Kumamoto J. Sci. (Math.)
Vol. 17, 1~4 March (1986)

ISOTOPY GROUPS OF CIRCLE-BOUNDED MANIFOLDS

David J. SPROWS

(Received July 31, 1985)

1. INTRODUCTION.

Let M be a compact, connected 2-manifold with boundary and let A be a subset of M. The group of all homeomorphisms of M which are the identity on A is denoted by $H(M,A)$. If $H(M,A)$ is given the compact-open topology, then the arc component of the identity, denoted $H_0(M, A)$, is a normal subgroup of $H(M,A)$. The quotient group $H(M,A)/H_0(M,A)$ is called the isotopy group (*rel* A) of M.

In [2] and [7] various isotopy groups (*rel* A) are investigated for the case M equal to a 2-sphere with boundary holes. Results similar to those contained in [2] are given in [6] for orientable, bounded 2-manifolds other than the 2-sphere with holes. In each of these papers the set A is taken to be either a subset of the boundary of M or a finite set of points in the interior of M and the corresponding isotopy group (*rel* A) is related to the isotopy group of M itself, denoted $\pi_0(H(M))$. In particular, knowledge of the structure of $\pi_0(H(M))$ is critical to the understanding of the structure of any of the other isotopy groups.

In this paper we consider the structure of the isotopy groups of arbitrary 2-manifolds with one boundary component. It is shown that if the given manifold is not a disk or a Moebius band, then the isotopy group of the manifold is isomorphic to the group of automorphisms of the fundamental group of the closed manifold obtained by sewing a disk to the boundary of the given manifold.

2. NOTATION.

Given a compact, connected 2-manifold M with x_0 in the interior of M, let *Aut* $\pi_1(M,x_0)$ denote the group of automorphisms of $\pi_1(M,x_0)$ and let *End* $\pi_1(M,x_0)$ denote the set of endomorphisms of $\pi_1(M,x_0)$ with the operation of composition. Let $[(M,x_0), (M,x_0)]$ denote the set of homotopy classes (*rel* x_0) of maps from (M,x_0) to (M,x_0) and let $[f]$ denote the homotopy class (*rel* x_0) of a mapping f

from (M, x_0) to (M, x_0). Finally, following the notation of [2], let $\pi_0(H(M, x_0))$ denote the isotopy group (*rel* x_0) of M.

3. ISOTOPY GROUPS.

The following lemmas will be used in the proof of the main result of this paper.

Define $\Phi: [(M, x_0), (M, x_0)] \to End\ \pi_1(M, x_0)$ *by* $\Phi([f]) = f*$ where $f*([\alpha]) = [f\alpha]$.

LEMMA 1. *If M is a closed 2-manifold with $\pi_2(M, x_0) = 1$, then Φ is a bijection which preserves the operation of composition.*

Proof of Lemma 1.

Note that without loss of generality it can be assumed that M is triangulated.

(1) Clearly Φ is well-defined and $\Phi([f \circ g]) = \Phi([f]) \circ \Phi([g])$.

(2) Claim Φ is a surjection.

Let $F \in End\ \pi_1(M, x_0)$. We define $f: (M, x_0) \to (M, x_0)$ with $f* = F$ as follows:

Take x_0 to be a vertex of the triangulation of M and let T be a maximal tree in M. If $x \in T$, we let $f(x) = x_0$. Now suppose s is a 1-simplex not in T with $h: [0,1] \cong s$. Let γ_i be a path in T from x_0 to $h(i)$ for $i = 0, 1$. Define the loop α_s at x_0 by letting $\alpha_s(t) = \gamma_0(t)$ if $-1 \leq t \leq 0$, $\alpha_s(t) = h(t)$ if $0 \leq t \leq 1$ and $\alpha_s(t) = \gamma_1^{-1}(t)$ if $1 \leq t \leq 2$. If $F[(\alpha_s]) = [B]$ we let $f/s = B \circ h^{-1}$. This defines f on the 1-skeleton of M.

If Δ is a 2-simplex with edges s_1, s_2, and s_3, then $[\alpha_{s_1} * \alpha_{s_2} * \alpha_{s_3}] = 1$. This means that $f/\partial\Delta$ is null-homotopic, *i. e.* f extends to Δ. Thus the mapping f defined on the 1-skeleton as above, extends to a mapping defined on all of the 2-dimensional manifold M. Note that by construction $f*([\alpha_s]) = F([\alpha_s])$ and since $\{[\alpha_s]$: s is a 1-simplex of $M\}$ generates $\pi_1(M, x_0)$, we have $f* = F$.

(3) Claim Φ is an injection.

Suppose $f* = g*$. As in Part (2), let T be a maximal tree. f/T is homotopic (*rel* x_0) to a map which sends T to x_0 (just use the retraction of T to x_0). Hence by the homotopy extension property f is homotopic (*rel* x_0) to a map f' with $f'(T) = x_0$. Therefore, we can assume $f(T) = g(T) = x_0$. In particular, for each 1-simplex s, f/s and g/s are loops at x_0. Since $f* = g*$, this means f/s

is homotopic to g/s (*rel* x_0). Thus for each 2-simplex Δ we have a map $H: \partial(\Delta\times I)\to M$ where $H/\Delta\times 0=f/\Delta$, $H/\Delta\times 1=g/\Delta$ and for each 1-simplex s in $\partial\Delta$, $H/s\times I$ is a homotopy from f/s to g/s. Since $\pi_2(M, x_0)=1$ and $\partial(\Delta\times I)\cong S^2$, this map H can be extended to all of $\Delta\times I$. Fitting together each of these $H's$ we get a homotopy (*rel* x_0) from f to g, *i. e.* $[f]=[g]$.

LEMMA 2. *If M is a closed 2-manifold and $h: (M, x_0)\to(M, x_0)$ is a homeomorphism which is homotopic to the identity (rel x_0), then h is isotopic to the identity (rel x_0).*
Proof. This is a special case of Theorem 6.3 of [1].
LEMMA 3. *If M is a closed 2-manifold with $x_0\in M$ and G is an automorphism of $\pi_1(M, x_0)$, then there exists a homeomorphism $h:(M, x_0)\to(M, x_0)$ with $h*=G$.*
Proof. This result is proved in [3].
THEOREM. *Let Y be a compact, connected 2-manifold with one boundary component and let X be the closed 2-manifold obtained by sewing a disk to the boundary of Y. If Y is not a disk or a Moebius band, then*

$$\pi_0(H(Y))\cong Aut\ \pi_1(X, x_0) \text{ where } x_0\in X-Y.$$

Proof. By Theorem 6 of [5], the group $\pi_0(H(Y))$ is isomorphic to $\pi_0(H(X, x_0))$. Thus it suffices to show that $\pi_0(H(X, x_0))$ and $Aut\ \pi_1(X, x_0)$ are isomorphic. By Lemma 2, the function from $\pi_0(H(X, x_0))$ to $[(X, x_0), (X, x_0)]$ which sends the isotopy class (*rel* x_0) of a homeomorphism to its homotopy class (*rel* x_0) is an injection. By Lemma 1, the composition

$$\pi_0(H(X, x_0) \longrightarrow [(X, x_0), (X, x_0)] \xrightarrow{\Phi} End\ \pi_1(X, x_0)$$

is a monomorphism of the group $\pi_0(H, X, x_0))$ onto a subgroup of $Aut\ \pi_1(X, x_0)$. Finally, Lemma 3 shows that this monomorphism is an isomorphism onto $Aut\ \pi_1(X, x_0)$.

Remark. If Y is a disk or a Moebius band, then $\pi_0(H(Y))$ is not isomorphic to $Aut\ \pi_1(X, x_0)$. In the case Y is a disk, so that $x=S^2$, we have $Aut\ \pi_1(X, x_0)=1$ while $\pi_0(H(Y))\cong Z_2$ (see Theorem 4.2 of [4]). In the case Y is a Moebius band, so that $X=P^2$, we have $Aut\ \pi_1(X, x_0)=1$ while $\pi_0(H(Y))=Z_2$ (see Theorem 8.1 of [4]).

REFERENCES

[1] D. B. A. Epstein, Curves on 2-manifolds and isotopies, Acta. Math. **115** (1966), 83-107.

[2] J. P. Lee, Isotopy groups of 2-sphere with boundary holes, Kumamoto Journal of Science, Math. **14** (1980), 1-8.

[3] J. Nielson, Untersuchugen zur topologie der geschlossenen zweiseitigen flächen, Act. Math. **50** (1927), 189-358.

[4] L. V. Quintas, Solved and unsolved problems in the computation of homeotopy groups of 2-manifolds, Trans. N. Y. Academy of Sciences, **30** (1968), 919-938.

[5] D. J. Sprows, Homeotopy groups of compact 2-manifolds, Fund. Math. **90** (1975), 99-103.

[6] D. J. Sprows, Isotopy groups of bounded 2-manifolds, Kumamoto Journal of Science, Math. **15** (1983), 73-77.

[7] D. J. Sprows, Homeotopy groups of punctured spheres with holes, Fund. Math. **115** (1983), 207-212.

Department of Mathematical Sciences
Villanova University
Villanova, PA 19085
U. S. A.