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COMPLEX SURFACES PROPERLY DOMINATED BY C^2

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0. Introduction.

We say a normal complex surface X is analytically compactifiable if there exists a normal compact analytic space S and an analytic subset C of S such that X is biholomorphic to $S-C$. By the resolution of singularities of S on C if necessary, we may assume that S is non-singular on the analytic subset C. Especially, if X is non-singular, then S is a compact complex manifold. The purpose of this paper is to prove the following

THEOREM. Let f: $C^2 \to X$ *be a proper holomorphic mapping of the complex affine plane* C^2 *onto an analytically compactifiable (resp. affine algebraic) normal complex surface* X. *Then, either* X *is biholomorphic (resp. biregular) to* C^2 *or* C^2/G, *where* G *is a small subgroup of* $\mathrm{GL}(2, C)$.

Our theorem can be considered as an analytic version of the theorem due to Miyanishi [13] (see also Gurjar-Shastri [5]).

In the proof, we use the results in [2] and [3], the theory of cluster sets due to Nishino-Suzuki [19], and some topological results due to Gurjar [4], Gurjar-Shastri [5].

1. Topological properties.

Let f: $C^2 \to X$ be a proper holomorphic mapping of C^2 onto an analytically compactifiable normal complex space. Since C^2 is a Stein manifold and f is a proper holomorphic mapping, X is also a (normal) Stein space by Narasimhan [17]. Then,

PROPOSITION 1 (Gurjar [4]) *X is topologically contractible.*

Let (S, C) be a (normal) analytic compactification. We may assume that any singular point of C is an ordinary double point, (ii) no non-singular rational irreducible component of C with the self-intersection number -1 has at most two intersection points with other components of C. Such a compactification is called the minimal normal compactification.

Let us consider the following exact sequence of cohomology group over $\boldsymbol{Z}$ of the pair (S, C):

$$\rightarrow H^i(S, C; \boldsymbol{Z}) \rightarrow H^i(S; \boldsymbol{Z}) \rightarrow H^i(C; \boldsymbol{Z}) \rightarrow H^{i+1}(S, C; \boldsymbol{Z}) \rightarrow$$

Since $H^i(S, C; \boldsymbol{Z}) \cong H_{4-i}(S-C; \boldsymbol{Z}) \cong H_{4-i}(X; \boldsymbol{Z})$, we have, by Proposition 1, $H^i(S; \boldsymbol{Z}) \cong H^i(C; \boldsymbol{Z})$ for $i \leqq 2$. Since $0 \rightarrow H^3(S; \boldsymbol{Z}) \rightarrow H^3(C; \boldsymbol{Z}) \cong 0$, we have the following

LEMMA 1. *$H^i(S; \boldsymbol{Z}) \cong H^i(C; \boldsymbol{Z})$ for $i \leqq 3$, if S is non-singular.*

Let T be a tubular neighbourhood of C in S (see [15], [20]) and ∂T be the boundary of T. Then, Gurjar has also the following

PROPOSITION 2 ([4]). *If X is a non-singular surface, then the fundamental group $\pi_1(\partial T)$ is either trivial or the binary icosahedral group $SL(2,5)$ of order 120.*

REMARK: In Proposition 2, the smoothness of X is essential. In fact, let us consider the following proper morphism $f: \boldsymbol{C}^2 \rightarrow \boldsymbol{C}^3$ given by $f(u, v) = (u^2, v^2, uv)$. We put $X = f(\boldsymbol{C}^2)$. Then $X \cong \{z^2 = xy\} \hookrightarrow \boldsymbol{C}^3$ and $\pi_1(\partial T) \cong \boldsymbol{Z}_2$.

2. A characterization of C^2

We shall first note some facts on rational ruled surfaces following to Suzuki [21].

Let M be a non-singular compact complex analytic surface, and D be a non-singular irreducible rational curve on M with the self-intersection number $(D)^2 = 0$. By Kodaira-Spencer [9] and Kodaira [7], we obtain a holomorphic mapping $\pi: M \rightarrow R$ of M onto a non-singular compact curve R which has D as a regular fibre. Thus M is a ruled surface. Let us assume that there exists another rational curve $D' \neq D$ on M which intersects D. Then the base curve R is isomorphic to a projective line $\boldsymbol{P}^1$ and M is a rational ruled surface. Thus we have the following

LEMMA 2. *Assume that there exists a non-singular rational curve C with the self-intersection number $(C)^2 > 0$ in a non-singular compact complex surface M. Then M is a rational ruled surface.*

By Nagata [16], M can be obtained from a geometrically ruled surface F_n ($n \geqq 0$) over P^1 by a finite succession of quadratic transformations $Q_{p_1}, .. Q_{p_r} (r>0)$, in such a way that

$$M = Q_{p_r} \cdot \ldots \cdot Q_{p_1}(F_n) \text{ and } \pi = \pi_0 Q_{p_1}^{-1} \ldots Q_{p_r}^{-1},$$

where $\pi_0: F_n \to P^1$ is the projection.
Therefore, each fibre $\pi^{-1}(z)$ $(z \in P^1)$ is a curve with no loop composed of non-singular rational curves crossing normally. Since rank $H^2(F_n; Z)=2$, we have rank $H^2(M; Z)=r+2$. On the other hand, if one denotes the number of irreducible components of $\pi^{-1}(z)$ by $1+\alpha(z)$ for each $z \in P^1$, we have $\sum_{z \in P^1} \alpha(z) = r$.
Thus we have

LEMMA 3. *rank* $H^2(M; Z) = r+2$ *and* $\sum_{z \in P^1} \alpha(z) = r$.

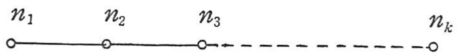
Now let M be a non-singular compact complex surface and C be a connected analytic subset of M consisting of non-singular rational curves $C_i (1 \leqq i \leqq k)$ crossing normally. With each collection C of the curve $C_i (1 \leqq i \leqq k)$, we associate a dual graph $\Gamma(C)$. Each vertex of the graph represents a non-singular rational curve C_i. Adjacent to each vertex we write the self-intersection number $(C_i)^2$ of the curve C_i. These are called weights. Two vertices are joined by a segment if the two rational curve they represent intersect.

Then we have a following characterization of C^2.

PROPOSITION 3. *Let* (M, C) *be as above. We put* $V = M - C$. *Suppose that*

(i) $H_i(V, Z) = 0$ for $i > 0$.

(ii) *the dual graph* $\Gamma(C)$ *of the curve* C *is of the form,*



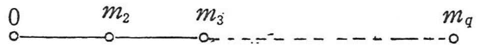
, *max* $\{n_i\} \geqq 0$ $(n_i \neq -1)$

(iii) *V contains no compact complex analytic curve. Then M is a rational surface and V is isomorphic to C^2. More precisely, there exists a birational mapping of M to P^2 which maps V onto $P^2 - \{a \text{ line}\}$ biregularly.*

PROOF. By the argument similar to Lemma 1, $H^1(M; Z)=0$ and $H^2(M; Z)=Z^k$. Since $max\{n_i\} \geqq 0$ and $H^1(M; Z)=0$, by Lemma 2, M is a rational (ruled) surface. Thus $H^1(M; \mathcal{O}^*) \cong H^2(M; Z)$ and $H^1(M; \mathcal{O}^*)$ is generated by the line bundle $[C_i]$ defined by C_i over Z $(1 \leqq i \leqq k)$.

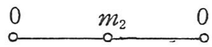
(The case of $k=1$): Since M is rational and $H^2(M;\boldsymbol{Z})\cong\boldsymbol{Z}$ is generated by the first Chern class $c_1([C])$ of the line bundle $[C]$, $M\cong\boldsymbol{P}^2$ and C is a line on $\boldsymbol{P}^2$.

(The case of $k\geqslant 2$): Since $max\ \{n_i\}\geqslant 0$, performing elementary transformations on C, it is easy to see that we may assume that the graph $\Gamma(C)$ is of the form



Where $m_2\leqslant -2$ and $m_j\neq -1$ for $j\geqslant 3$.

Assume that $q\geqslant 3$. If $max\ \{m_3,\dots,m_q\}\geqslant 0$, then by Lemma 6 in [21], the graph $\Gamma(C)$ must be of the form



Since M is a rational surface, M can be obtained from $\boldsymbol{F}_n$ by a finite succession of quadratic transformations. By Lemma 1 and 2, $r+2=rank\ H^2(C;\boldsymbol{Z})=3$, hence $r=1$. This contradicts the assumption (iii). If $max\ \{m_3,\dots,m_q\}<0$, then we may assume that the graph $\Gamma(C)$ is of the form

where $m_j\geqq 2$ $(j\geqq 2)$. Let C_1, C_j $(j\geqq 2)$ be the irreducible components of C associated with the vertices with weights 0, $-m_j(j\geqq 2)$, respectively. Since M is a rational ruled surface, we have $H^1(M;\mathcal{O}^*)\cong H^2(M;\boldsymbol{Z})\cong H(C;\boldsymbol{Z})\cong\boldsymbol{Z}^q$. Then we have

$$K_M=\sum_{i=1}^{q}\lambda_i C_i,\ \text{where}\ \lambda_i\in\boldsymbol{Z}.$$

By the adjunction formula for each component of C, we have the equality

$$[m_2,\dots m_q]+[m_3,\dots,m_q]\ (\lambda_1+1)+1=0,$$

that is,

$$m_2+\lambda_1+1=\frac{[m_4,\dots,m_q]-1}{[m_3,\dots\dots,m_q]}\ \in\boldsymbol{Z},$$

where $[m_i,m_{i+1},\dots,m_q]$ $(1\leqq i\leqq q)$ represent the integers defined inductively in the following way: $[m_q]=m_q$, $[m_{q-1},m_q]=m_{q-1}[m_q]-1$, $[m_{q-2},m_{q-1},m_q]=m_{q-2}[m_{q-1},m_q]-[m_q],\dots,[m_1,m_2,\dots,m_q]=m_1[m_2,m_3,\dots,m_q]-[m_3,\dots,m_q]$. Since $[m_3,\dots m_q]>[m_4,\dots m_q]$ if $m_j\geqq 2$, we have

$$m_2+\lambda_1+1=\frac{[m_4,\ldots,m_q]-1}{[m_3,\ldots\ldots,m_q]}=0,$$

hence $q=3$ and $\lambda_1=-(m_2+1)$. By Lemma 1 and 2, $r+2=rank\ H^2(M;Z)=rank\ H^2(C;Z)=q=3$, thus $r=1$. Therefore $V\cong M-C$ contains an exceptional curve of first kind, since $m_3\geqq 2$. This is a contradiction. Consequently, we have $q=2$. In this case of $q=2$, the graph $\Gamma(C)$ is of the form

$(m_1\geqq 2)$.

Performing again elementary transformations on C, we have a surface (M',C') such that the graph $\Gamma(C')$ is of the form $\overset{1}{\circ}$ and $V\cong M'-C'$. Then we have seen that M' is isomorphic to $\boldsymbol{P}^2$ and C' is a line in $\boldsymbol{P}^2$. This completes the proof.

Q.E.D.

3. Determination of the curve C.

Let $f:C^2\to X$ and (S,C) be as in section 1. Then we have the following

LEMMA 4 (*see Lemma* 3 *in* [2]). *For each* C_i, *there exists a holomorphic mapping* ϕ_i: $C\to S-C$ *of the complex line* C *into* $S-C$ *such that*

$$C_i\subset\phi_i(\infty:S)\subset C,$$

where $\phi_i(\infty:S)=\bigcap_{R>0}\overline{\phi_i(\Delta_R)}$, $\Delta_R=\{z\in C;|z|>R\}$ *and* $\overline{\phi_i(\Delta_R)}$ *is the closure of* $\phi_i(\Delta_R)$ *in* S

By Théorème 5 of [19], we have the following

LEMMA 5. *The curve* C *must be one of the type from* (α) *to* (ε) *in Table* I *below, in which, for types* (β_r) $(r\geqq 2)$, (γ), (γ'), (δ), (ε), each *irreducible component of* C *is a non-singular rational curve and assigned Figure* (1−5) *represent the weighted dual graph* $\Gamma(C)$ of C.

Table I

	Name of type	Explication of C
(α)	$\alpha(n)$	an irreducible non-singular elliptic curve with the self-intersection number $(C^2)=n\geqq 0$.
(β)	$\beta(n)$	an irreducible rational curve with only one ordinary double point and $(C^2)=n\geqq 0$.
(β_r)	$\beta(n_1,\dots,n_r)(r\geqq 2)$	Figure 1, all $n_i=-2$ or $max\ \{n_i\}\geqq 0$.
(γ)	$\gamma(n_1,\dots n_r)(r\geqq 1)$	Figure 2, all $n_i=-2$ or $max\ \{n_1+1,\dots,n_{r-1}, n_r+1\}\geqq 0$
(γ')	$\gamma(n_1,\dots,n_r)$ $(r\geqq 1)$	Figure 3, $max\ \{n_1+1, n_2,\dots,n_r\}\geqq 0$.
(δ)	$\delta(n_0 \| q_1/l_1, q_2/l_2, q_3/l_3)$	Figure 4, (i) $n_0\geqq -2$, (ii) $(l_1,l_2,l_3)=(3,3,3)$, $(2,4,4)$, or $(2,3,6-m)$, with $m=0,1,2,3$, (iii) for each $i=1,2,3$, (q_i,l_i) is a pair of coprime integers such that $0<q_i<l_i$ and that $l_i/q_i=n_{i,1}-\cfrac{1}{n_{i,2}-\cfrac{1}{\ddots -\cfrac{1}{n_{i,r_i}}}}$ (continued fraction expansion) where $n_{i,j}\geqq 2$ are integers appearing in Figure 4.
(ε)	$\varepsilon(n_1,n_2,\dots,n_r)$	Figure 5, $max\ \{n_1,\dots,n_r\}\geqq 0$.

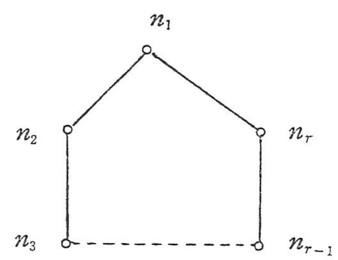
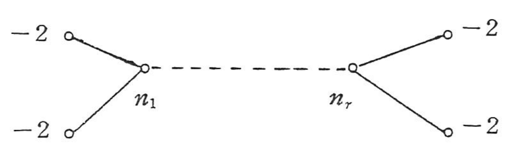


Figure 1



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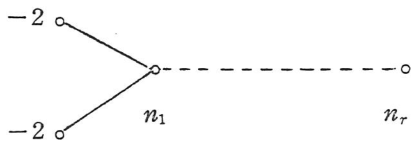


Figure 3

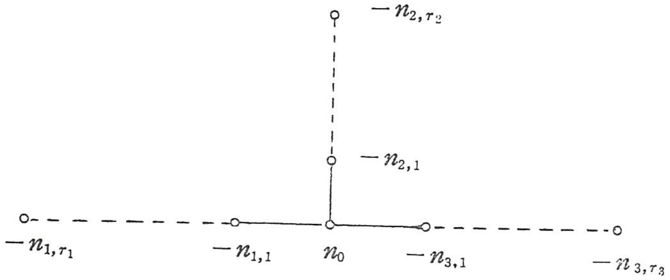


Figure 4

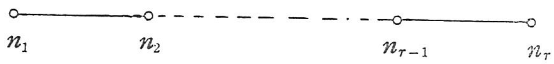


Figure 5

LEMMA 6. *The possible type of the curve C is* (γ'), (δ) *or* (ε).

PROOF. In the care of tupes (α), (β), (β_r), (γ), the fundamental group $\pi_1(\partial T)$ can not be a finite group (see (4.1) and (4.2) of 4 in [2]).

PROPOSITION 4. *Assume that X is non-singular. Then the curve C must be of the type* (ε).

PROOF. In the case of type (γ'), we see that there exists a normal subgroup N of $\pi_1(\partial T)/N$ is isomorphic to the dihedral group D_α of order 2α, where $\alpha=[n_2, n_3, \dots, n_r]$. Therefore $\pi_1(\partial T)$ is not trivial, and thus, it is isomorphic to $SL(2,5)$ by Proposition 2. But the quotient group of $SL(2;5)$ by a proper normal subgroup is isomorphic to the alternating group $\mathfrak{A}_5$ of order 60 or $SL(2,5)$, none of which is isomorphic to the dihedral group. We have thus a contradiction.

Assume that C is of the type (δ). If S is neither rational nor a ruled surface, by Corollaire 2 in [19], the possible types of the curve C are the following

(i) $\delta\left(-1\,\middle|\,\frac{1}{2}, \frac{1}{3}, \frac{1}{6}\right)$ (ii) $\delta\left(-1\,\middle|\,\frac{1}{2}, \frac{1}{4}, \frac{1}{4}\right)$

(iii) $\delta\left(-1\,\middle|\,\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)$ (iv) $\delta\left(-2\,\middle|\,\frac{1}{2},\frac{1}{3},\frac{1}{6}\right)$

(v) $\delta\left(-2\,\middle|\,\frac{1}{2},\frac{1}{4},\frac{1}{4}\right)$ (vi) $\delta\left(-2\,\middle|\,\frac{2}{3},\frac{2}{3},\frac{2}{3}\right)$.

We can verify that, in each case, the fundamental group $\pi_1(\partial T)$ is an infinite group (see also [p.68, 1]). This contradicts Proposition 2. Therefore S is rational or a ruled surface. Since $H^1(S;\boldsymbol{Z})\cong H^1(C;\boldsymbol{Z})\cong 0$ by Lemma 1, S is a rational surface. Thus $Pic\ (S)\cong H^2(S;\boldsymbol{Z})$ $(\cong H^2(C;\boldsymbol{Z}))$. Let K_S be the canonical divisor on S. Then we have

$$K_S=a_0C_0+\sum_{i=1}^{3}\left(\sum_{j_i=1}^{r_i}a_{i,j_i}\cdot C_{i,j_i}\right),$$

where a_0, a_{ij}, $\in\boldsymbol{Z}$ and C_0, C_{i,j_i} $(1\leqq i\leqq 3,\ 1\leqq j_i\leqq r_i)$ are the irreducible components of C associated with the vertices

$$\overset{n_0}{\circ}\quad\text{and}\quad\overset{-n_{i,j_i}}{\circ}$$

in Figure 4 respectively. By the adjunction formula, we have

$$\begin{cases} -2-n_0=\sum_{i=1}^{3}a_{i,1}+a_0n_0 \\ -2+n_{i,j_i-1}=a_{i,j_i-2}-a_{i,j_i-1}\cdot n_{i,j_i-1}+a_{i,j_i} \\ -2+n_{i,r_i}=a_{i,r_i-1}-a_{i,r_i}\cdot n_{i,r_i}, \end{cases}$$

wher $1\leqq i\leqq 3$, $1\leqq j_i\leqq r_i$.
We put $l_i=[n_{i,1},\ldots,n_{i,r_i}]$ and $q_i=[n_{i,2},\ldots,n_{i,r_i}]$.
We have then

$$\frac{q_i}{l_i}=\cfrac{1}{n_{i,1}-\cfrac{1}{n_{i,2}-\cfrac{1}{\ddots-\cfrac{1}{n_{i,r_i}}}}}$$

and $l_i\geqq r_i+1, l_i>q_i>0$, since $n_{i,j_i}\geqq 2$.
We can verify easily that

$$(*)\begin{cases} l_i(a_{i,1}+1)-q_i(a_0+1)=1 & (1\leqq i\leqq 3) \\ (a_0+1)n_0+\sum_{i=1}^{3}(a_{i,1}+1)=1. \end{cases}$$

Thus we have finally

$$(**)\quad (a_0+1)\left(n_0+\frac{q_1}{l_1}+\frac{q_2}{l_2}+\frac{q_3}{l_3}\right)+\frac{1}{l_1}+\frac{1}{l_2}+\frac{1}{l_3}=1.$$

Since $\pi_1(\partial T)$ is a finite group, we must have

$$\frac{1}{l_1}+\frac{1}{l_2}+\frac{1}{l_3}>1$$

(see [p. 35, 1]). Therefore $a_0+1=0$ and $\left(n_0+\frac{q_1}{l_1}+\frac{q_2}{l_2}+\frac{q_3}{l_3}\right)=0.$ Further,

$$-(a_0+1)=\left(\frac{1}{l_1}+\frac{1}{l_2}+\frac{1}{l_3}-1\right)\Big/\left(n_0+\frac{q_1}{l_1}+\frac{q_2}{l_2}+\frac{q_3}{l_3}\right)\in \boldsymbol{Z}.$$

Since $\frac{q_i}{l_i}>\frac{1}{l_i}$, we have $n_0<0$. Since $n_0\geqq -2$, we have $n_0=-1$ or -2.

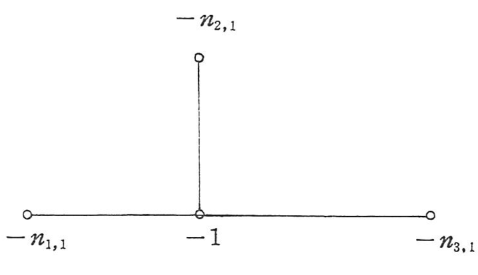
(Case 1) $n_0=-1$. We see that

$$\left(\frac{1}{l_1}+\frac{1}{l_2}+\frac{1}{l_3}-1\right)\Big/\left(\frac{q_1}{l_1}+\frac{q_2}{l_2}+\frac{q_3}{l_3}-1\right)$$

is an integer if and only if $q_1=q_2=q_3=1$. Thus $-(a_0+1)=1$, that is, $a_0=-2$. By the equation $(*)$ $a_{i,1}=-1$ $(1\leqq i\leqq 3)$, since $l_i\neq 0$. Since $[n_{i,2},\dots,n_{i,r_i}]=q_i=1$, we must have $r_i=1$. Therefore

$$K_S=-2C_0-C_{1,1}-C_{2,1}-C_{3,1},$$

and the graph $\Gamma(C)$ is of form



Since $b_1 = rank\ H^1(C; \mathbf{Z}) = 0$, $q = dim\ H^1(S; \mathcal{O}) = 0$. Thus by Noether's formula, we have

$$10 + 12p_g = K_S^2 + b_2,$$

where $p_g = dim\ H^2(S; \mathcal{O})$ and $b_2 = rank\ H^2(S; \mathbf{Z})$ $(= rank\ H^2(C; \mathbf{Z}))$. Since $b_2 = 4$ and $p_g = 0$, we have $K_S^2 = 6$. On the other hand, $K_S^2 = (2C_0 + C_{1,1} + C_{2,1} + C_{3,1})^2 = 8 - (n_{1,1} + n_{2,1} + n_{3,1})$. Thus $n_{1,1} + n_{2,1} + n_{3,1} = 3$. This is a contradiction, since $n_{i,1} \geqq 2$ $(1 \leqq i \leqq 3)$.

(Case 2) $n_0 = -2$. Since

$$\sum_{i=1}^{3} \frac{q_i + 1}{l_i} \leqq \sum_{i=1}^{3} \frac{l_i}{l_i} = 3,$$

we have

$$\sum_{i=1}^{3} \frac{q_i}{l_i} - 2 = 1 - \sum_{i=1}^{3} \left(1 - \frac{q_i}{l_i}\right) \leqq 1 - \sum_{i=1}^{3} \frac{1}{l_i} < 0.$$

Therefore the intersection matrix $(C_{i,j_i} \cdot C_{k,j_k})$ is negative definite. This is a contradiction, since $S - C$ is a Stein manifold. Therefore the proof of our proposition is completed. Therefore the curve C must be of the type (ε).

Q. E. D.

4. Proof fo Theorem.

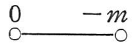
(4.1) The case where X is non-singular. By Proposition 1, we have $H_i(X; \mathbf{Z}) = 0$. for $i > 0$. Since X is a Stein manifold, X contains no compact analytic curve. By Proposition 4, the curve C is of the type (ε) in Table 1. Thus, (S, C) and X satisfy the assumptions (i), (ii) and (iii) in Proposition 3. Therefore X is biholomorphic (biregular) to $\mathbf{C}^2$.

(4.2) The case where X has singularities. Let $p = \{p_1, \dots, p_k\}$ $(k \geqq 1)$ be the set of singular points of X. Let U_i be a sufficiently small Stein neighbourhood of p_i in X and denote by ∂U_i the boundary of U_i. We put $U = \bigcup_{i=1}^{k} U_i$ and $\partial U = \bigcup_{i=1}^{k} \partial U_i$. Since f: $\mathbf{C}^2 \to X$ is proper finite, we can see that the fundamental group $\pi_1(\partial U_i)$ is a finite group, and thus $\pi_1(\partial U)$ is also a finite group. Therefore each p_i is a quotient singularity by Brieskorn. By Lemma 6, the curve C is of the type (γ'), (δ), (ε).

PROPOSITION 5. *If $H_1(\partial U; Z)=0$, then C is of the type (δ).*

PROOF, Since $\pi_1(\partial U_i)$ is a finite group and $H_1(\partial U_i; Z)=0$. each singularity p_i is the E_8-singularity,and thus a rational double point. Then the canonical divisor K_S on S can be defined. Moreover, $H^2(S; Z)$ is generated by the irreducible components of C.

First, assume that C is of the type (ε). As we have seen in the proof of Proposition 3, performing elementary transformations on C, we may assume that the groph $\Gamma(C)$ of C is of the form



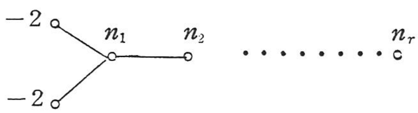
$(m \geqq 2)$

Let C_1(resp. C_2) be the irreducible component of C associated with the vertex $\overset{0}{\circ}$ (resp. $\overset{-m}{\circ}$). Let $\tilde{S}$ be the minimal resolution of S. Since p_i's are rational double points, $K_{\tilde{S}}=K_S$ can be written as follow

$$K_{\tilde{S}}=-(m+2)C_1-2C_2$$

Since p_i is the E_8-singularity and $b_2(\tilde{S})=2$, we hawe $b_2(S)=8k+2$. By the Noether formula, $(K_{\tilde{S}})^2=10-(8k+2)=4(m+2)-4m=8$. This is a contradiction, since $k \geqq 1$. Therefore C is not of the type (ε).

Next, assume that C is of the type (γ'), that is,



, $max\ \{n_1+1, n_2, \ldots, n_r\} \geqq 0$.

(i) The case of $max\{n_2, \ldots, n_r\} < 0$. Then $n_1+1 \geqq 0$. Let $C_0^{(1)}$, $C_0^{(2)}$ be the curves associated to the vertex $\overset{-2}{\circ}$ and C_i $(i \geqq 2)$ be that of the vertex $\overset{n_i}{\circ}$. Then $K_{\tilde{S}}$ can be written as follow

$$K_{\tilde{S}}=\alpha C_0^{(1)}+\beta C_0^{(2)}+\sum_{i=1}^{r} \alpha_i C_i,$$

where $\alpha, \beta, \alpha_i \in Z$.

We put $m_1=n_1$, $m_i=-n_i$ $(i \geqq 2)$. By the adjunction formula, we have

$$(*)\quad\begin{cases} \alpha_1 = 2\alpha = 2\beta \\ -m_1 - 2 = \alpha + \beta + m_1\alpha_1 + \alpha_2 \\ m_2 - 2 = \alpha_1 - m_2\alpha_2 + \alpha_3 \\ \\ m_r - 2 = \alpha_{r-1} - m_r\alpha_r \end{cases}$$

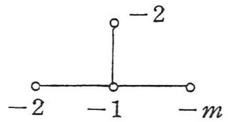
From (*), we have the relation

$$((m_1+1)[m_r, \dots, m_2] + [m_r, \dots, m_3])(\alpha_1 + 1) = -1.$$

Since $m_1 + 1 \geqq 0$, $m_j \geqq 2$ $(j \geqq 2)$ and $\alpha_1 + 1 \in \boldsymbol{Z}$, we must have

$$(m_1+1)[m_r, \dots m_2] + [m_r, \dots . m_3] = 1.$$

Therefore we must have $m_1 = -1$ and $r = 2$. Then $\Gamma(C)$ is the form



$(m \geqq 2)$

Further, blowing down the exceptional curve of first kind on C, we may finally assume that $C = C_1 \cup C_2$, where $(C_1)^2 = 0$, $(C_2)^2 = -m+2$ and $(C_1 \cdot C_2) = 2$. Since the topological type of X is preserved under the elementary transformation, we have $b_2(S) = 2$ and $b_2(\tilde{S}) = 8k+2$. We have then $-K_S = C_1 + C_2$. Let D be an irreducible exceptional curve on S with $D \neq C_2$. Then, by the adjunction formula, we have

(a) D is a non-singular rational curve with $(D)^2 = -2$ and $D \cdot (C_1 + C_2) = 0$, or

(b) D is an exceptional curve of first kind with $D \cdot (C_1 + C_2) = 1$.

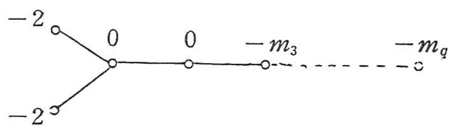
Let $\nu \colon \tilde{S} \to \boldsymbol{P}^1$ be a proper holomorphic mapping which has C_1 as a regular fiber (see 2). By the above (a), we find that the exceptional curve B associated with the resolution $\tilde{S} \to S$ is contained in the singular fibres $F_1, \dots, F_d$. Let $1 + e_i$ (resp. g_i) be the number of irreducible components of F_i (resp. those of F_i which are not contained in B). Then, we have

$$\begin{cases} 2 + \sum_{i=1}^{d} e_i = b_2(\tilde{S}) = 8k+2 \\ 2 + \sum_{i=1}^{d} (1 + e_i - g_i) = b_2(C) + b_2(B) = 8k+2 \end{cases}$$

Thus we have $g_i = 1$ $(i = 1, 2, \dots, d)$.

Since each singular fiber F_i contains an exceptional curve of first kind, we can see that there exists only one exceptional curve of first kind and the other irreducible components of F_i are all those of B. Taking account that each irreducible component of B is a non-singular rational curve with self-intersection number -2, the singular fiber F_i can be completely determined. This implies that each singularity p_i is the D_n-singularity. This is a contradiction.

(ii) The case of $max\{n_2, \ldots, n_r\} > 0$. Performing the elementary transformations on the curves associated to the vertices $\overset{n_i}{\circ}$ $(i \geqq 1)$, we may assume that $\Gamma(C)$ is of the form



By Lemma 5 in [21], we must have max $\{-m_3, -m_4, \ldots -m_q\} < 0$, that is, $m_i \geqq 2$ $(i \geqq 3)$. Let $C_0^{(1)}$, $C_0^{(2)}$ be the curves associated to the vertex $\overset{-2}{\circ}$, C_1, C_2 (resp. C_i $(i \geqq 3)$) be the curves associated to the vertex $\overset{0}{\circ}$ (resp. $\overset{m_i}{\circ}$). Then,

$$K_{\tilde{S}} = \alpha\, C_0^{(1)} + \beta\, C_0^{(2)} + \sum_{i=1}^{q} \alpha_i\, C_i, \text{ where}$$

$\alpha, \beta, \alpha_i \in \boldsymbol{Z}$.

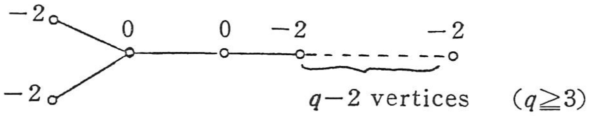
By the adjunction formula, we have

$$(**) \begin{cases} 0 = \alpha_1 = 2\alpha = 2\beta \\ -2 = \alpha + \beta + \alpha_2 \\ -2 = \alpha_1 + \alpha_3 \\ m_3 - 2 = \alpha_2 - m_3\, \alpha_3 + \alpha_4 \\ \\ m_q - 2 = \alpha_{q-1} - m_q\, \alpha_q \end{cases}$$

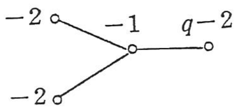
From (**), we have

$$([m_q, \ldots, m_3] - [m_q, \ldots, m_4])(\alpha_2 + 1) = 1$$

Since $m_i \geqq 2$, we must have $m_3 = m_4 = \cdots = m_q = 2$, and $\alpha_2 = 0$.
Then $\Gamma(C)$ is of the form



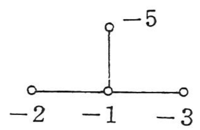
Performing elementary transformations on C, we may finally assume that $\Gamma(C)$ is of the from



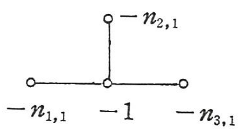
Then $(K_{\tilde{S}})^2=2+q\geqq 5$. On the other hand, since $b_2(S)=4$ and $q_2(\tilde{S})=8k+4$, by Noether's formula, we have $(K_{\tilde{S}})^2=6-8k<0$. This is a contradiction. Therefore the cueve C is of the type (δ). The proof is completed.

Q. E. D.

COROLLARY 1. *If $H_1(\partial U; Z)=0$, then $\Gamma(C)$ is of the form*



PROOF. By Proposition 5, C is of the type (δ). By the same argument as in the proof of Proposition 4 (Case 2), $\Gamma(C)$ must be of the form

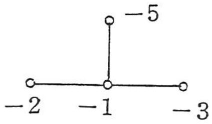


By Noether's formula, $10-(8k+4)=(K_{\tilde{S}})^2=8-(n_{1,1}+n_{2,1}+n_{3,1})$. Since $\pi_1(\partial T)$ is a finite group, we have

$$\frac{1}{n_{1,1}}+\frac{1}{n_{2,1}}+\frac{1}{n_{3,1}}>1.$$

Thus, $(n_{1,1}, n_{2,1}, n_{3,1})=(2,3,5)$ or $(2,2,n)$ $(n\geqq 2)$.
As we have seen in the proof of Proposition 5, the case $(2,2,n)$ can not occur.

Therefore $\Gamma(C)$ is of the form



and then $k=1$. In this case, the $\pi_1(\partial T)\cong SL(2,5)$ (the binary icosahedral group).

Q. E. D.

PROPOSITION 6 ([5]). *If $\pi_1(X-p)=\{1\}$, then $p=\phi$.*

REMARK: This proposition is proved in [5] under the assumption $\pi_1(\partial T)=SL(2,5)$. In fact, since $\pi_1(X-p)=\{1\}$ implies $H_1(\partial U;\, \boldsymbol{Z})=0$ by the Mayer-Vietoris sequence, as we have seen in Proposition 5 and Corollary 1, we conclude that $\pi_1(\partial T)\cong SL(2,5)$.

We will continue the proof of Theorem. Since $p\neq\phi$, $\pi_1(X-p)\neq 1$ by Proposition 6. Since $f: C^2\to X$ is proper finite and $C^2-f^{-1}(p)$ is simply connected, $\pi_1(X-p)$ is a finite group. Let us denote by $\widetilde{X-p}$ the universal covering space of $X-p$. Then we have (see [5] or [3]) that there exists a normal complex (affine) surface $\tilde{X}$ which contains $\widetilde{X-p}$ and a propor holomorphic mapping $\mu: \tilde{X}\to X$ such that the following diagram

$$\begin{array}{ccc} C^2 & \xrightarrow{\psi} & \tilde{X} \\ & f\searrow \quad \swarrow\mu & \\ & X & \end{array}$$

is commutative, where $\psi: C^2\to\tilde{X}$ is a proper holomorphic mapping. Moreover, $G=\pi_1(X-p)$ can be extended to $\tilde{X}$ as a group of analytic automorphisms of $\tilde{X}$ which has no pseudo-reflection. Since $\psi: C^2\to\tilde{X}$ is proper and $\pi_1(\tilde{X}-\mu^{-1}(p))=\pi_1(\widetilde{X-p})=1$, $\tilde{X}$ is non-singular, and thus $\tilde{X}$ is biholomorphic (biregular) to C^2 and $X=C^2/G$.

PROPOSITION 7 ([3]). *Let G be a finite groups of analytic automorphisms of C^2, and $\pi: C^2\to X=C^2/G$ the projection. Let $\Sigma\hookrightarrow X$ the branch locus of the finite covering π. Assume that (i) X is complex analytically compactifiable (ii) the closure $\bar{\Sigma}$ of Σ in an analytic compactification $\bar{X}$ of X is also an analytic subset of $\bar{X}$. Then G is conjugate with a finite subgroup of $GL(2, C)$.*

COROLLARY 2. *Let G be a finite group of polynomial automorphisms of* C^2. *Then G is conjugate with a finite subgroup of* $GL(2, C)$.

In our case, since X is analytically compactifiable and the branch locus is the point set p, we can apply Proposition 7. Therefore G is conjugate with a finite subgroup of $GL(2, C)$. This completes the proof of Theorem.

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References

[1] Coxeter, H. M. S. and W. O. J. Moser, Generators and relations for discrete groups, four edition, Ergebnisse der Math. und ihrer Grenzgebiete **14**, Springer-Verlag Berlin, Heidelberg, New York 1980.
[2] Furushima, M., Finite groups of polynomial automorphisms in the complex affine plane (I), Mem. Fac. Sci. Kyushu Univ., **36**(1982), 85-105.
[3] Furushima, M., Finite groups of polynomial automorphisms in C^n, Tohoku Math. J., **35**(1983), 415-424.
[4] Gurjar, R. V., Affine varieties dominated by C^2, Comment. Math. Helv., **55**(1980), 378-389.
[5] Gurjar, R. V. and A. R. Shastri, The fundamental group at infinity of affine surfaces, Comment. Math. Helv., **59**(1984), 459-484.
[6] Kobayashi, S. and T. Ochiai, Characterizations of complex projective spaces and hyperquadrics, J. Math. Kyoto Univ., **13**(1973), 31-47.
[7] Kodaira, K., On compact complex analytic surface I, Ann. Math., **71**(1960), 111-152.
[8] Kodaira, K., On the structure of compact complex analytic surface I, Amer. J. Math., **86** (1964), 751-758.
[9] Kodaira, K. and D. C. Spencer, A theorem of completeness of characteristic systems of complete continuous systems, Amer. J. Math., **81**(1959), 447-500.
[10] Milnor, J., Groups which act on S^n without fixed points, Amer, J. Math., **79**(1957), 623-630.
[11] Miyanishi, M. and T. Sugie, Affine surfaces containing cylinderlike open set, J. Math. Kyoto Univ., **20**(1980), 11-42.
[12] Miyanishi, M., Regular subrings of a polynomial ring, Osaka J. Math., **17**(1980), 329-338.
[13] Miyanishi, M., Normal affine subalgebras of a polynomial ring, preprint.
[14] Morrow, J., Minimal normal compactifications of C^2, Proceedings of the Conference of Complex Analysis, Rice Univ. Studies, **59**(1973), 97-112.
[15] Mumford, D., The topology of normal singularities of an algebraic surface and a criterion for simplicity, Publ. math. I. H. E. S., No. **9**(1961), 229-246.
[16] Nagata, M., On rational Surfaces II, Irreducible curves of arithmetic genus 0 or 1, Mem. Coll. Sci. Univ. Kyoto, Ser. A, **32**(1960), 351-370.
[17] Narasimhan, R., A note on Stein spaces and their normalization, Ann. Scuola Norm. Sup. Pisa, (3) **16** (1962), 327-333.
[18] Narasimhan, R., On the homology groups of Stein spaces, Invent. math., **2** (1966-67), 377-385.
[19] Nishino, T. and M. Suzuki, Sur les singularites essentielles et isolees des applications

holomorphes a valeurs dans une surface complexe, Publ. Res. Inst. Math. Soc. Kyoto Univ., **16** (1980), 461-497.
[20] Ramanujam, C. P., A topological characterization of the affine plane as an algebraic variety, Ann. Math., (2) **94** (1971), 69-88.
[21] Suzuki, M., Compactifications of $\boldsymbol{C}\times\boldsymbol{C}^*$ and $(\boldsymbol{C}^*)^2$, Tohoku Math. J., **30** (1979), 453-468.
[22] Suzuki, M., Compactifications of $\boldsymbol{C}^2$, $\boldsymbol{C}\times\boldsymbol{C}^*$, $(\boldsymbol{C}^*)^2$, private note (Japanese), 1979.

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