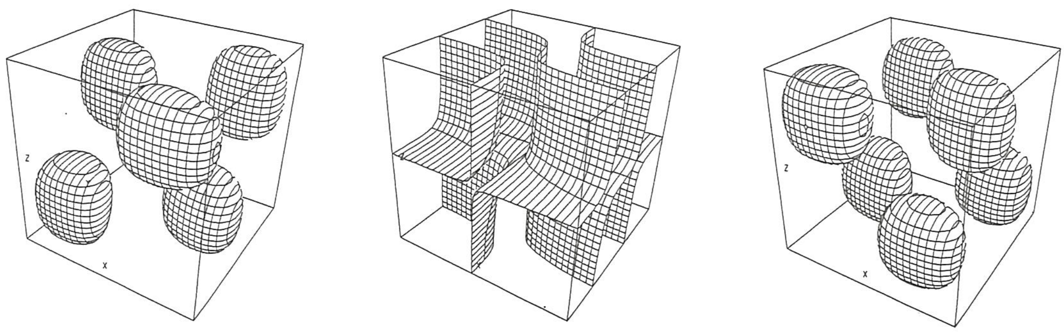

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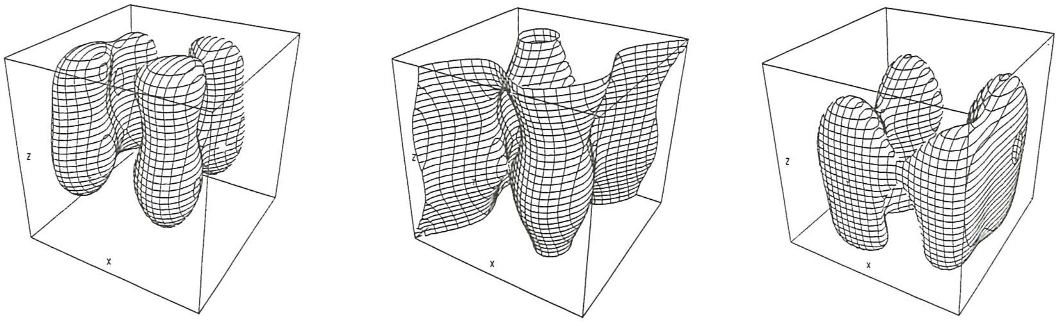

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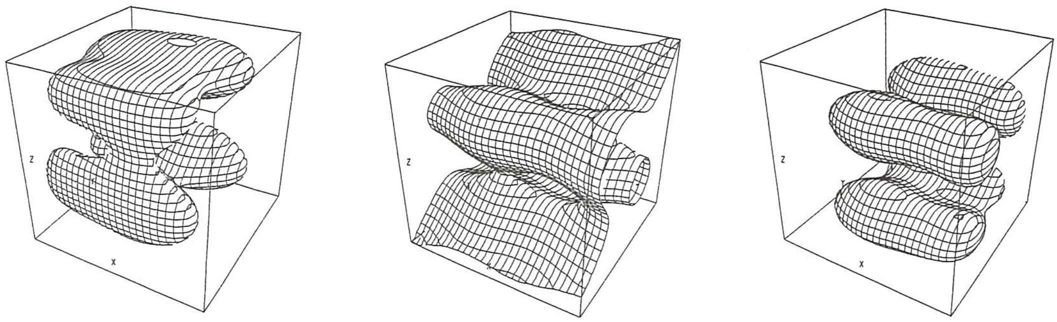

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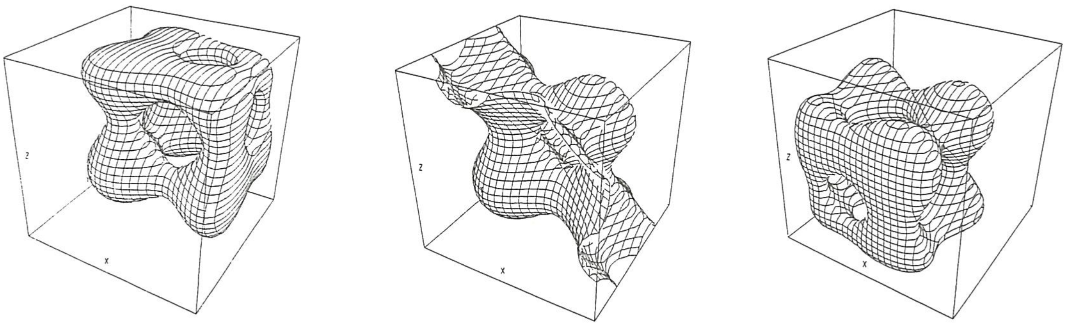

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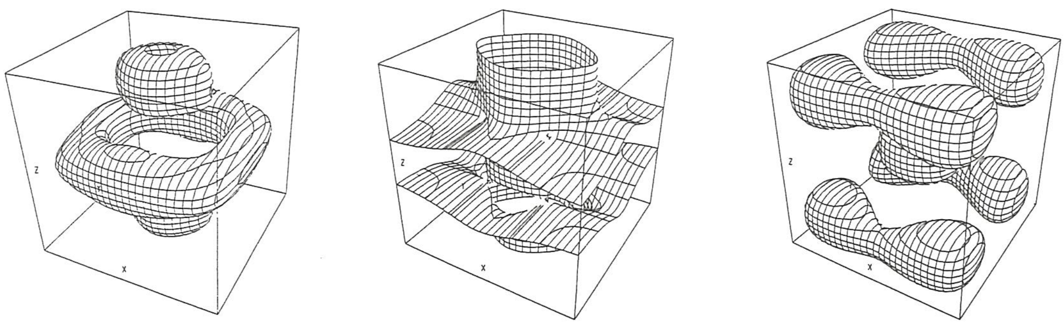

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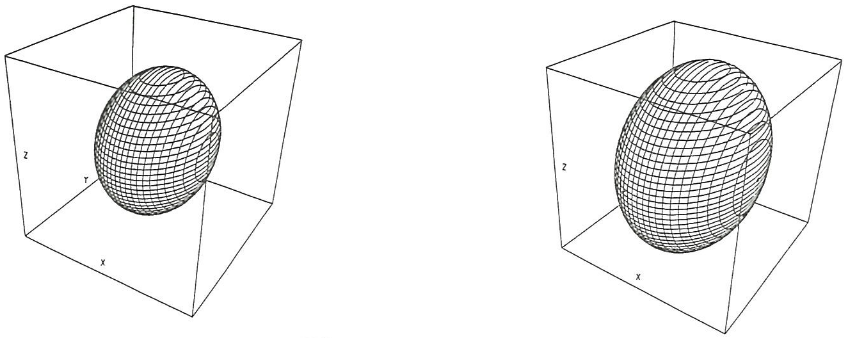

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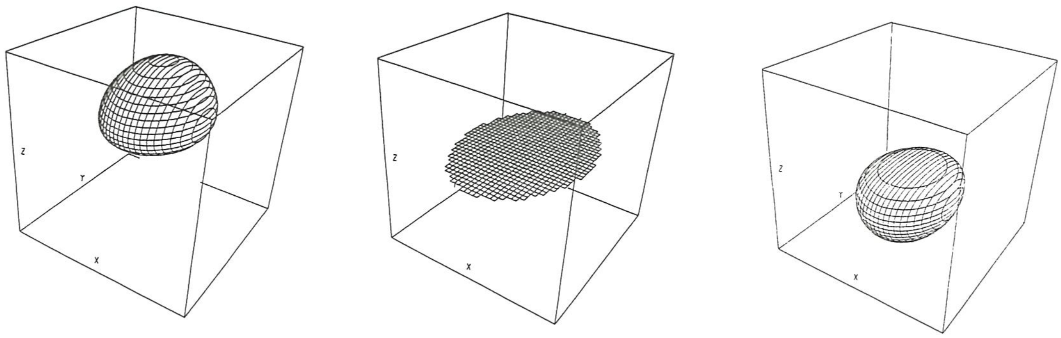

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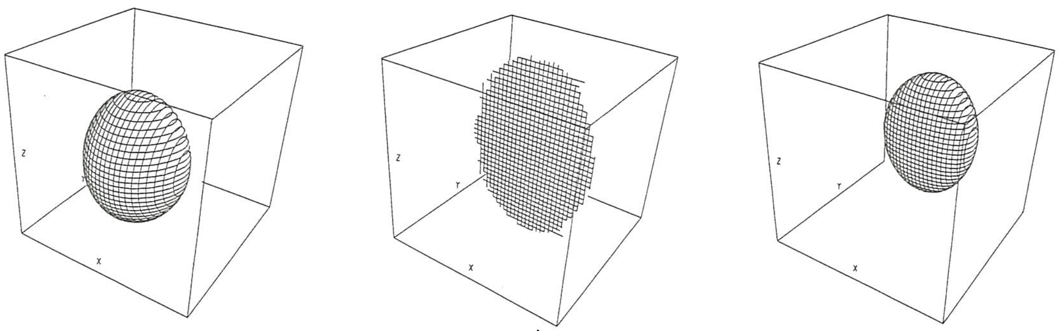

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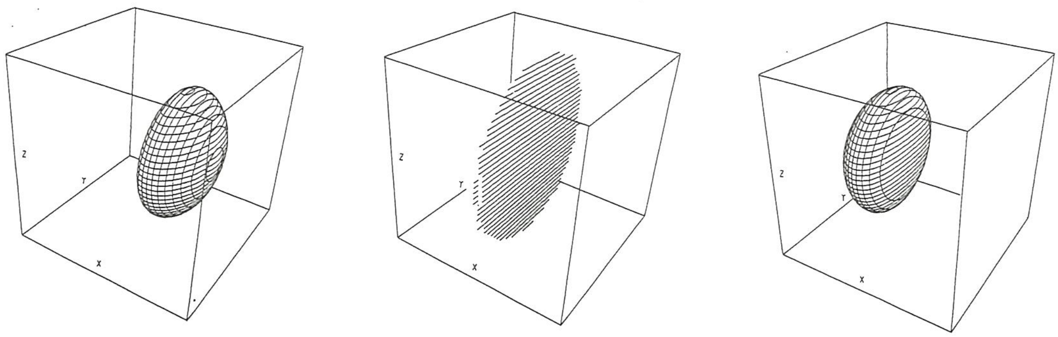

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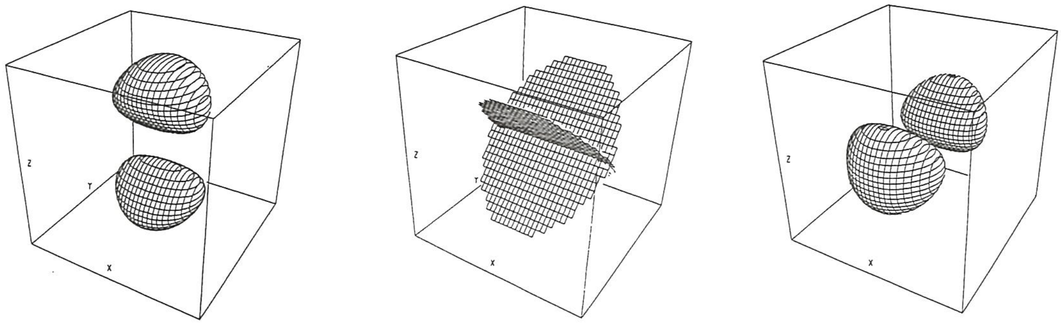

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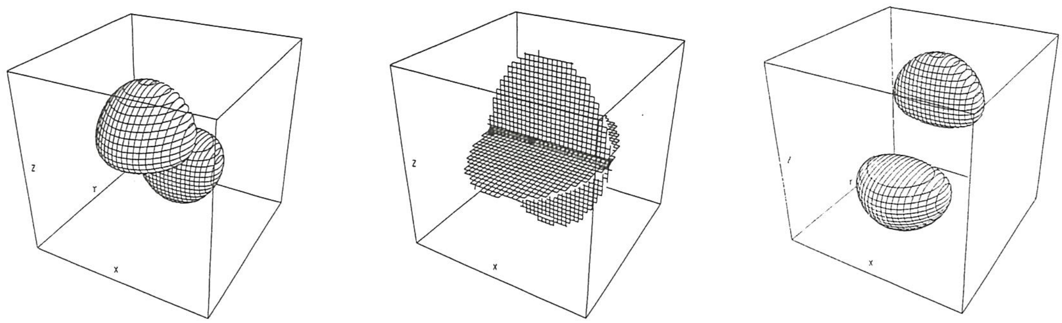

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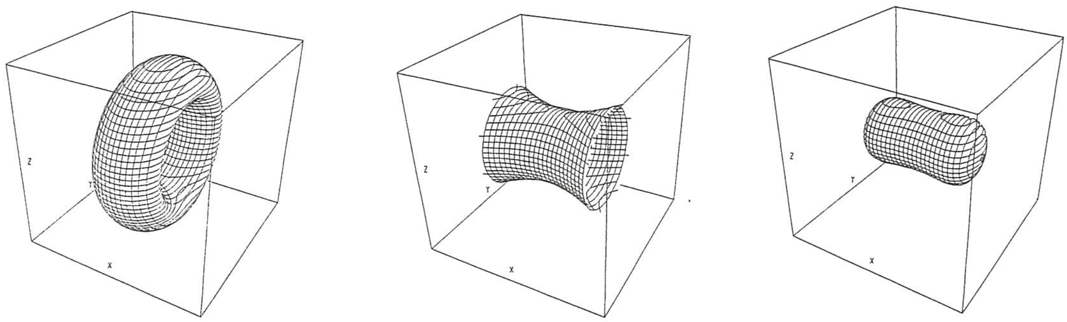

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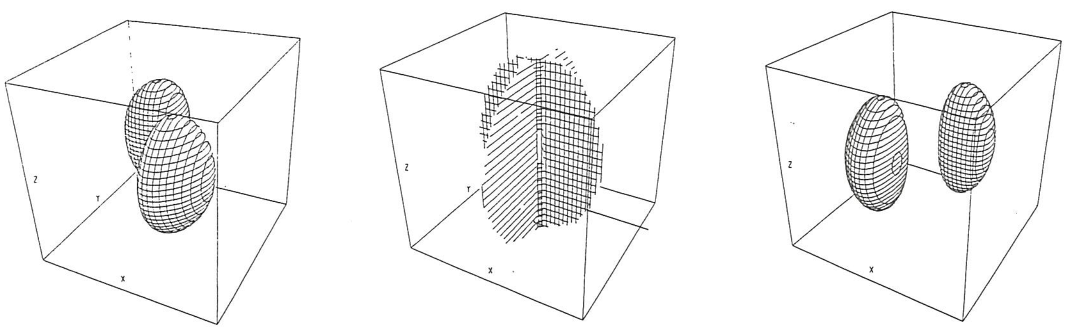

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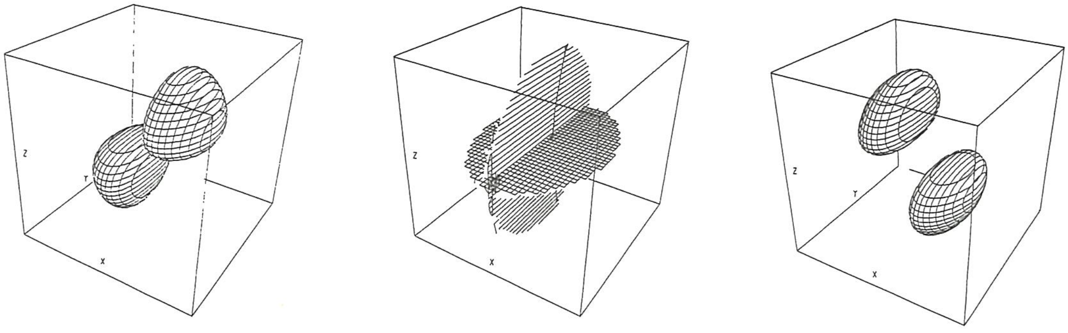

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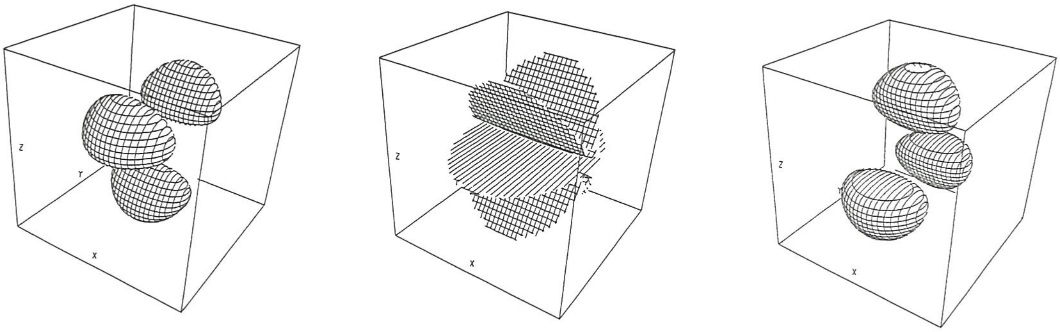

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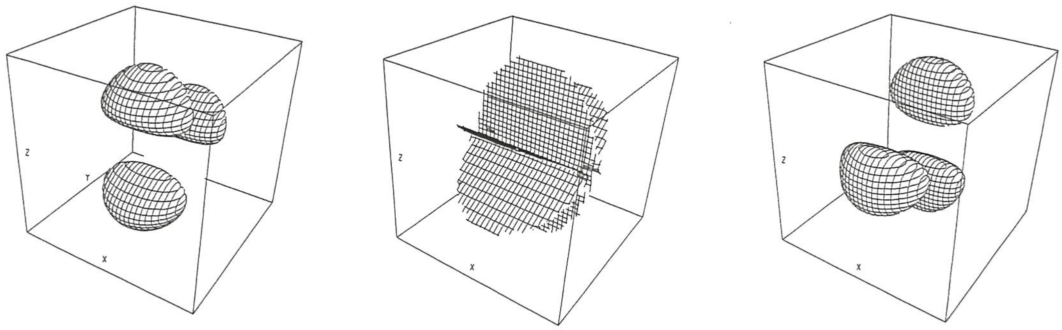

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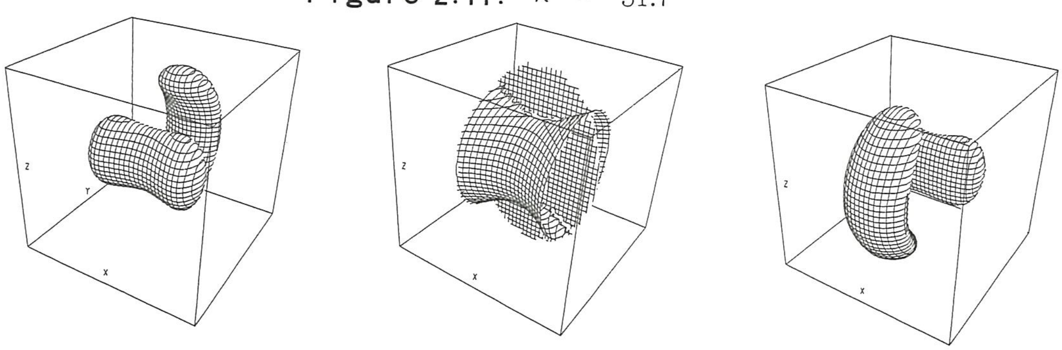

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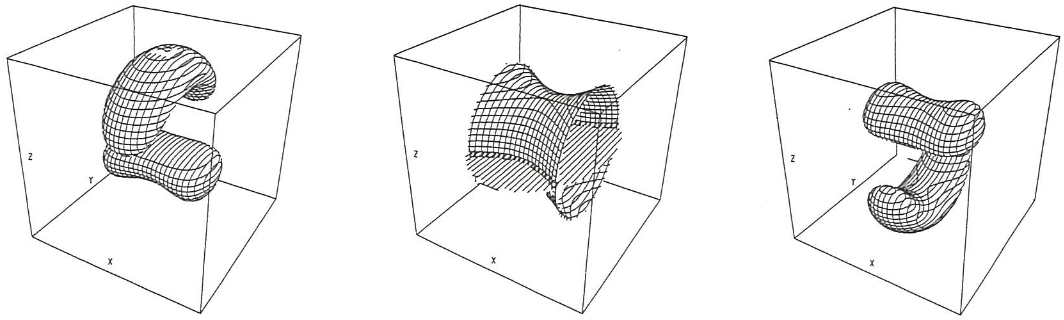

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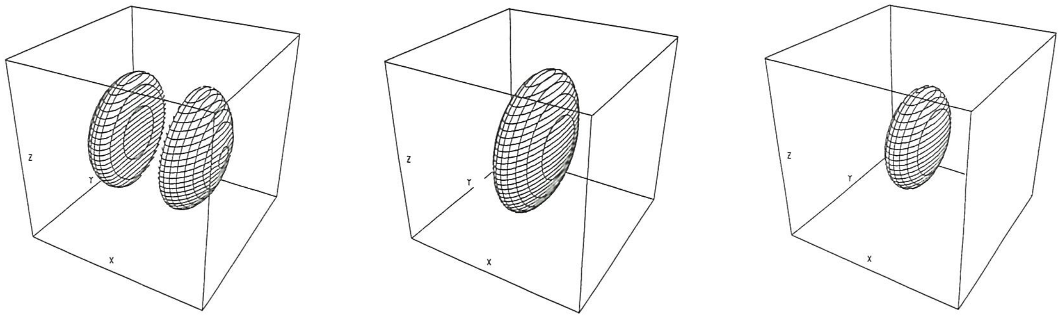

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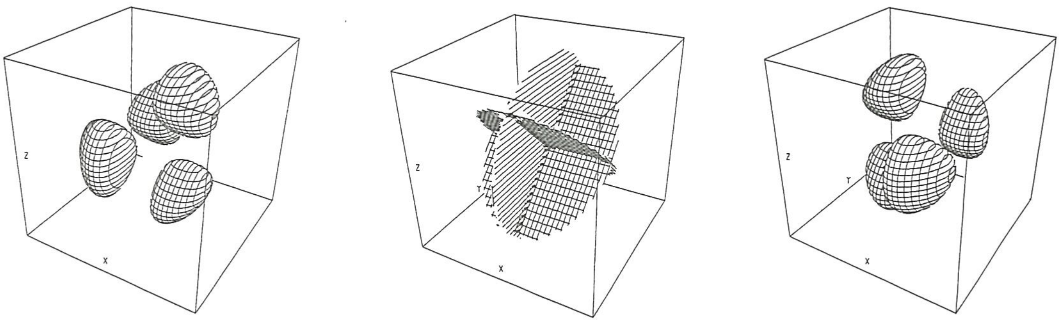
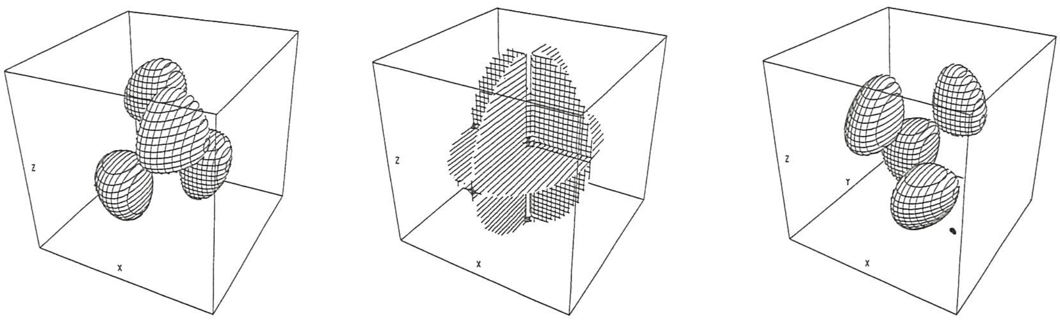


Figure 2.15. λ = 61.4



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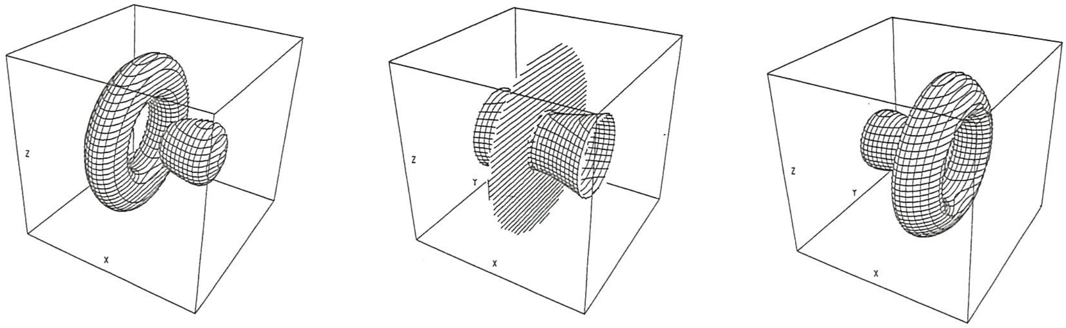

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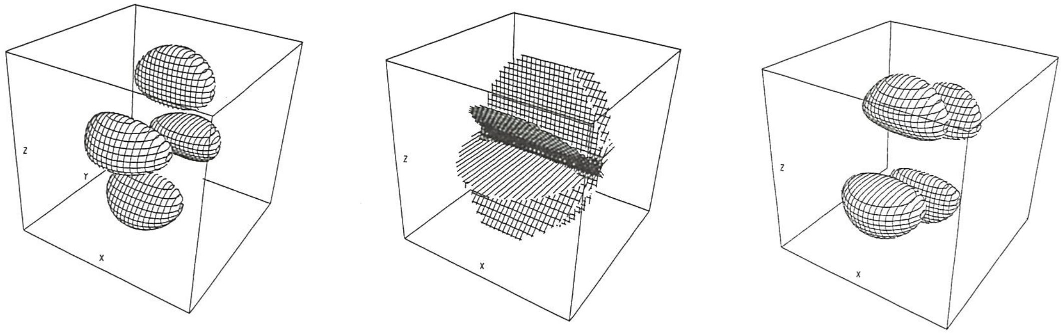

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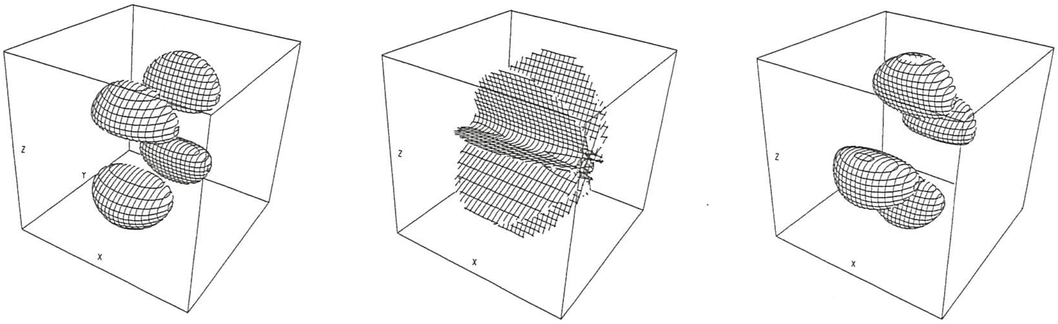

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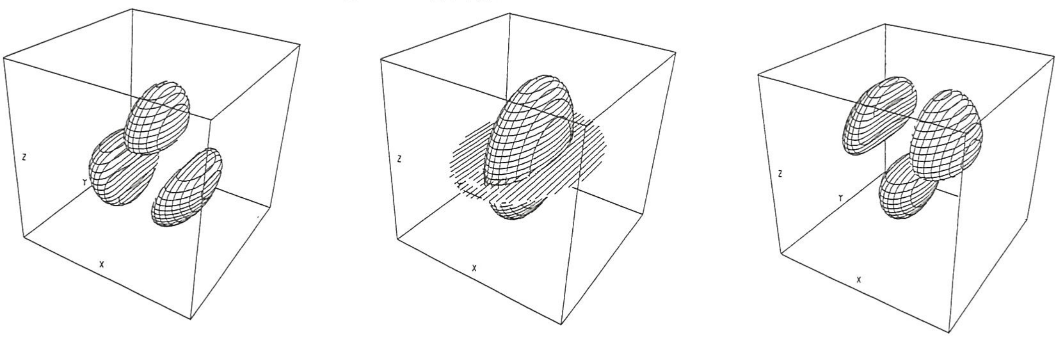

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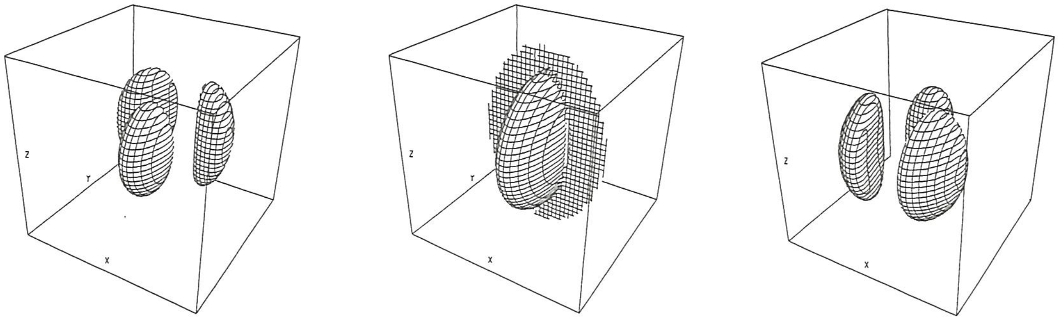

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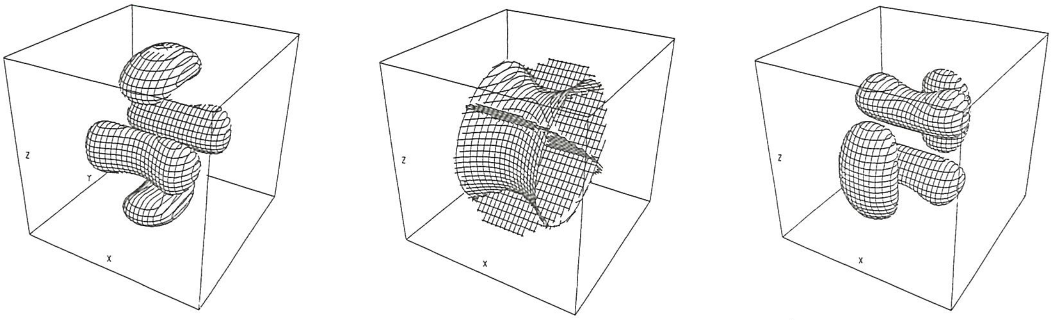

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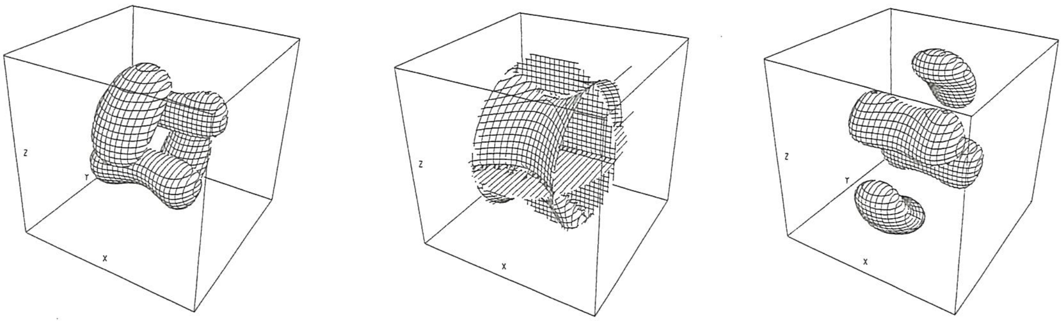

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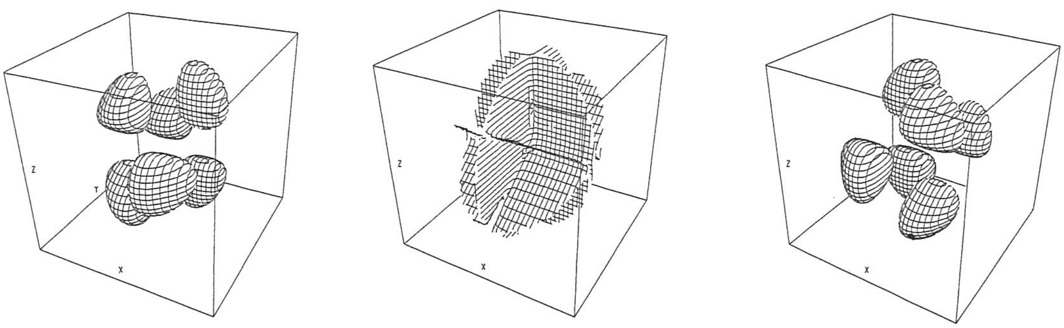

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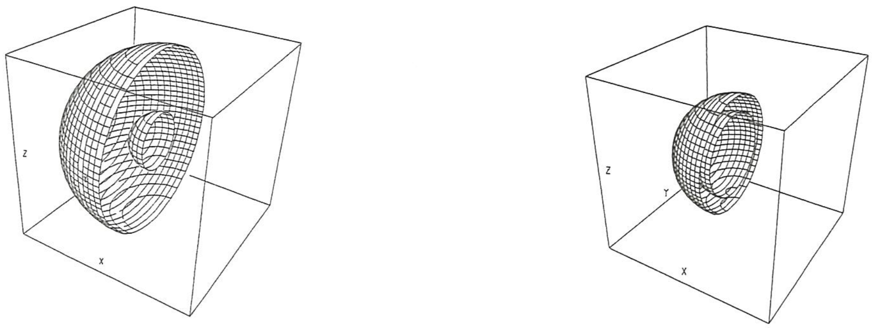

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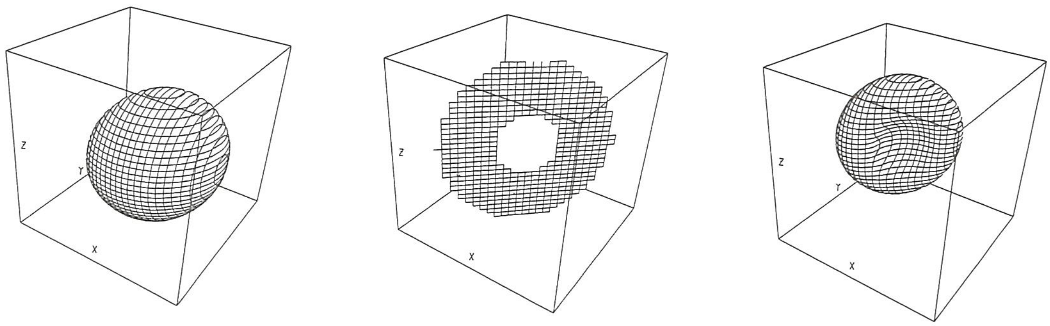

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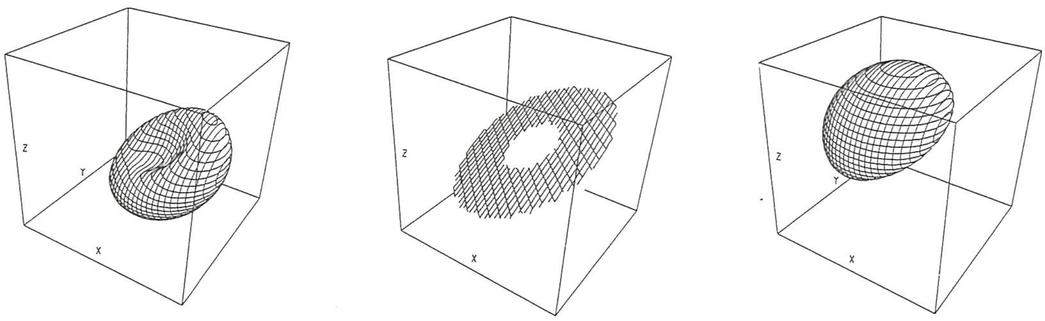

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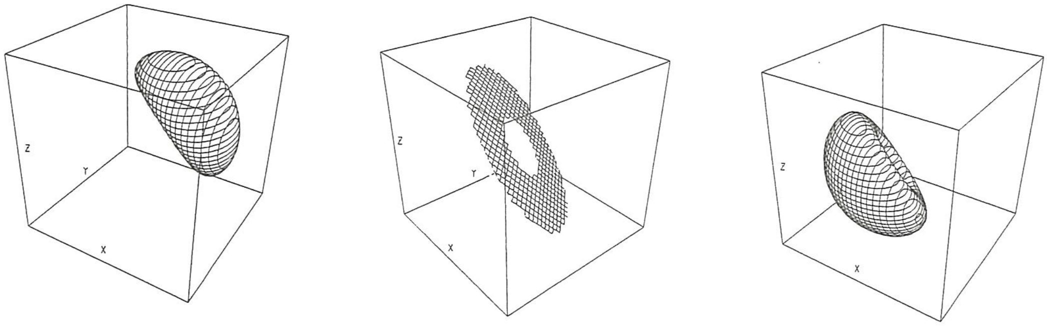

Figure 3.4. λ = 24.01

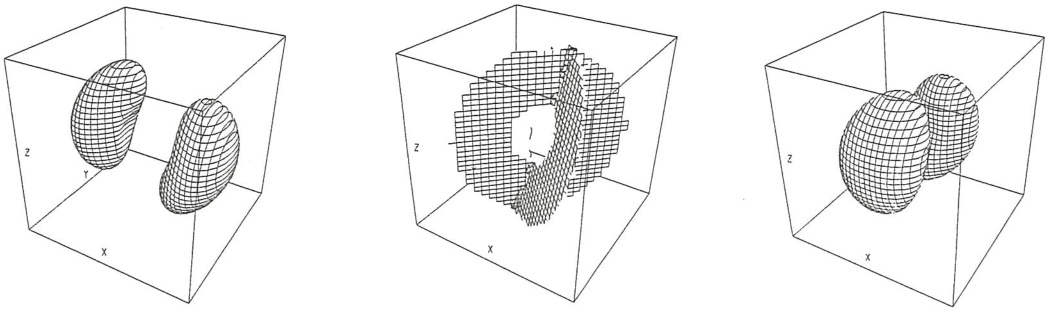

Figure 3.5. λ = 33.8

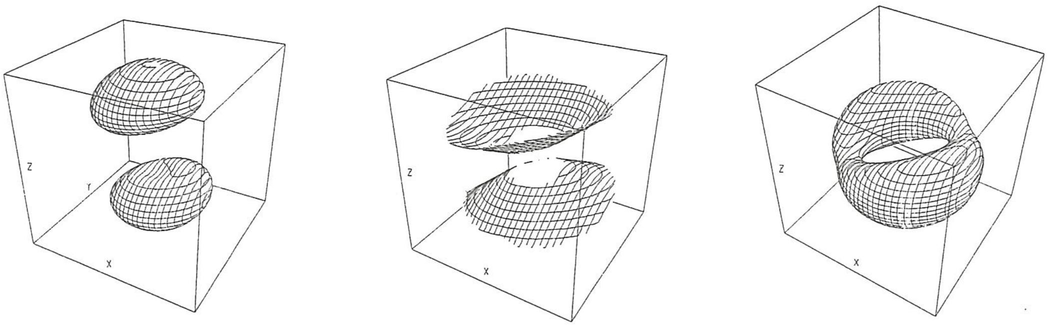

Figure 3.6. λ = 33.8

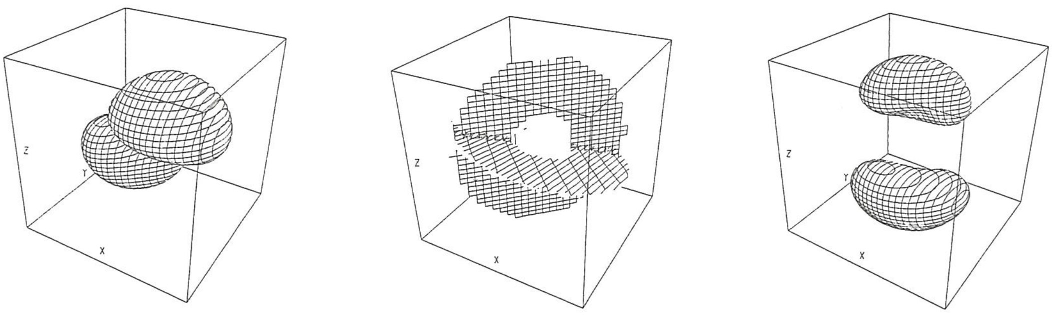

Figure 3.7. λ = 33.8

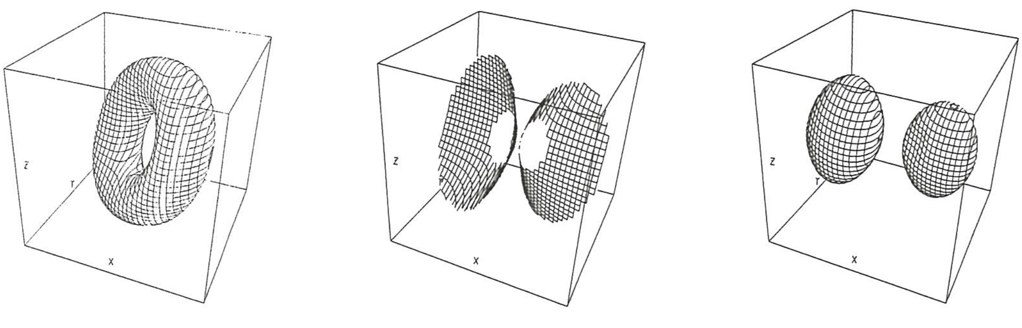

Figure 3.8. λ = 33.8

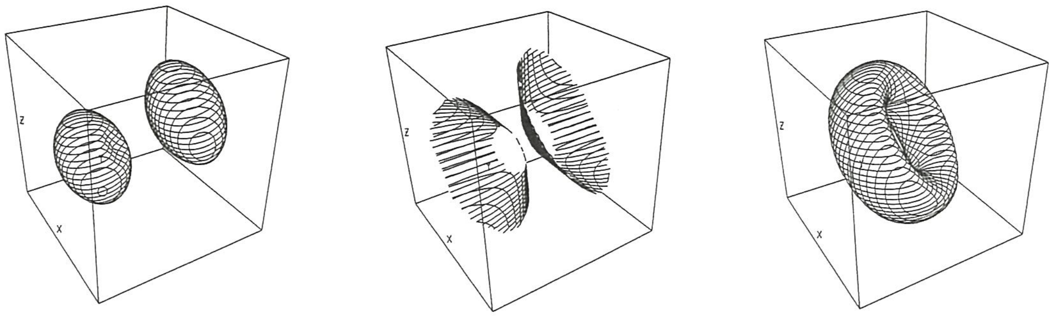

Figure 3.9. λ = 33.8

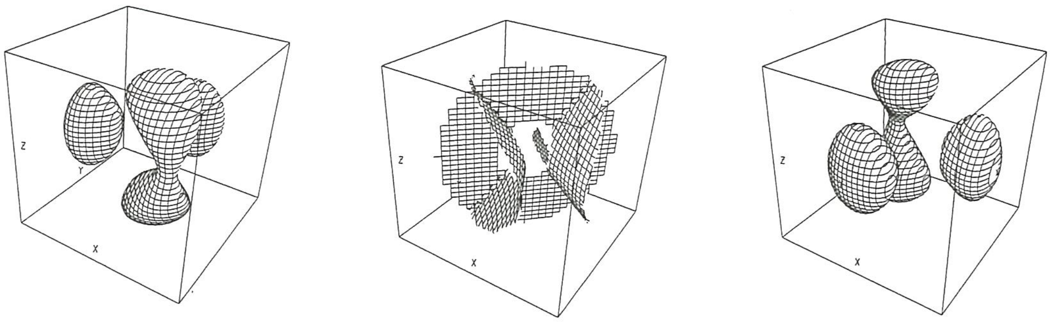

Figure 3.10. λ = 47.4

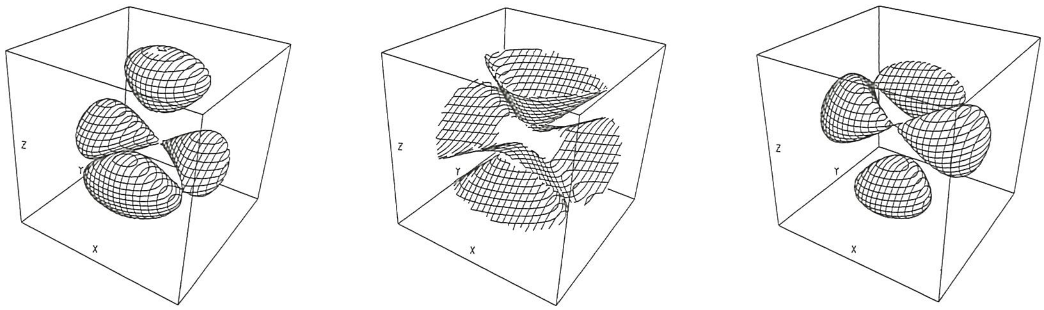

Figure 3.11. λ = 47.41

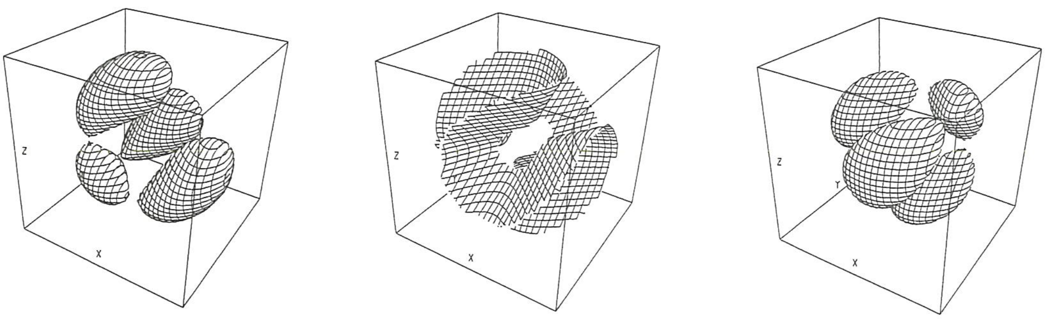

Figure 3.12. λ = 47.41

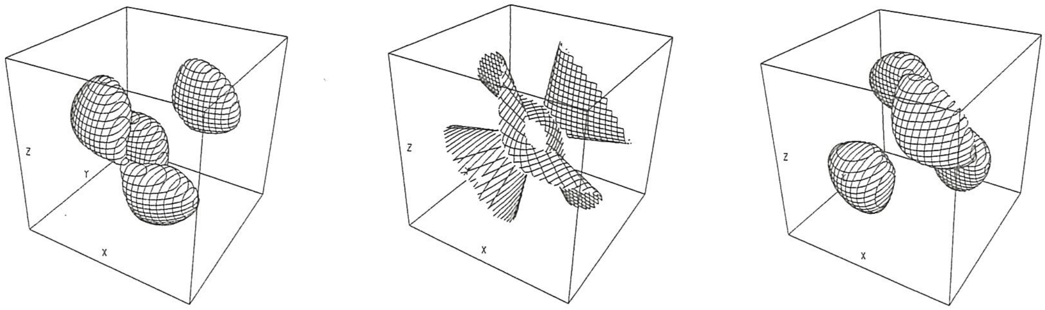

Figure 3.13. λ = 47.5

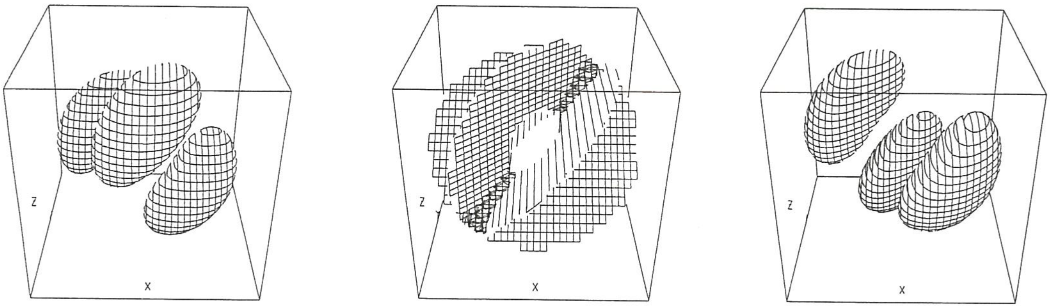

Figure 3.14. λ = 47.6

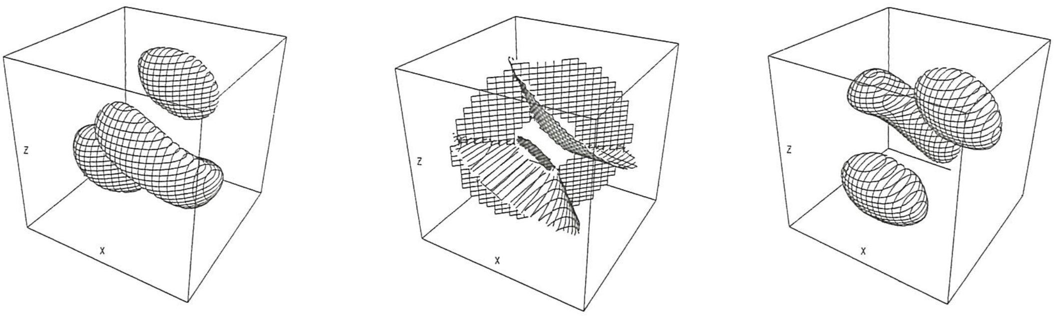

Figure 3.15. λ = 47.6

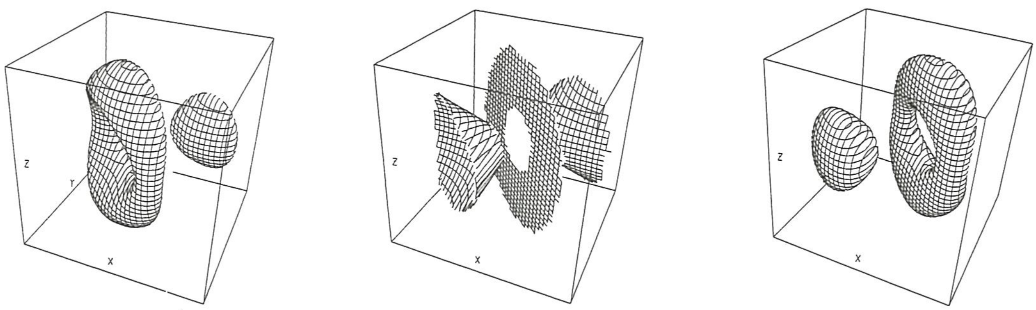

Figure 3.16. λ = 47.6

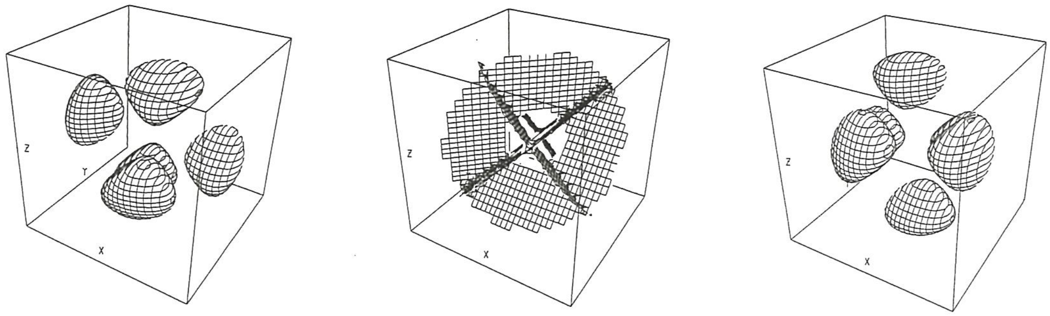

Figure 3.17. λ = 64.0

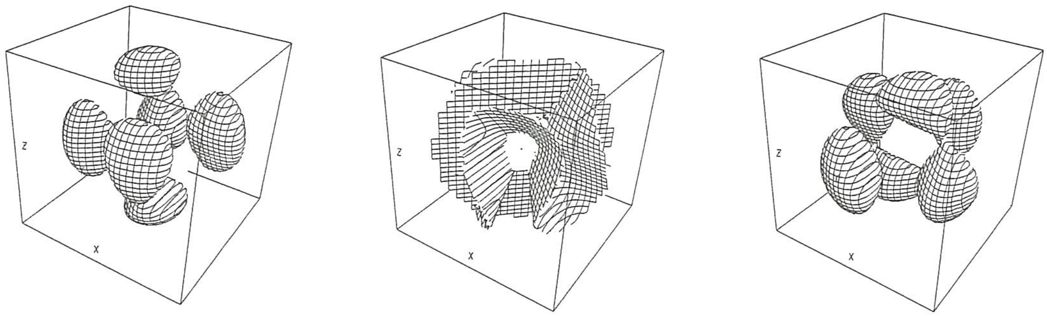

Figure 3.18. λ = 64.0

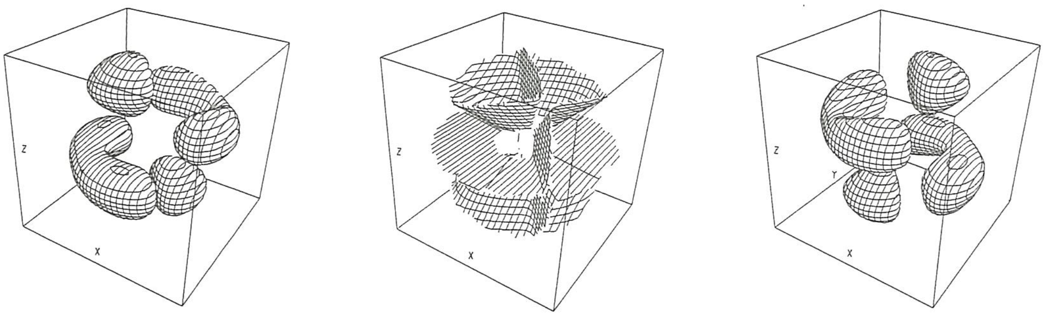

Figure 3.19. λ = 64.0

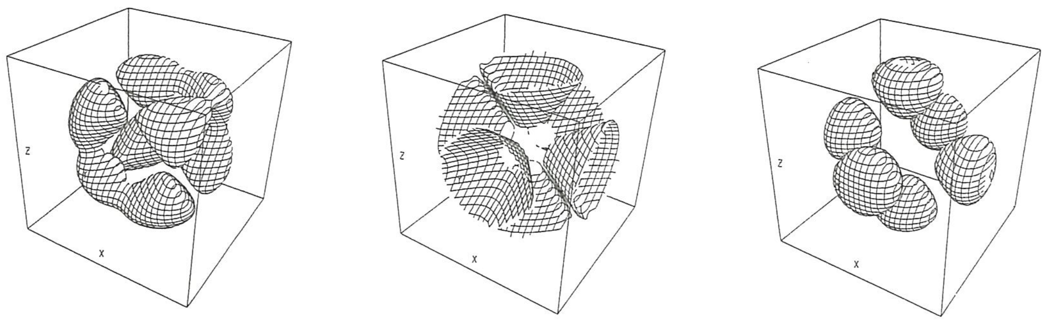

Figure 3.20. λ = 64.1

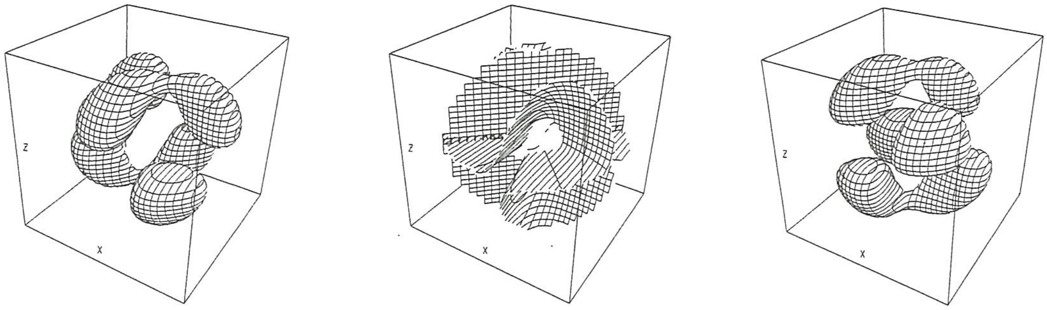

Figure 3.21. λ = 64.3

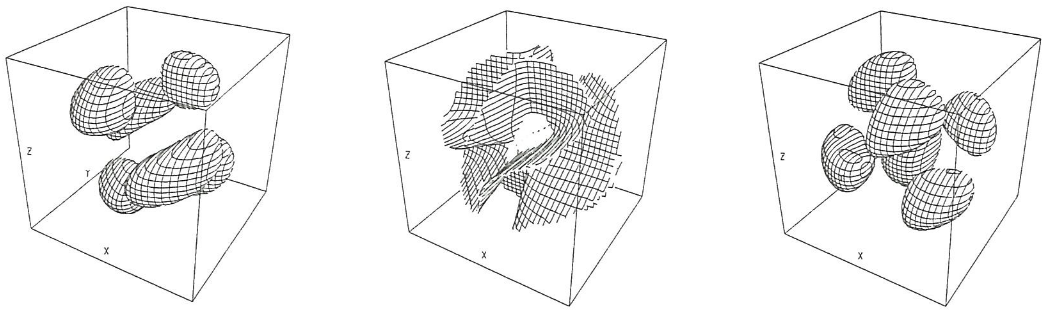

Figure 3.22. λ = 64.3

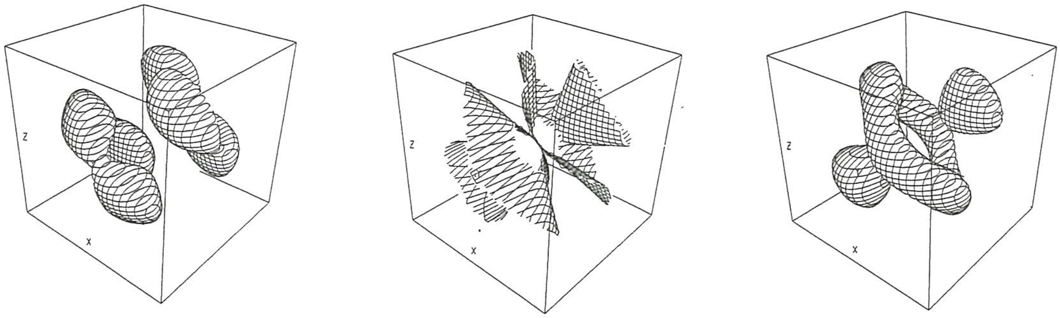

Figure 3.23. λ = 64.4

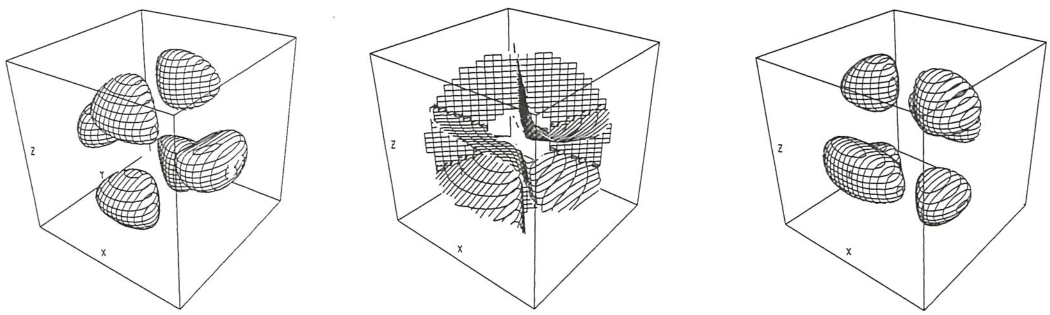

Figure 3.24. λ = 64.4

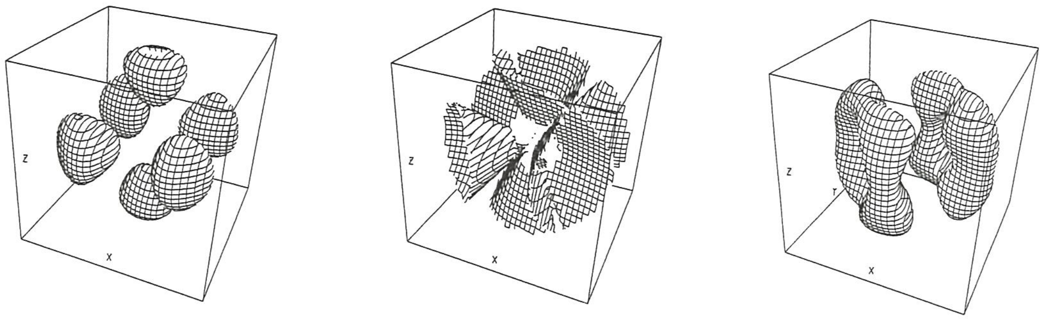

Figure 3.25. λ = 64.4

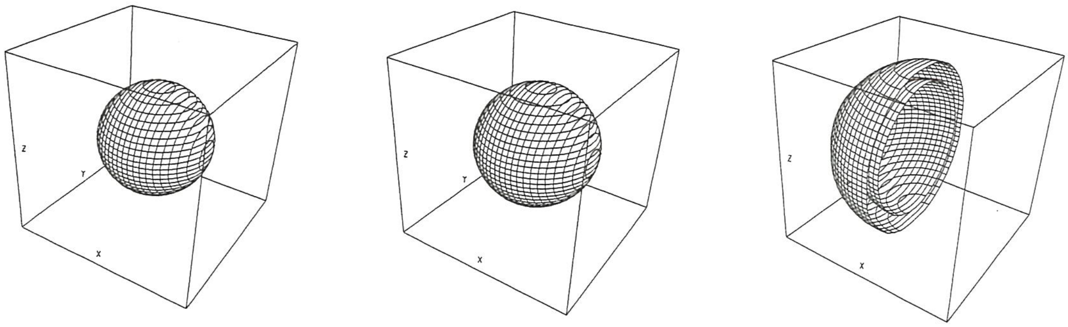

Figure 3.26. λ = 74.4

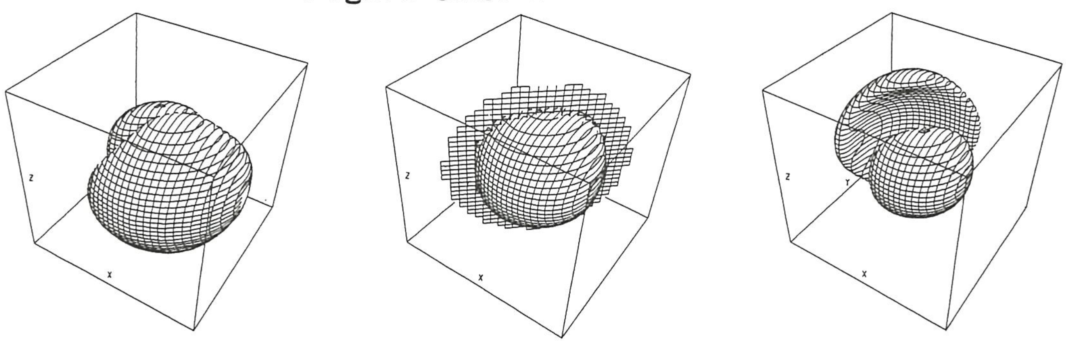

Figure 3.27. λ = 80.3

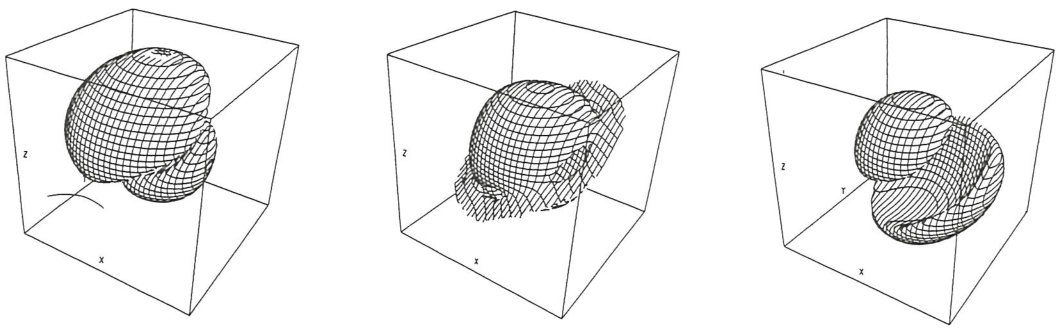

Figure 3.28. λ = 80.3

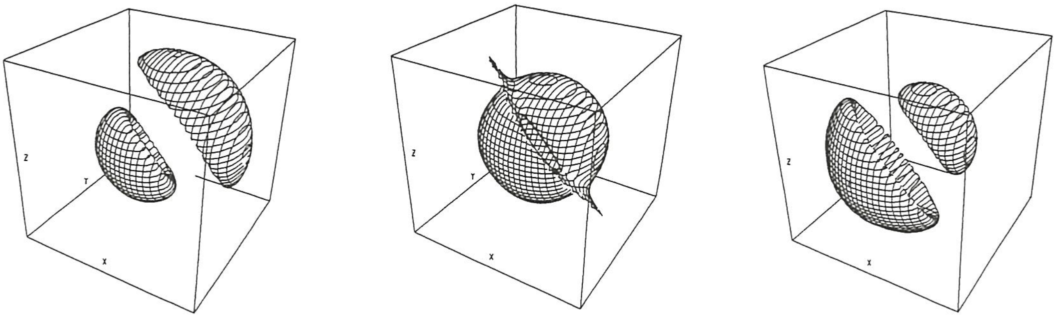

Figure 3.29. λ = 80.4

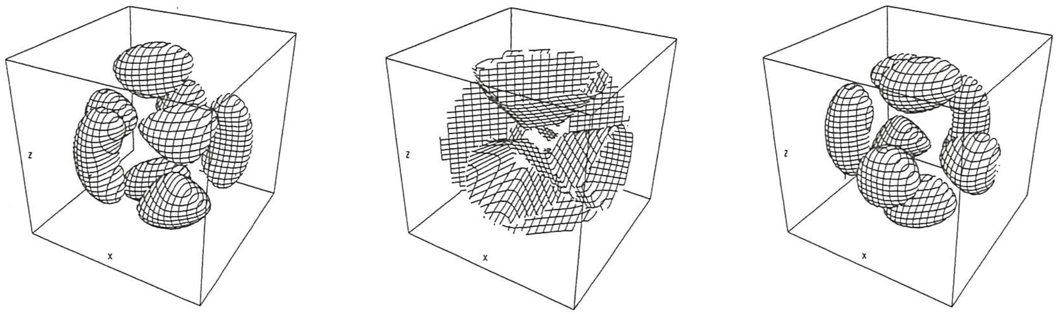

Figure 3.30. λ = 83.2

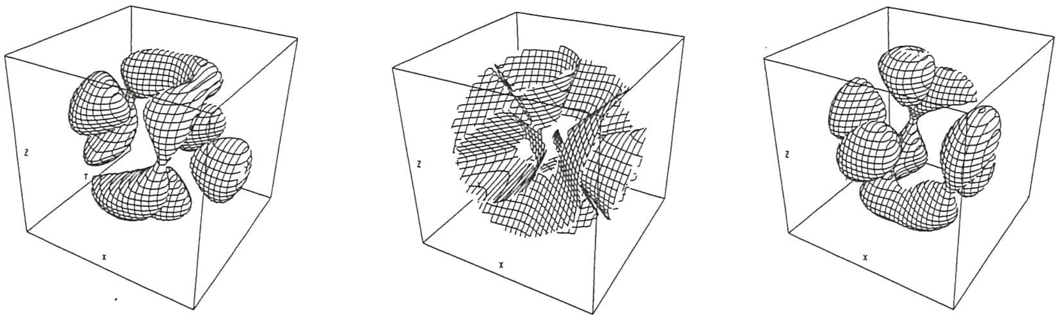

Figure 3.31. λ = 83.2

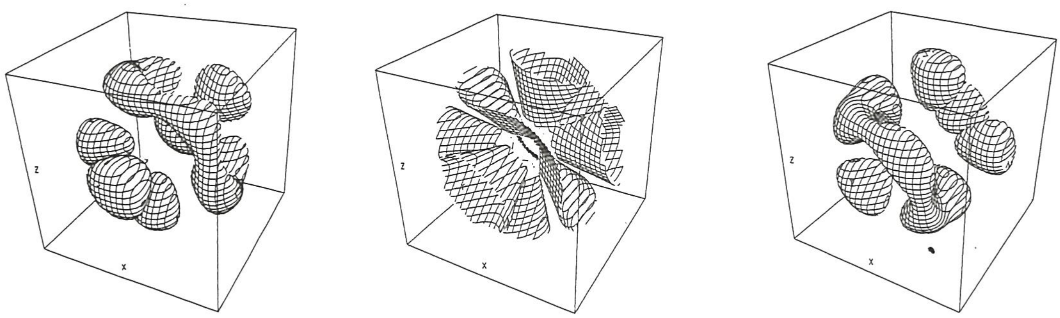

Figure 3.32. λ = 83.2

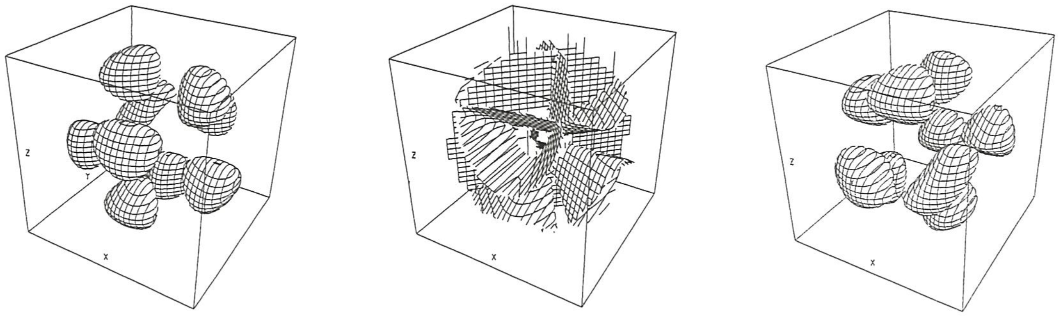

Figure 3.33. $\lambda = 83.4$

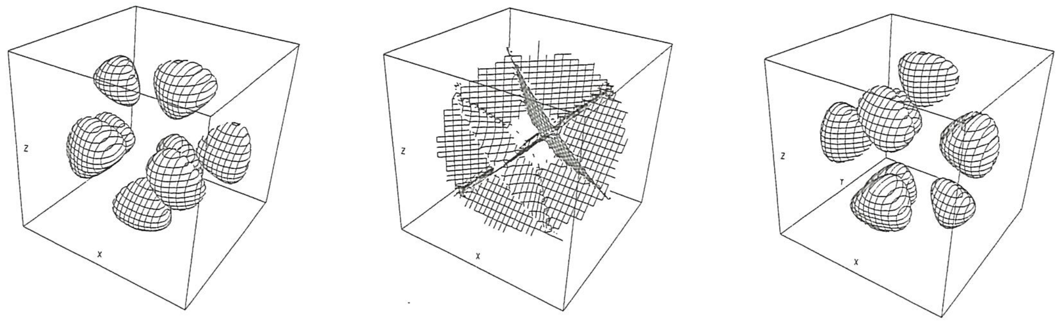

Figure 3.34. $\lambda = 83.4$

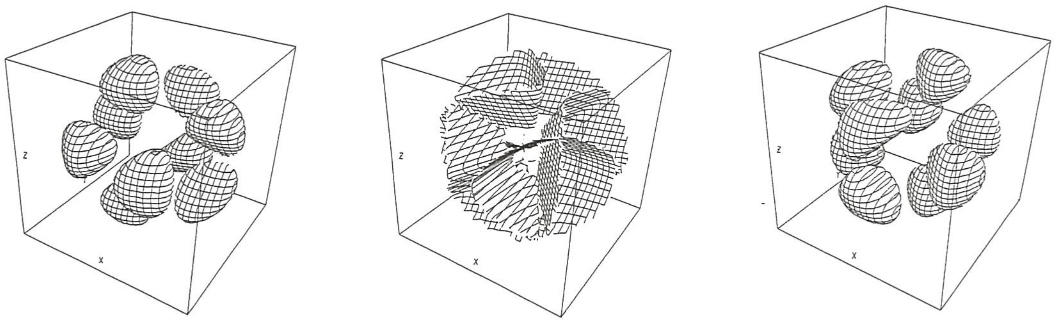

Figure 3.35. $\lambda = 83.4$

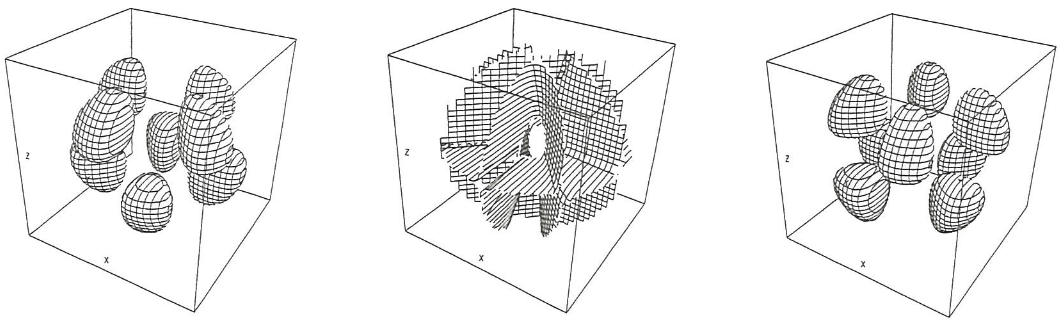

Figure 3.36. $\lambda = 83.5$

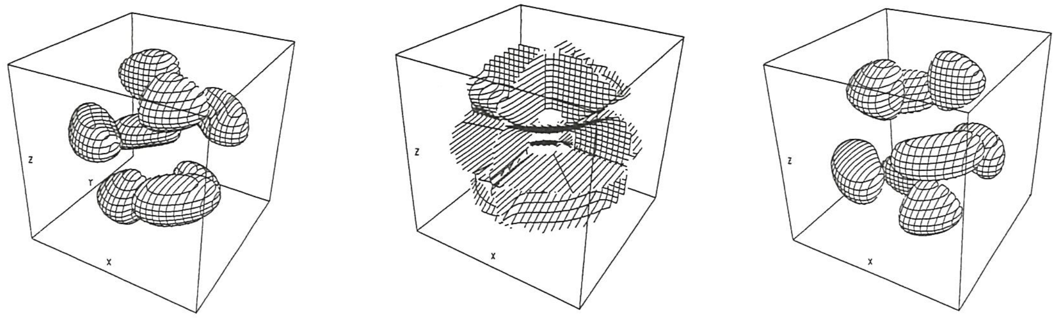

Figure 3.37. λ = 83.5

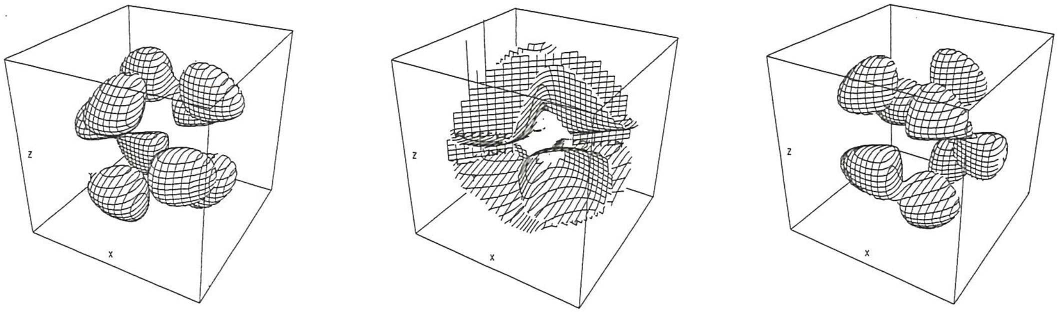

Figure 3.38. λ = 83.8

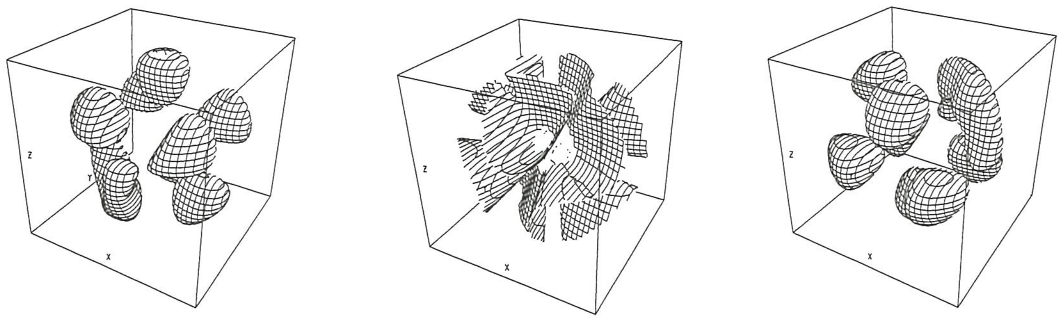

Figure 3.39. λ = 83.8

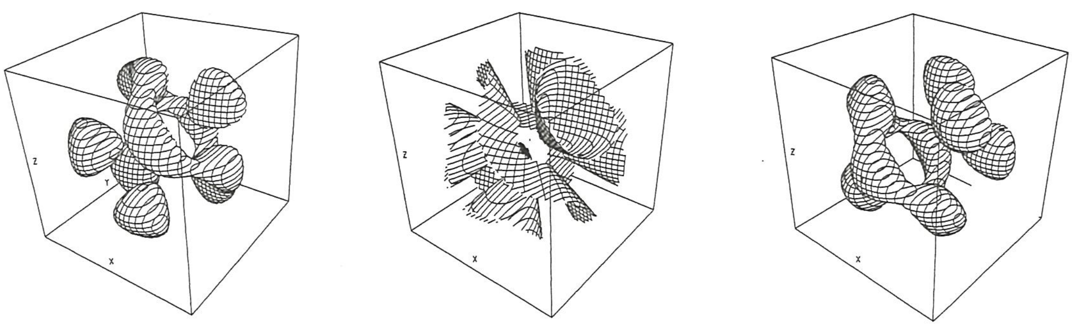

Figure 3.40. λ = 83.8

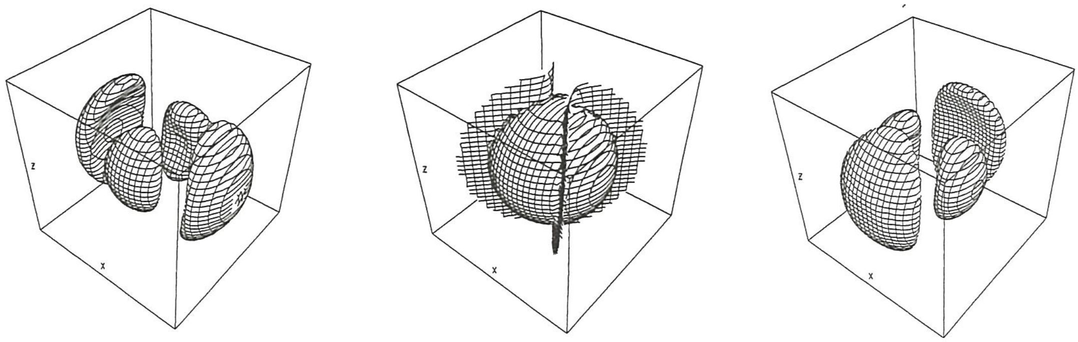

Figure 3.41. λ = 92.2

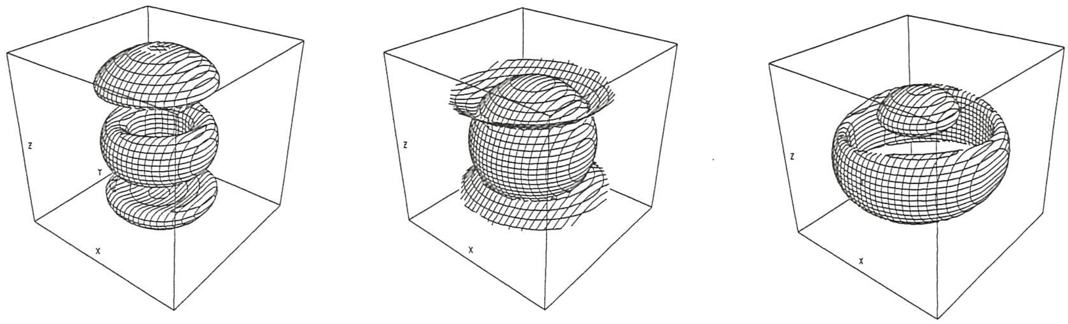

Figure 3.42. λ = 92.2

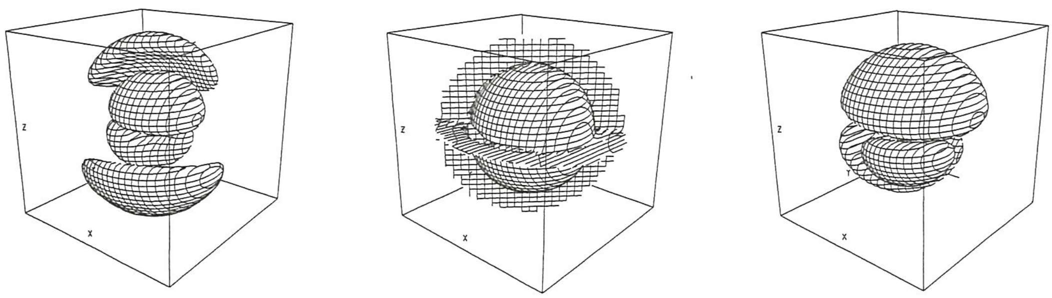

Figure 3.43. λ = 92.4

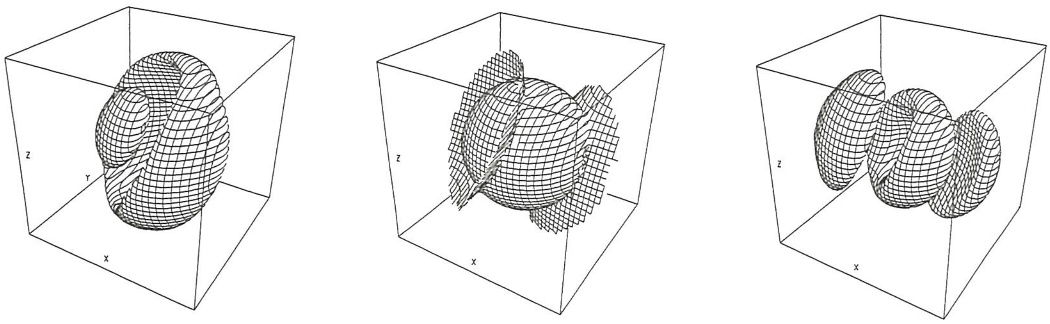

Figure 3.44. λ = 92.4

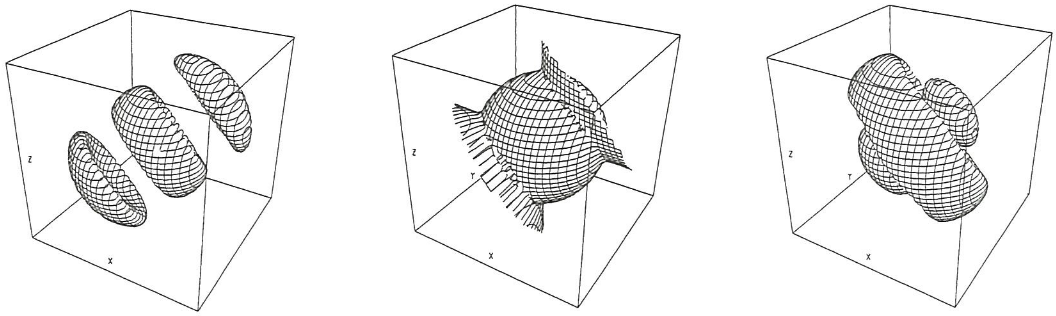

Figure 3.45. λ = 92.5

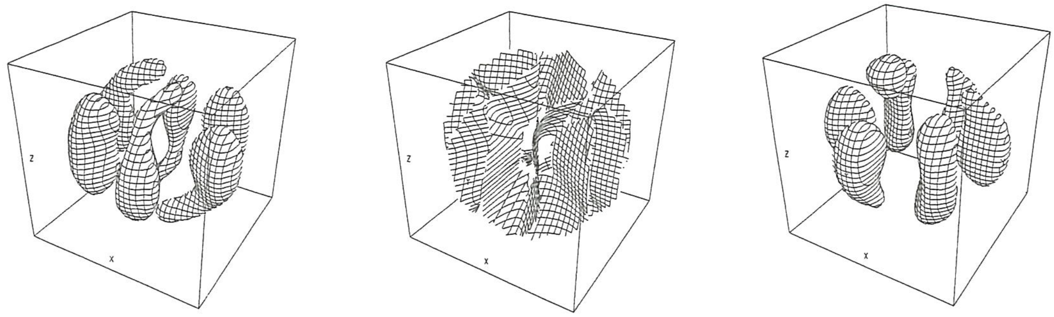

Figure 3.46. λ = 104.5i

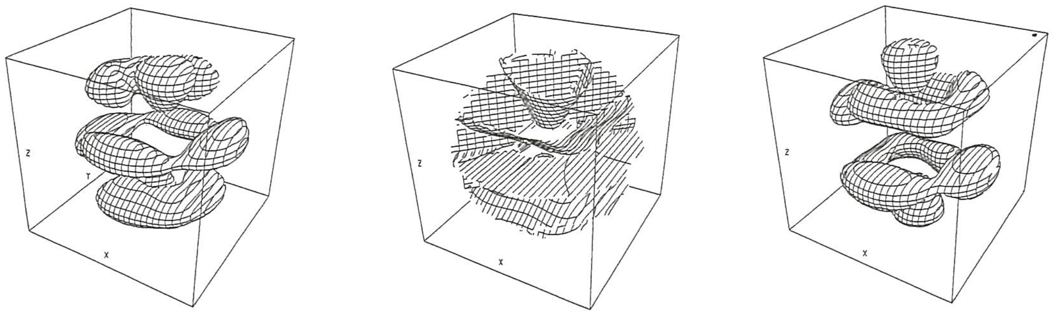

Figure 3.47. λ = 104.5

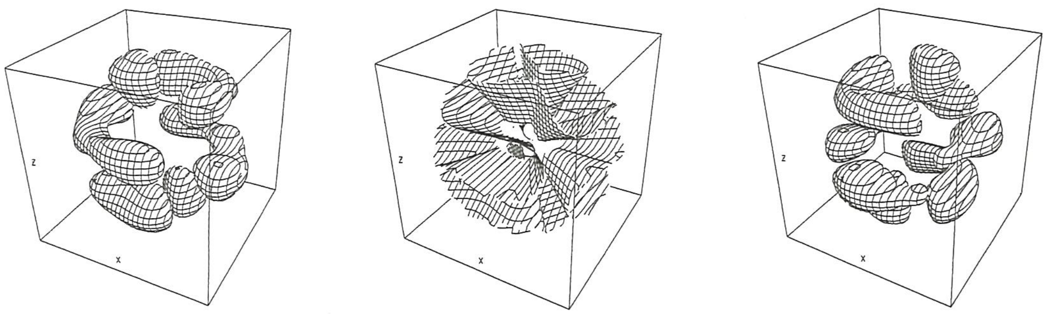

Figure 3.48. λ = 104.5