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REDUCTION OF SINGLE FUCHSIAN DIFFERENTIAL EQUATIONS TO HYPERGEOMETRIC SYSTEMS

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1. Introduction

Linear differential equations, which have only regular singular points in the whole complex plane, are called Fuchsian equations. Let $t=\lambda_j$ $(j=1, 2, \cdots, p)$ and $t=\infty$ *) be regular singularities. Then Fuchsian equation can be written in the form

$$(1.1) \qquad \phi^N y^{(N)} = \sum_{l=1}^{N} A_l(t)\phi^{N-l} y^{(N-l)},$$

where $\phi = \prod_{j=1}^{p}(t-\lambda_j)$ and the coefficients $A_l(t)$ $(l=1, 2, \cdots, N)$ are polynomials of degree at most $(p-1)l$. Differentiating both sides of (1.1) in $N(p-1)$ times and recalling the Leibniz rule, one can immediately obtain a differential equation of the form

$$(1.2) \qquad P_n(t)y^{(n)} = P_{n-1}(t)y^{(n-1)} + \cdots + P_1(t)y' + P_0(t)y,$$

where $n=Np$, $P_n(t)=\phi^N$ and $P_i(t)$ $(i=n-1, n-2, \cdots, 0)$ are polynomials of degree at most i. (1.2) is just of the extended form of Gauβ' hypergeometric equation.

$$t(t-1)y'' + \{(\alpha+\beta+1)t-\gamma\}y' + \alpha\beta y = 0.$$

So (1.2), regarded as a general form of Fuchsian differential equations, is merely called a *hypergeometric equation* after K. Okubo [6]. In the above paper K. Okubo showed without proof that (1.2) is equivalent to a system of differential equations called the *hypergeometric system*

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*) When a differential equation has only finite regular singularities, we can transfer one of them to infinity by a linear transformation of the independent variable $t'=1/(t-\lambda)$.

(1.3) $$(t-B)\frac{dX}{dt}=AX,$$

where $B=diag(\lambda_1, \lambda_2, \cdots, \lambda_n)$ and A is an n by n constant matrix. M. Hukuhara [4] gave a short proof of the equivalence in a distinct case, where $\lambda_i \neq \lambda_k (i \neq k)$, together with the consideration of relations between classes of solutions of (1.2) and (1.3). Later, K.Okubo published a monograph [7], in which he gave a complete and detailed proof by an algebraic method. However, both [4] and [7] give no final result on a form of the constant matrix A. In 1981 − 1982, the first author had a chance to give a lecture on "Connection Problems" at Professor R. Gérard's seminor of Strasbourg University, and then tried to prove the equivalence for the purpose of determining the explicit form of A and of seeing what types of hypergeometric equations correspond to the vanishing of elements or the degeneracy of eigenvalues of A. The first part (§1−§3) is based on the lecture notes made at that time, in which how to determine A in terms of coefficients of the polynomials $P_i(t)$ $(i=n-1, n-2, \cdots, 0)$ is shown explicitly in a constructive manner. After that, in order to use a computer in the actual calculation, the first author asked the second author to make a program for such an algorithm. In the last section, the program will be shown together with some examples illustrating its application.

Before the last section, for finding out good examples of hypergeometric equations and for the future plan of use, we supplement to this paper one section §4, where we show a method of reduction of systems of partial differential equations to systems of total differential equations.

2. Distinct case

We first consider the case in which $P_n(t)=\prod_{k=1}^{n}(t-\lambda_k)=\phi_n$, where $\lambda_i \neq \lambda_k (i \neq k)$, in (1.2), i.e., we shall prove that the hypergeometric equation

(2.1) $$\phi_n y^{(n)}=P_{n-1}(t)y^{(n-1)}+\cdots+P_0(t)y,$$

$P_i(t)$ being polynomials of degree at most i, can be reduced to the hypergeometric system (1.3) with

(2.2) $$B=diag(\lambda_1, \lambda_2, \cdots, \lambda_n)$$

and

(2.3)
$$A=\begin{bmatrix} \alpha_{11} & 1 & & & \\ \alpha_{21} & \alpha_{22} & 1 & & \mathbf{0} \\ \vdots & & \ddots & \ddots & \\ & & & & 1 \\ \alpha_{n1} & \alpha_{n2} & \cdots & & \alpha_{nn} \end{bmatrix}$$

In this case, using the notation

$$\phi_j=\prod_{k=1}^{j}(t-\lambda_k) \quad (i=1, 2, \cdots, n),$$

we put

(2.4)
$$\begin{cases} y_1=y, \\ y_2=\phi_1 y'+a_{2,0}(t)y, \\ \vdots \\ y_j=\phi_{j-1}y^{(j-1)}+a_{j,j-2}(t)y^{(j-2)}+\cdots+a_{j,0}(t)y, \\ \vdots \\ y_n=\phi_{n-1}y^{(n-1)}+a_{n,n-2}(t)y^{(n-2)}+\cdots+a_{n,0}(t)y, \end{cases}$$

the coefficients $a_{j,k}(t)$ being polynomials of degree at most k, and determine the polynomials $a_{j,k}(t)$ and constants α_{ij} so that the column vector $X=(y_1, y_2, \cdots, y_n)*$ satisfies the hypergeometric system with (2.2) and (2.3).

Consider the j-th element y_j:

(2.5)
$$\begin{aligned}(t-\lambda_j)y_j' = \phi_j y^{(j)} & \\ +(t-\lambda_j)\ (\phi_{j-1}'+a_{j,j-2}(t))y^{(j-1)} & \\ +(t-\lambda_j)\ (a_{j,j-2}'(t)+a_{j,j-3}(t))y^{(j-2)} & \\ \vdots & \\ +(t-\lambda_j)\ (a_{j,1}'(t)+a_{j,0}(t))y' . &\end{aligned}$$

Substituting

$$\phi_j y^{(j)}=y_{j+1}-a_{j+1,j-1}(t)y^{(j-1)}-\cdots-a_{j+1,0}(t)y$$

into the right hand side of (2.5), we have

(2.6)
$$\begin{aligned}(t-\lambda_j)y_j' &= y_{j+1} \\ &+[(t-\lambda_j)\ (\phi_{j-1}'+a_{j,j-2}(t))-a_{j+1,j-1}(t)]y^{(j-1)} \\ &+[(t-\lambda_j)\ (a_{j,j-2}'(t)+a_{j,j-3}(t))-a_{j+1,j-2}(t)]y^{(j-2)}\end{aligned}$$

$$
\begin{aligned}
&+\\
&\vdots\\
&+[(t-\lambda_j)(a'_{j,1}(t)+a_{j,0}(t))-a_{j+1,1}(t)]y'\\
&-a_{j+1,0}(t)y
\end{aligned}
$$

and then put

$$
(t-\lambda_j)\,(\phi'_{j-1}+a_{j,j-2}(t))-a_{j+1,j-1}(t)=\alpha_{jj}\phi_{j-1}\,, \tag{2.7}
$$

which determines the constant α_{jj} and the polynomial $a_{j,j-2}(t)$ from $a_{j+1,j-1}(t)$, i.e.,

$$
\begin{cases}
\alpha_{jj}=[-a_{j+1,j-1}(t)/\phi_{j-1}]_{t=\lambda_j}=-a_{j+1,j-1}(\lambda_j)/\prod_{k=1}^{j-1}(\lambda_j-\lambda_k),\\
a_{j,j-2}(t)=-\phi'_{j-1}+(\alpha_{jj}\phi_{j-1}(t)+a_{j+1,j-1}(t))/(t-\lambda_j).
\end{cases}
$$

Next, substituting

$$
\phi_{j-1}y^{(j-1)}=y_j-a_{j,j-2}(t)y^{(j-2)}-\cdots-a_{j,0}(t)y
$$

into the right hand side of (2.6), we have

$$
\begin{aligned}
(t-\lambda_j)y'_j=&\,y_{j+1}+\alpha_{jj}y_j\\
&+[(t-\lambda_j)(a'_{j,j-2}(t)+a_{j,j-3}(t))-a_{j+1,j-2}-\alpha_{jj}a_{j,j-2}(t)]y^{(j-2)}\\
&+\\
&\vdots\\
&+[(t-\lambda_j)(a'_{j,1}(t)+a_{j,0}(t))-a_{j+1,1}(t)-\alpha_{jj}a_{j,1}(t)]y'\\
&-[a_{j+1,0}(t)+\alpha_{jj}a_{j,0}(t)]y
\end{aligned}
$$

and then put

$$
(t-\lambda_j)(a'_{j,j-2}(t)+a_{j,j-3}(t))-a_{j+1,j-2}(t)-\alpha_{jj}a_{j,j-2}(t)=\alpha_{j,j-1}\phi_{j-2}\,, \tag{2.8}
$$

which determines the constant $\alpha_{j,j-1}$ and the polynomial $a_{j,j-3}(t)$ from $a_{j+1,j-2}(t)$, $a_{j,j-2}(t)$ and α_{jj}, i.e.,

$$
\begin{cases}
\alpha_{j,j-1}=-(a_{j+1,j-2}(\lambda_j)+\alpha_{jj}a_{j,j-2}(\lambda_j))/\prod_{k=1}^{j-2}(\lambda_j-\lambda_k),\\
a_{j,j-3}(t)=-a'_{j,j-2}(t)+(\alpha_{j,j-1}\phi_{j-2}+\alpha_{jj}a_{j,j-2}(t)+a_{j+1,j-2}(t))/(t-\lambda_j).
\end{cases}
$$

Continuing the above procedure, we have

(2.9) $$(t-\lambda_j)(a'_{j,j-k}(t)+a_{j,j-k-1}(t))=a_{j+1,j-k}(t)+\sum_{l=0}^{k-2}\alpha_{j,j-l}a_{j-l,j-k}(t)+\alpha_{j,j-k+1}\phi_{j-k}$$

for $k=2, 3, \cdots, j-1$, successively, which determines the constant $\alpha_{j,j-k+1}$ and the polynomial $a_{j,j-k-1}(t)$ for $k=2, 3, \cdots, j-1$, successively, and lastly we put

(2.10) $$-[a_{j+1,0}(t)+\sum_{l=0}^{j-2}\alpha_{j,j-l}a_{j-l,0}(t)]=\alpha_{j1}.$$

In particular, for the n-th element y_n we use the hypergeometric equation (2.1) in the first stage of above calculations:

$$\begin{aligned}(t-\lambda_n)y'_n&=\phi_n y^{(n)}+(t-\lambda_n)(\phi'_{n-1}+a_{n,n-2}(t))y^{(n-1)}+\cdots\\&=[(t-\lambda_n)(\phi'_{n-1}+a_{n,n-2}(t))+P_{n-1}(t)]y^{(n-1)}\\&\quad+[(t-\lambda_n)(a'_{n,n-2}(t)+a_{n,n-3}(t))+P_{n-2}(t)]y^{(n-2)}\\&\quad+\\&\quad\vdots\\&\quad+[(t-\lambda_n)(a'_{n,1}(t)+a_{n,0}(t))+P_1(t)]y'\\&\quad+P_0(t)y.\end{aligned}$$

Hence, we have the required formulas for $j=n$ by replacing the $a_{n+1,n-k}(t)$ by $-P_{n-k}(t)$ $(k=1, 2, \cdots, n)$ in (2.7), (2.9) and (2.10).

We here summarize above results in the following:

(2.11) $$\begin{cases}(\mathrm{i})_j\quad (t-\lambda_j)(a_{j,j-2}(t)+\phi'_{j-1})=a_{j+1,j-1}(t)+\alpha_{jj}\phi_{j-1}\\ \qquad\qquad (j=2, 3, \cdots, n-1),\\ (\mathrm{ii})_j\quad (t-\lambda_j)(a_{j,j-k-1}(t)+a'_{j,j-k}(t))\\ \qquad =a_{j+1,j-k}(t)+\sum_{l=0}^{k-2}\alpha_{j,j-l}a_{j-l,j-k}(t)+\alpha_{j,j-k+1}\phi_{j-k}\\ \qquad\qquad (k=2, 3, \cdots, j-1;\ j=3, 4, \cdots, n-1)\\ (\mathrm{iii})_j\quad -a_{j+1,0}(t)-\sum_{l=0}^{j-2}\alpha_{j,j-l}a_{j-l,0}(t)=\alpha_{j1}\\ \qquad\qquad (j=1, 2, \cdots, n-1),\end{cases}$$

(2.12) $$\begin{cases}(\mathrm{i})_n\quad (t-\lambda_n)(a_{n,n-2}(t)+\phi'_{n-1})=-P_{n-1}(t)+\alpha_{nn}\phi_{n-1},\\ (\mathrm{ii})_n\quad (t-\lambda_n)(a_{n,n-k-1}(t)+a'_{n,n-k}(t))\\ \qquad =-P_{n-k}(t)+\sum_{l=0}^{k-2}\alpha_{n,n-l}a_{n-l,n-k}(t)+\alpha_{n,n-k+1}\phi_{n-k}\\ \qquad\qquad (k=2, 3, \cdots, n-1),\\ (\mathrm{iii})_n\quad P_0(t)-\sum_{l=0}^{n-2}\alpha_{n,n-l}a_{n-l,0}(t)=\alpha_{n1}.\end{cases}$$

We interpret these formulas as follows: The right hand side of $(\mathrm{i})_j$ or $(\mathrm{ii})_j$ includes a factor $(t-\lambda_j)$, and hence the constant α_{jj} or $\alpha_{j,j-k+1}$ can be determined by putting $t=\lambda_j$ in the right hand side. After that, the right hand side divided by $(t-\lambda_j)$ gives the polynomial $a_{j,j-2}(t)$ or $a_{j,j-k-1}(t)$.

Now we shall explain the ordcr of calculation of the constants α_{ij} and the polynomials $a_{i,j}(t)$. First, using $(\mathrm{i})_n$ and $(\mathrm{i})_j$ $(j=n-1, n-2, \cdots, 2)$, we can determine

$$\alpha_{jj} \text{ and } a_{j,j-2}(t) \quad (j=n, n-1, \cdots, 2)$$

uniquely one after another and then, from $(\mathrm{iii})_1$, we have α_{11}. Next, using $(\mathrm{ii})_n$ and $(\mathrm{ii})_j$ $(j=n-1, n-2, \cdots, 3)$, where we put $k=2$, we can determine

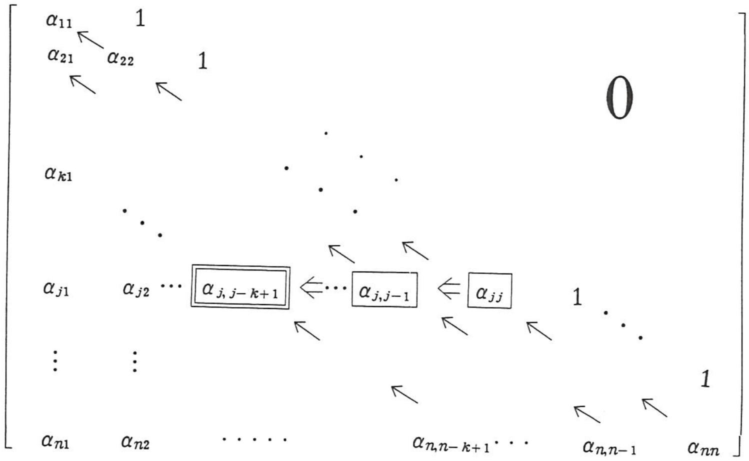
$$\alpha_{j,j-1} \text{ and } a_{j,j-3}(t) \quad (j=n, n-1, \cdots, 3)$$

uniquely one after another and then, from $(\mathrm{iii})_2$, we have α_{21}. We continue the above procedure by putting $k=3, 4, \cdots, n-1$ in $(\mathrm{ii})_n$ and $(\mathrm{ii})_j$, successively, obtaining

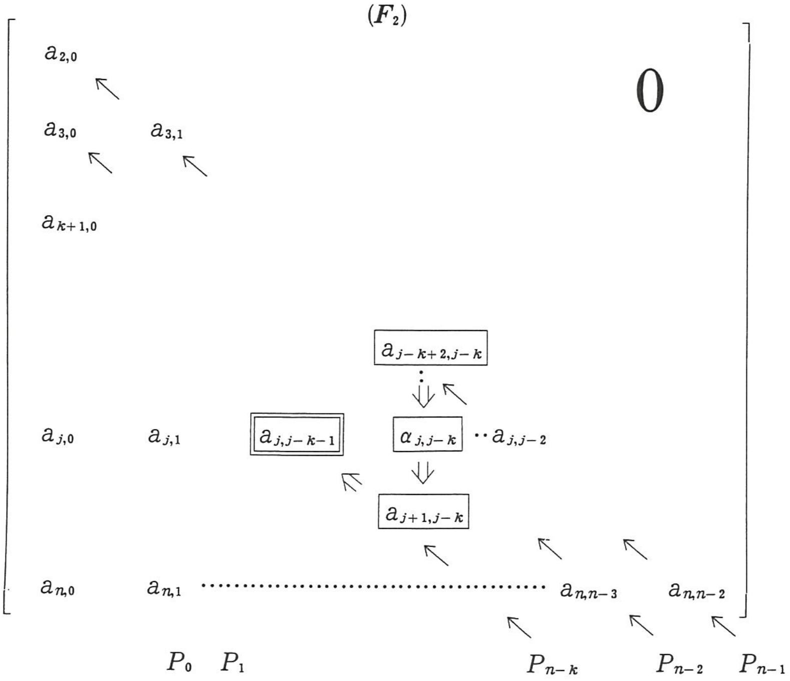
$$\alpha_{j,j-k+1} \text{ and } a_{j,j-k-1}(t) \ (j=n, n-1, \cdots, k+1),$$

together with α_{k1}. And at last, we have α_{n1} from $(\mathrm{iii})_n$. We illustrate the above order of determination in the figure, where the values are determined according to the arrow.

(F_1)



(F_2)



The values surrounded by a double rectangular are determined in terms of those surrouned by a simple rectangular.

Thus we have determined $n(n+1)/2$ constants of the hypergeometric system uniquely from the same number of constants in $P_j(t)(j=0, 1, \cdots, n-1)$. As a matter of course, the transformation from (2.1) to (1.3) with (2.2) and (2.3) is composed of polynomials in t, and hence it brings about no change of global behavior of solutions. In fact, to see this, we have only to verify that the characteristic exponents at all regular singularities cannot be changed modulo integers.

The hypergeometric equation (2.1) has the characteristic exponents at each regular singularity $t=\lambda_j (j=1, 2, \cdots, n)$ given by roots of the equation

$$[\rho]_n=\left[\frac{P_{n-1}(t)}{\phi_n}\right]_{t=\lambda_j}[\rho]_{n-1}$$

where $[\rho]_j=\rho(\rho-1)\cdots(\rho-j+1)$, i.e.,

$$\hat{\rho}_j=\frac{P_{n-1}(\lambda_j)}{\phi_n'(\lambda_j)}+n-1, 0, 1, \cdots, n-2 \quad (j=1, 2, \cdots, n). \tag{2.13}$$

This implies that near $t=\lambda_j$ there exist $(n-1)$ holomorphic solutions and one non-holomorphic solution with the characteristic exponent $\hat{\rho}_j$.

Near $t=\infty$ there exists n non-holomorphic solutions of the form

$$y^k(t)=t^{-\hat{\mu}_k}\sum_{s=0}^{\infty}h_k(s)t^{-s} \quad (k=1, 2, \cdots, n),$$

where the characteristic exponents $\hat{\mu}_k$ are given by roots of the equation

$$[-\mu]_n=\left[\frac{P_{n-1}(t)}{\phi_n}t\right]_{t=\infty}[-\mu]_{n-1}+\cdots+\left[\frac{P_0(t)}{\phi_n}t^n\right]_{t=\infty},$$

i.e.,

$$[-\mu]_n=P_{n-1}^0[-\mu]_{n-1}+P_{n-2}^0[-\mu]_{n-2}+\cdots+P_0^0, \tag{2.14}$$

where $P_j(t)=P_j^0t^j+\cdots \quad (j=0, 1, \cdots, n-1)$.

Here we make a remark on the Fuchs relation. From (2.13) we have

$$\begin{aligned}\sum_{j=1}^{n}\hat{\rho}_j&=n(n-1)+\sum_{j=1}^{n}\frac{P_{n-1}(\lambda_j)}{\phi_n'(\lambda_j)}\\&=n(n-1)+\frac{1}{2\pi i}\oint\frac{P_{n-1}(t)}{\phi_n(t)}dt\\&=n(n-1)+P_{n-1}^0.\end{aligned}$$

From (2.14) we also have

$$\sum_{k=1}^{n}\hat{\mu}_k=-\frac{n(n-1)}{2}-P_{n-1}^0.$$

Hence we obtain

$$\sum_{j=1}^{n}\hat{\rho}_j+\sum_{k=1}^{n}\hat{\mu}_k=\frac{n(n-1)}{2}, \tag{2.15}$$

which is just the Fuchs relation.

On the other hand, as for the hypergeometric system, it is well-known that near each singularity $t=\lambda_j$ $(j=1, 2, \cdots, n)$ there exist $(n-1)$ holomorphic solutions and one non-holomorphic solution whose characteristic exponent is equal to the corresponding diagonal element of A, i.e., a_{jj}, and near $t=\infty$ there exist n non-holomorphic solutions with the characteristic exponents μ_k $(k=1, 2, \cdots, n)$, which are eigenvalues of A, i.e., roots of

$$\det|A+\mu I|=0. \tag{2.16}$$

The α_{jj} have been determined by $(\mathrm{i})_n$, $(\mathrm{i})_j$ $(j=n-1, n-2, \cdots, 2)$ and $(\mathrm{iii})_1$. Now, multiplying $(\mathrm{i})_j$ by $\prod_{k=j+1}^{n}(t-\lambda_k)$, we have

$$\begin{cases} -\prod_{k=2}^{n}(t-\lambda_k)a_{2,0}(t)=\alpha_{11}\prod_{k=2}^{n}(t-\lambda_k), \\ -\prod_{k=j}^{n}(t-\lambda_k)a_{j,j-2}(t)+\prod_{k=j}^{n}(t-\lambda_k)\phi'_{j-1}=\prod_{k=j+1}^{n}(t-\lambda_k)a_{j+1,j-1}(t)+\alpha_{jj}\prod_{k=j+1}^{n}(t-\lambda_k)\phi_{j-1} \\ \qquad\qquad (j=2, 3, \cdots, n-1), \\ P_{n-1}(t)+(t-\lambda_n)\phi'_{n-1}+(t-\lambda_n)a_{n,n-2}(t)=\alpha_{nn}\phi_{n-1}. \end{cases}$$

Summing them up, we have

$$\begin{aligned} &P_{n-1}(t)+(t-\lambda_n)\phi'_{n-1}+\cdots+\prod_{k=j}^{n}(t-\lambda_k)\phi'_{j-1}+\cdots+\prod_{k=2}^{n}(t-\lambda_k)\phi'_1 \\ &=\alpha_{nn}\phi'_{n-1}+\cdots+\alpha_{jj}\prod_{k=j+1}^{n}(t-\lambda_k)+\cdots+\alpha_{11}\prod_{k=2}^{n}(t-\lambda_k), \end{aligned}$$

whence it immediately follows that

$$\alpha_{jj}=\frac{P_{n-1}(\lambda_j)}{\phi'_n(\lambda_j)}+n-j=\hat{\rho}_j-(j-1) \quad (j=1, 2, \cdots, n). \tag{2.17}$$

In order to calculate explicit values of eigenvalues of (2.16), we pick up the coefficients of the highest degree from the identities $(\mathrm{i})_j$, $(\mathrm{ii})_j$ and $(\mathrm{iii})_j$ $(j=1, 2, \cdots, n)$: We have

$$\begin{cases} P^0_{n-1}+(n-1)+a^0_{n,n-2}=\alpha_{nn}, \\ P^0_{n-k}+(n-k)a^0_{n,n-k}+a^0_{n,n-k-1}=\sum_{l=0}^{k-2}\alpha_{n,n-l}a^0_{n-l,n-k}+\alpha_{n,n-k-1} \\ \qquad\qquad (k=2, 3, \cdots, n-1), \\ P^0_0=\sum_{l=0}^{n-2}\alpha_{n,n-l}a^0_{n-l,0}+\alpha_{n1}, \end{cases}$$

and for $j=n-1, n-2, \cdots, 2$,

$$\begin{cases} a^0_{j,j-2}+(j-1)=a^0_{j+1,j-1}+\alpha_{jj}, \\ a^0_{j,j-k+1}+(j-k)a^0_{j,j-k}=a^0_{j+1,j-k}+\sum_{l=0}^{k-2}\alpha_{j,j-l}a^0_{j-l,j-k}+\alpha_{j,j-k+1} \\ \qquad\qquad (k=2, 3, \cdots, j-1), \\ -a^0_{j+1,0}-\sum_{l=0}^{j-2}\alpha_{j,j-l}a^0_{j-l,0}=\alpha_{j1}, \end{cases}$$

together with

$$-a^0_{2,0}=\alpha_{11},$$

where $a_{j,i}(t)=a^0_{j,i}t^i+\cdots$.

From above relations one can see that there holds

$$\begin{pmatrix} \alpha_{11}+\mu & 1 & & & & \mathbf{0} \\ \alpha_{21} & \alpha_{22}+\mu & 1 & & & \\ \vdots & & \ddots & \ddots & & \\ \alpha_{j1} & \alpha_{j2} & \cdots & \alpha_{jj}+\mu & 1 & \\ \vdots & \vdots & & & \ddots & 1 \\ \alpha_{n1} & \alpha_{n2} & \cdots\cdots & & \cdots & \alpha_{nn}+\mu \end{pmatrix} \begin{pmatrix} 1 & & & & & \mathbf{0} \\ a^0_{2,0} & 1 & & & & \\ \vdots & \ddots & \ddots & & & \\ a^0_{j,0} & a^0_{j,1} & \cdots & a^0_{j,j-2} & 1 & \\ \vdots & \vdots & & & \ddots & \\ a^0_{n,0} & a^0_{n,1} & \cdots\cdots & & a^0_{n,n-2} & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \mu & 1 & & & \mathbf{0} \\ \mu a^0_{2,0} & a^0_{2,0}+(1+\mu) & 1 & & \\ \vdots & & \ddots & \ddots & \\ \mu a^0_{j,0} & a^0_{j,0}+(1+\mu)a^0_{j,1} & \cdots\ a^0_{j,j-2}+(j-1+\mu) & 1 & \\ \vdots & & & \ddots & 1 \\ P^0_0+\mu a^0_{n,0} & P^0_1+a^0_{n,0}+(1+\mu)a^0_{n,1} & \cdots\cdots & & P^0_{n-1}+a^0_{n,n-2}+(n-1+\mu) \end{pmatrix}$$

$$= \begin{pmatrix} 1 & & & & & \mathbf{0} \\ a^0_{2,0} & 1 & & & & \\ \vdots & \ddots & \ddots & & & \\ a^0_{j,0} & a^0_{j,1} & \cdots & a^0_{j,j-2} & 1 & \\ \vdots & & & & \ddots & \\ a^0_{n,0} & a^0_{n,1} & \cdots & & a^0_{n,n-2} & 1 \end{pmatrix} \begin{pmatrix} \mu & 1 & & & & \mathbf{0} \\ 0 & \mu+1 & 1 & & & \\ & & \ddots & \ddots & & \\ & \mathbf{0} & & \mu+j-1 & 1 & \\ & & & & \ddots & 1 \\ P^0_0 & P^0_1 & \cdots & & \cdot & P^0_{n-1}+\mu+n-1 \end{pmatrix},$$

which implies that

$$\det|A+\mu I|=\det\begin{vmatrix} \mu & 1 & & & \mathbf{0} \\ & \mu+1 & 1 & & \\ \mathbf{0} & & \ddots & \ddots & \\ & & & \ddots & 1 \\ P^0_0 & P^0_1 & \cdots & \cdot & P^0_{n-1}+\mu+n-1 \end{vmatrix}$$

$$=(-1)^n\{[-\mu]_n-P^0_{n-1}[-\mu]_{n-1}-\cdots-P^0_0\}=0,$$

i.e.,

$$\mu_k=\hat{\mu}_k \qquad (k=1,2,\cdots,n). \tag{2.18}$$

For the hypergeometric system the Fuchs relation is just the trace relation, which can be verified from (2.15) as follows:

$$\sum_{j=1}^n a_{jj}=\sum_{j=1}^n \hat{\rho}_j-\frac{n(n-1)}{2}=-\sum_{k=1}^n \hat{\mu}_k$$
$$=-\sum_{k=1}^n \mu_k.$$

Consequently, we see from (2.17) and (2.18) that solutions of the reduced hypergeometric system behave exactly like those of the hypergeometric equation.

3. General case

Now we shall consider a general case in which, as seen in the example of the Fuchsian equation (1.1), $P_n(t)$ has multiple roots. Suppose that

$$P_n(t)=(t-\lambda_1)^{n_1}(t-\lambda_2)^{n_2}\cdots(t-\lambda_q)^{n_q}, \tag{3.1}$$

where it may be assumed without loss of generality that

$$\begin{cases} n_1+n_2+\cdots+n_q=n & (1\leqq q\leqq n), \\ 1\leqq n_q\leqq n_{q-1}\leqq\cdots\leqq n_1\leqq n. \end{cases}$$

In this case, in order that $t=\lambda_\nu$ $(1\leqq\nu\leqq q)$ is a regular singular point of (1.2), the functions

$$(t-\lambda_\nu)^i P_{n-i}(t)/P_n(t) \quad (i=1,2,\cdots,n)$$

must be holomorphic at $t=\lambda_\nu$, that is, the $P_{n-i}(t)$ must include the factor $(t-\lambda_\nu)^{n_\nu-i}$ for $1\leqq i\leqq n_\nu$. From this fact, we have

$$\begin{cases} P_{n-i}(t)=[\prod_{\nu=1}^{q}(t-\lambda_\nu)^{n_\nu-i}]\hat{P}_{n-i}(t) & (0<i\leqq n_q), \\ P_{n-i}(t)=[\prod_{\nu=1}^{k-1}(t-\lambda_\nu)^{n_\nu-i}]\hat{P}_{n-i}(t) & (n_k<i\leqq n_{k-1};k=q,q-1,\cdots,2), \\ P_{n-i}(t)=\hat{P}_{n-i}(t) & (n_1<i\leqq n), \end{cases} \tag{3.2}$$

where for $n_k < i \leqq n_{k-1}$ $(k=1, 2, \cdots, q+1; n_0=n,\ n_{q+1}=0)$ the $\hat{P}_{n-i}(t)$ are polynomials of degree at most

$$(n-i)-(n_1+n_2+\cdots+n_{k-1}-i(k-1))=n-N_{k-1}+i(k-2).$$

In the above and hereafter use is made of the notation

$$\begin{cases} N_k=n_1+n_2+\cdots+n_k \quad (k=1, 2, \cdots, q), \\ N_0=0,\ N_q=n. \end{cases}$$

Now we shall show that by means of the transformation (2.4) the hypergeometric equation (1.2) with (3.1) and (3.2) can be reduced to the hypergeometric system (1.3) with

$$B=diag(\overbrace{\lambda_1, \cdots, \lambda_1}^{n_1}, \overbrace{\lambda_2, \cdots, \lambda_2}^{n_2}, \cdots, \overbrace{\lambda_q, \cdots, \lambda_q}^{n_q}) \tag{3.3}$$

and

$$A=\left[\begin{array}{c|c|c|c} A_1 & 1 & & \mathbf{0} \\ \hline & A_2 & 1 & \\ \hline & & \ddots & 1 \\ \hline a_{ij} & & & A_q \end{array}\right], \tag{3.4}$$

where each A_k $(k=1, 2, \cdots, q)$ is n_k by n_k matrix and is so-called companion matrix of the form

$$A_k=\begin{bmatrix} 0 & 1 & & & \mathbf{0} \\ & 1 & 1 & & \\ & & \ddots & \ddots & \\ \mathbf{0} & & & n_k-2 & 1 \\ \beta_{k1} & \beta_{k2} & \cdots\cdots & & \beta_{kn_k}+n_k-1 \end{bmatrix}. \tag{3.5}$$

The hypergeometric equation includes

$$\sum_{k=1}^{q+1}\sum_{i=n_k+1}^{n_{k-1}}\{n-N_{k-1}+i(k-2)+1\}$$
$$=\sum_{k=1}^{q+1}\{(n+1-N_{k-1})(n_{k-1}-n_k)+(k-2)(n_{k-1}+n_k+1)(n_{k-1}-n_k)/2\}$$
$$=n(n+2)/2-(\sum_{k=1}^{q}n_k^2)/2$$

constants, which determines uniquely the same number of constants of A.

To see this, we here introduce the following notation:

$$(3.6)\qquad \begin{cases} f_k^i=\prod_{\nu=1}^{n}(t-\lambda_\nu)^{n_\nu-i}, \\ f_k^i=f_k^{i+1}\psi_k, \text{ where } \psi_k=\prod_{\nu=1}^{k}(t-\lambda_\nu), \\ (f_k^i)'=f_k^{i+1}g_k^i, \text{ where } g_k^i=\sum_{\nu=1}^{k}(n_\nu-i)\prod_{\substack{\nu'=1\\ \nu'\neq\nu}}^{k}(t-\lambda_{\nu'}), \\ \qquad\qquad (k=1,2,\cdots,q). \end{cases}$$

Then (3.2) can be written in the form

$$(3.7)\qquad \begin{cases} P_{n-i}(t)=f_{k-1}^i\hat{P}_{n-i}(t) \quad (n_k<i\leqq n_{k-1};\ k=q+1,q,\cdots,2), \\ P_{n-i}(t)=\hat{P}_{n-i}(t) \quad (n_1<i\leqq n), \end{cases}$$

and in (2.4) the ϕ_j can be written as follows:

$$(3.8)\qquad \begin{cases} \phi_j=f_k^0(t-\lambda_k)^{j-N_k}=f_k^1\psi_k(t-\lambda_k)^{j-N_k}=f_k^1\psi_{k-1}(t-\lambda_k)^{j+1-N_k}, \\ \phi_j'=f_k^1(t-\lambda_k)^{j-N_k}\{(t-\lambda_k)g_{k-1}^0+(j-N_{k-1})\psi_{k-1}\} \end{cases}$$
$$(N_{k-1}<j\leqq N_k;\ k=1,2,\cdots,q).$$

Using the above notation, we first calculate α_{jj} and $a_{j,j-2}(t)$ from $(\mathrm{i})_j$ $(j=n,n-1,\cdots,2)$. From $(\mathrm{i})_n$ we have

$$a_{n,n-2}(t)=(t-\lambda_q)^{-1}\{\alpha_{nn}\phi_{n-1}-P_{n-1}(t)\}-\phi_{n-1}'$$
$$=(t-\lambda_q)^{-1}\{\alpha_{nn}f_q^0(t-\lambda_q)^{-1}-f_q^1\hat{P}_{n-1}(t)\}$$
$$-f_q^1(t-\lambda_q)^{-1}\{(t-\lambda_q)g_{q-1}^0+(n_q-1)\psi_{q-1}\}$$
$$=f_q^1(t-\lambda_q)^{-1}\{(\alpha_{nn}-n_q+1)\psi_{q-1}-\hat{P}_{n-1}(t)-(t-\lambda_q)g_{q-1}^0\}$$

whence we here put $t=\lambda_q$ in the brackets of the right hand side, obtaining

$$(3.9)\qquad \begin{cases} \alpha_{nn}=n_q-1+\hat{P}_{n-1}(\lambda_q)/\psi_{q-1}(\lambda_q), \\ a_{n,n-2}(t)=f_q^1\hat{a}_{n,n-2}(t), \end{cases}$$

where $\hat{a}_{n,n-2}(t)$ is a polynomial of the form

$$(3.10)\qquad \hat{a}_{n,n-2}(t)=(t-\lambda_q)^{-1}\{(\alpha_{nn}-n_q+1)\psi_{q-1}-\hat{P}_{n-1}(t)-(t-\lambda_q)g_{q-1}^0\}.$$

Next, let $N_{q-1}<j\leqq N_q-1$. In $(\mathrm{i})_j$ of (2.7) we put

$$(3.11)\qquad a_{j,j-2}(t)=f_q^1(t-\lambda_q)^{j-N_q}\hat{a}_{j,j-2}(t)$$

and then, using (3.8), obtain

$$\begin{aligned} &f_q^1(t-\lambda_q)^{j-N_q}\hat{a}_{j,j-2}(t) \\ &=f_q^1(t-\lambda_q)^{j-N_q}\{\hat{a}_{j+1,j-1}(t)-g_{q-1}^0\} \\ &+f_q^1(t-\lambda_q)^{j-1-N_q}\psi_{q-1}\{\alpha_{jj}-(j-N_{q-1}-1)\} \end{aligned}$$

whence it follows that

$$(3.12)\qquad \begin{cases} \alpha_{jj}=j-N_{q-1}-1, \\ \hat{a}_{j,j-2}(t)=\hat{a}_{j+1,j-1}(t)-g_{q-1}^0=\hat{a}_{n,n-2}(t)-(n-j)g_{q-1}^0 \end{cases}$$

$$(N_{q-1}<j\leqq N_q-1).$$

For $j=N_{q-1}$, since $a_{j+1,j-1}(t)=f_{q-1}^1\hat{a}_{j+1,j-1}(t)$ from (3.11) and

$$\begin{cases} \phi_{j-1}=f_{q-1}^0(t-\lambda_{q-1})^{-1}=f_{q-1}^1\psi_{q-1}(t-\lambda_{q-1})^{-1}=f_{q-1}^1\psi_{q-2}, \\ \phi_{j-1}'=f_{q-1}^1(t-\lambda_{q-1})^{-1}\{(t-\lambda_{q-1})g_{q-2}^0+(n_{q-1}-1)\psi_{q-2}\} \end{cases}$$

from (3.8), we have from $(\mathrm{i})_j$

$$a_{j,j-2}(t)=f_{q-1}^1(t-\lambda_{q-1})^{-1}\{\hat{a}_{j+1,j-1}(t)+(\alpha_{jj}-(n_{q-1}-1))\psi_{q-2}\}-f_{q-1}^1g_{q-2}^0,$$

which implies that

$$(3.13)\qquad \begin{cases} \alpha_{jj}=n_{q-1}-1-\hat{a}_{j+1,j-1}(\lambda_{q-1})/\psi_{q-2}(\lambda_{q-1}), \\ a_{j,j-2}(t)=f^1_{q-1}\hat{a}_{j,j-2}(t) \quad (j=N_{q-1}), \end{cases}$$

where

$$(3.14)\qquad \hat{a}_{j,j-2}(t)=(t-\lambda_{q-1})^{-1}\{\hat{a}_{j+1,j-1}(t)+(\alpha_{jj}-(n_{q-1}-1))\psi_{q-2}\}-g^0_{q-2} \quad (j=N_{q-1}).$$

From the above one can guess that for $N_{k-1}<j\leqq N_k$ $(k=q-1, q-2, \cdots, 1)$

$$(3.15)\qquad a_{j,j-2}(t)=f^1_k(t-\lambda_k)^{j-N_k}\hat{a}_{j,j-2}(t).$$

This is indeed the case. In fact, for $N_{k-1}<j\leqq N_k$, $(\mathrm{i})_j$, i.e.,

$$a_{j,j-2}(t)=(t-\lambda_k)^{-1}\{a_{j+1,j-1}(t)+\alpha_{jj}\phi_{j-1}\}-\phi'_{j-1}$$

can be reduced to

$$\hat{a}_{j,j-2}(t)=\hat{a}_{j+1,j-1}(t)-g^0_{k-1}+(t-\lambda_k)^{-1}\psi_{k-1}(\alpha_{jj}-(j-N_{k-1}-1)).$$

Hence, by putting

$$(3.16)\qquad \alpha_{jj}=j-N_{k-1}-1,$$

we can determine the polynomial

$$(3.17)\qquad \begin{aligned} \hat{a}_{j,j-2}(t)&=\hat{a}_{j+1,j-1}(t)-g^0_{k-1} \\ &=\hat{a}_{N_k,N_k-2}(t)-(N_k-j)g^0_{k-1} \quad (N_{k-1}<j\leqq N_k). \end{aligned}$$

Moreover, for $j=N_{k-1}$, since $a_{j+1,j-1}(t)=f^1_{k-1}\hat{a}_{j+1,j-1}$, and

$$\begin{cases} \phi_{j-1}=f^1_{k-1}\psi_{k-2}, \\ \phi'_{j-1}=f^1_{k-1}(t-\lambda_{k-1})^{-1}\{(t-\lambda_k)g^0_{k-2}+(n_k-1)\psi_{k-2}\}, \end{cases}$$

we have from $(\mathrm{i})_j$

$$a_{j,j-2}(t)=f^1_{k-1}(t-\lambda_{k-1})^{-1}\{\hat{a}_{j+1,j-1}(t)+(\alpha_{jj}-n_{k-1}+1)\psi_{k-2}\}-f^1_{k-1}g^0_{k-2},$$

whence we can determine

$$\alpha_{jj}=n_{k-1}-1-\hat{a}_{j+1,j-1}(\lambda_{k-1})/\psi_{k-2}(\lambda_{k-1}), \tag{3.18}$$

obtaining

$$a_{j,j-2}(t)=f^1_{k-1}\hat{a}_{j,j-2}(t) \quad (j=N_{k-1}), \tag{3.19}$$

where

$$\hat{a}_{j,j-2}(t)=(t-\lambda_{k-1})^{-1}\{\hat{a}_{j+1,j-1}(t)+(\alpha_{jj}-n_{k-1}+1)\psi_{k-2}\}-g^0_{k-2}. \tag{3.20}$$

We have thus determined the polynomials $a_{j,j-2}(t)$ $(j=n, n-1, \cdots, 2)$, which are of the form (3.15), and also the constants α_{jj} $(j=n, n-1, \cdots, 2)$, together with $\alpha_{11}=-a_{2,0}(t)=0$, uniquely from $P_{n-1}(t)$. That is, in the figures $(\boldsymbol{F}_1)$ $(\boldsymbol{F}_2)$ in §2 we have determined the principal diagonal parts.

Now we proceed to the determination of the first subdiagonal parts, the second subdiagonal parts and so on in $(\boldsymbol{F}_1)$ $(\boldsymbol{F}_2)$. And then we can see that the polynomials $a_{j,j-i}(t)$ are expressed as

$$a_{j,j-i}(t)=f^{i-1}_{k}(t-\lambda_k)^{j-N_k}\hat{a}_{j,j-i}(t) \tag{3.21}$$
$$(N_{k-1}<j\leqq N_k;\ k=1, 2, \cdots, q),$$

where we understand that the factor $(t-\lambda_\nu)$ to the power non-positive integers is equal to 1, i.e.,

$$\begin{cases} (t-\lambda_k)^{n_k-N_k+j+1-i}\equiv 1 & (n_k-N_k+j+1\leqq i), \\ (t-\lambda_\nu)^{n_\nu-i+1}\equiv 1 & (n_\nu+1\leqq i). \end{cases}$$

For $i=2$, (3.12) are just the formulas (3.15), and moreover for $j=n+1$, (3.7) correspond to (3.12), i.e.,

$$a_{n+1,n+1-i}(t)=P_{n-i+1}(t)=f^{i-1}_q\hat{P}_{n-i+1}(t).$$

So, to see (3.21), we have only to carry out the proof by mathematical induction according to the subdiagonal order i. We here consider the rows $a_{j,j-i}(t)$ for

$N_{k-1}<j\leqq N_k$. In this case $(\mathrm{ii})_j$ can be written as

(3.22)
$$(t-\lambda_k)(a_{j,j-i-1}(t)+a'_{j,j-i}(t))$$
$$=a_{j+1,j-i}(t)+\sum_{l=0}^{i-2}\alpha_{j,j-l}a_{j-l,j-i}(t)+\alpha_{j,j-i+1}\phi_{j-i}$$
$$(N_{k-1}<j\leqq N_k).$$

Assuming that (3.21) are valid for $j=N_k+1$, we can verify by induction in i ($i=2,3,\cdots$) that the n_k by n_k matrix $\{\alpha_{j,j-i}\ i=0,1,\cdots,n_k-1\}$ is a companion matrix. In fact, for $i=2$ we have

(3.23)
$$(t-\lambda_k)a_{j,j-3}(t)-a_{j+1,j-2}(t)$$
$$=f_k^2(t-\lambda_k)^{j-N_k+1}[\{(\alpha_{jj}-j+N_k)\psi_{k-1}+g_k^1\}\hat{a}_{j,j-2}(t)-\psi_k\hat{a}'_{j,j-2}(t)]$$
$$+\alpha_{j,j-1}f_k^2(t-\lambda_k)^{j-N_k}\psi_{k-1}^2.$$

Since from the assumption $a_{j+1,j-2}(t)=f_k^2\hat{a}_{j+1,j-2}(t)$ for $j=N_k$, we have to put

(3.24)
$$\alpha_{j,j-1}=-\hat{a}_{j+1,j-2}(\lambda_k)/\psi_{k-1}^2(\lambda_k) \qquad (j=N_k)$$

and then we see that $a_{j,j-3}(t)$ can be determined in the form

(3.25)
$$a_{j,j-3}(t)=f_k^2\hat{a}_{j,j-3}(t) \qquad (j=N_k).$$

Substituting this into (3.23) for $j=N_k-1$, we obtain $\alpha_{j,j-1}=0$ and (3.21) for $j=N_k-1$. In fact, for $N_k-1\geqq j\geqq N_{k-1}+2$ (3.23) can be reduced to

$$(t-\lambda_k)\{\hat{a}_{j,j-3}(t)-\hat{a}_{j+1,j-2}(t)\}$$
$$+(t-\lambda_k)[\{(\alpha_{jj}-j+N_k)\psi_{k-1}+g_k^1\}\hat{a}_{j,j-2}(t)-\psi_k\hat{a}'_{j,j-2}(t)]+\alpha_{j,j-1}\psi_{k-1}^2$$
$$(j=N_k-1,N_k-2,\cdots,N_{k-1}+2),$$

whence we consequently obtain

(3.26)
$$\alpha_{j,j-1}=0 \qquad (j=N_k-1,N_k-2,\cdots,N_{k-1}+2).$$

For $j=N_{k-1}+1$, $a_{j,j-2}(t)=f_{k-1}^1\hat{a}_{j,j-2}(t)$ and $\phi_{j-2}=f_{k-1}^1\psi_{k-2}$, which do not include the factor $(t-\lambda_k)$. So in this case $\alpha_{j,j-1}$ can be determined by putting $t=\lambda_k$ in (3.22). And we also see that $a_{j,j-3}(t)$ includes the factor f_{k-1}^2.

The above calculation will be carried out for all blocks ($N_{k-1}<j\leqq N_k$; $k=q, q-1,\cdots,1$), and hence the first subdiagonal in ($\boldsymbol{F}_1$) ($\boldsymbol{F}_2$) are determined by $P_{n-1}(t)$.

Now assume that (3.21) are valid up to $(i-2)$-th subdiagonal parts in ($\boldsymbol{F}_2$). We then prove that (3.21) holds for $(i-1)$-th subdiagonal part in ($\boldsymbol{F}_2$), together with the determination of the constants $\alpha_{j,j-i+1}$.

From the assumption we have for $N_{k-1}<j\leqq N_k$

$$(3.27)\qquad \begin{cases} a'_{j,j-i}(t)=f^i_{k-1}(t-\lambda_k)^{j-i-N_{k-1}} \\ \qquad\times[\{g^{i-1}_{k-1}(t-\lambda_k)+\psi_{k-1}(j-i+1-N_{k-1})\}\hat{a}_{j,j-i}+\psi_{k-1}(t-\lambda_k)\hat{a}'_{j,j-i}], \\ a_{j-l,j-i}(t)=f^{i-l-1}_{k-1}(t-\lambda_k)^{j-i+1-N_{k-1}}\hat{a}_{j-l,j-i}(t) \\ \qquad =f^i_{k-1}\psi^{l+1}_{k-1}(t-\lambda_k)^{j-i+1-N_{k-1}}\hat{a}_{j-l,j-i}(t), \end{cases}$$

and

$$(3.28)\qquad \phi_{j-i}=f^0_{k-1}(t-\lambda_k)^{j-i-N_{k-1}}=f^i_{k-1}\psi^i_{k-1}(t-\lambda_k)^{j-i-N_{k-1}}.$$

Let $2\leqq i\leqq n_k$. Then, for given $a_{j+1,j-i}(t)=f^i_{k-1}(t-\lambda_k)^{n_k-i}\hat{a}_{j+1,j-i}(t)$ $(j=N_k)$, we can easily see from (3.22) and (3.27−8) that

$$(3.29)\qquad \begin{cases} \alpha_{j,j-i+1}=-\hat{a}_{j+1,j-i}(\lambda_k)/\psi^i_{k-1}(\lambda_k), \\ a_{j,j-i-1}(t)=f^i_k\hat{a}_{j,j-i-1}(t) \qquad (j=N_k). \end{cases}$$

Then, substituting (3.29) into (3.22) and continuing this procedure, we can determine $a_{j,j-i-1}(t)$ as the form (3.21). In fact, for $N_k-1\geqq j\geqq N_{k-1}+i$, we have

$$(t-\lambda_k)\{\hat{a}_{j,j-i-1}(t)+(g^{i-1}_{k-1}(t-\lambda_k)+\psi_{k-1}(j-i+1-N_{k-1}))\hat{a}_{j,j-i}(t)\ +\psi_{k-1}(t-\lambda_k)\hat{a}'_{j,j-i}(t)\}$$

$$=(t-\lambda_k)\{\hat{a}_{j+1,j-i}(t)+\sum_{l=0}^{i-2}\alpha_{j,j-l}\psi^{l+1}_{k-1}\hat{a}_{j-l,j-i}(t)\}+\alpha_{j,j-i+1}\psi^i_{k-1},$$

whence it immediatly follows that

$$(3.30)\qquad \alpha_{j,j-i+1}=0 \qquad (j=N_k-1, N_k-2,\cdots,N_{k-1}+i)$$

and $a_{j,j-i-1}(t)$ can be determined uniquely.

We have therefore verified that

$$A_k=\{\alpha_{ji};\ i,j=N_{k-1}+1, N_{k-1}+2,\cdots,N_k\}$$

$$=\begin{bmatrix} \alpha_{N_{k-1}+1,N_{k-1}+1} & 1 & & & \\ & \cdot & 1 & & 0 \\ & & \cdot & \cdot & \\ & & & \cdot & \cdot \\ 0 & & & \cdot & 1 \\ \alpha_{N_k,N_{k-1}+1} & \alpha_{N_k,N_{k-1}+2} & \cdots & & \alpha_{N_k,N_k} \end{bmatrix}$$

is just a companion matrix

As will be easily seen from (3.27) and (3.28), if $N_{k-1}<j\leqq N_{k-1}+i-1$ or if $i>n_k$, then the right member of (3.22) no longer includes the factor $(t-\lambda_k)$. In these cases we can determine the constants $\alpha_{j,j-i+1}$ by putting $t=\lambda_k$ in (3.22), and then obtain $a_{j,j-i-1}(t)$ in terms of dividing the right hand side of (3.22) by $(t-\lambda_k)$. Here it is remarked that as long as $i<n_1$, the factor f^i_{k-1} (naturally $f^i_{k-1}=f^i_h$ if $i<n_h$) is handed over in the next block.

As a matter of course, the above calculation can be carried out for all blocks ($N_{k-1}<j\leqq N_k$; $k=q,q-1,\cdots,1$). Therefore the $(i-1)$-th subdiagonal parts in ($\boldsymbol{F}_1$) ($\boldsymbol{F}_2$) are determined by $P_{n-i}(t)$. We have thus completed the proof of mathematical induction.

Lastly we shall make a short remark on the characteristic exponents. The hypergeometric equation with (3.1) and (3.2) has a fundamental set of solutions near each regular singular point $t=\lambda_\nu$ $(\nu=1,2,\cdots,q)$, which consists of $(n-n_\nu)$ holomorphic solutions and n_ν non-holomorphic solutions. The characteristic exponents are given by roots of the equation

$$(3.31)\qquad \begin{cases} [\rho]_{n-n_\nu}=0, \text{ i.e., } \rho=0,1,\cdots,n-n_\nu-1, \\ [\rho-n+n_\nu]_{n_\nu}=\sum_{i=1}^{n_\nu}\left[\dfrac{P_{n-i}(t)}{P_n(t)}(t-\lambda_\nu)^i\right]_{t=\lambda_\nu}[\rho-n+n_\nu]_{n_\nu-i}, \end{cases}$$

whose roots are denoted by

$$\hat{\rho}^\nu_1,\ \hat{\rho}^\nu_2,\ \cdots,\ \hat{\rho}^\nu_{n_\nu}\qquad (\nu=1,2,\cdots,q).$$

The coefficients in (3.31) are written as follows:

$$(3.32)\qquad \left[\frac{f^i_k}{f^0_q}(t-\lambda_\nu)^i\hat{P}_{n-i}(t)\right]_{t=\lambda_\nu}$$

$$=\frac{f_k^i(\lambda_\nu)}{f_q^i(\lambda_\nu)}\frac{\hat{P}_{n-i}(\lambda_\nu)}{(\psi_q'(\lambda_\nu))^i}$$

$$=\frac{\hat{P}_{n-i}(\lambda_\nu)}{\prod_{\substack{\mu=k+1\\ \mu\neq\nu}}^{q}(\lambda_\nu-\lambda_\mu)^{n_\mu}\prod_{\mu=1}^{k}(\lambda_\nu-\lambda_\mu)^i} \qquad (n_{k-1}<i\leqq n_k\leqq n_\nu).$$

From (3.31) we easily obtain

$$\sum_{j=1}^{n_\nu}\hat{\rho}_j^\nu=\frac{n(n-1)}{2}-\frac{(n-n_\nu-1)(n-n_\nu)}{2}+\frac{\hat{P}_{n-1}(\lambda_\nu)}{\psi_q'(\lambda_\nu)},$$

whence

$$\sum_{\nu=1}^{q}\sum_{j=1}^{n_\nu}\hat{\rho}_j^\nu=\frac{n(2n-1)}{2}-\frac{1}{2}\sum_{\nu=1}^{q}n_\nu^2+\sum_{\nu=1}^{q}\frac{\hat{P}_{n-1}(\lambda_\nu)}{\psi_q'(\lambda_\nu)}$$

$$=\frac{n(2n-1)}{2}-\frac{1}{2}\sum_{\nu=1}^{q}n_\nu^2+P_{n-1}^0.$$

Combining this with the sum of the characteristic exponents $\hat{\mu}_k$ at infinity, we obtain the Fuchs relation

$$\sum_{\nu=1}^{q}\sum_{j=1}^{n_\nu}\hat{\rho}_j^\nu+\sum_{k=1}^{n}\hat{\mu}_k=\frac{n^2}{2}-\frac{1}{2}\left(\sum_{\nu=1}^{q}n_\nu^2\right). \tag{3.33}$$

On the other hand, the hypergeometric system, in which B is of the from (3.3), has $(n-n_\nu)$ holomorphic solutions with the characterstic exponent 0 and n_ν non-holomorphic solutions, whose characteristic exponents are given by eigen-values of the matrix A_ν, near each $t=\lambda_\nu$ $(\nu=1, 2, \cdots, q)$.

For example, A_q is a companion matrix of the form (3.5), whose diagonal elements are given by (3.9) and (3.12), and elements of the last row are given by (3.29) for $j=n$:

$$\alpha_{n,n-i+1}=\frac{\hat{P}_{n-i}(\lambda_q)}{\psi_{q-1}^i(\lambda_q)}=\frac{\hat{P}_{n-i}(\lambda_q)}{(\psi_q'(\lambda_q))^i} \qquad (1\leqq i\leqq n_q).$$

Hence the eigenvalues of A_q are roots ρ_j^q $(j=1, 2, \cdots, n_q)$ of the equation

$$[\rho]_{n_q}=\sum_{i=1}^{n_q}\alpha_{n,n-i+1}[\rho]_{n_q-i}$$

$$=\sum_{i=1}^{n_q}\frac{\hat{P}_{n-i}(\lambda_q)}{(\psi_q'(\lambda_q))^i}[\rho]_{n_q-i}.$$

Comparing this with (3.31), we have

$$\rho_j^q=\hat{\rho}_j^q-n+n_q \qquad (j=1, 2, \cdots, n_q).$$

Like this, we can also calculate the eigenvalues of A_ν and see that by the transformation (2.4) the characteristic exponents at all regular singularities are not changed modulo integers.

In order to verify the Fuchs relation (trace relation), we have only to calculate the exact values of α_{jj} for $j=N_k$ ($k=1, 2, \cdots, q$) which are only defined by (3.18). From (3.17) we have

$$\hat{a}_{N_{k+1},N_k-1}(t)=\hat{a}_{N_{k+1},N_{k+1}-2}(t)-(n_{k+1}-1)g_k^0 .$$

Substituting this into

$$(t-\lambda_k)\hat{a}_{j,j-2}(t)=\hat{a}_{j+1,j-1}(t)+(\alpha_{jj}-n_k+1)\psi_{k-1}-(t-\lambda_k)g_{k-1}^0 \quad (j=N_k),$$

we have the following relations

$$(3.34)\qquad (t-\lambda_k)\beta_k=\beta_{k+1}+(\hat{\alpha}_k-n_k+1)\psi_{k-1}-(n_{k+1}-1)g_k^0-(t-\lambda_k)g_{k-1}^0$$
$$(k=1, 2, \cdots, q-1),$$

$$(3.35)\qquad (t-\lambda_q)\beta_q=-\hat{P}_{n-1}(t)+(\hat{\alpha}_q-n_q+1)\psi_{q-1}-(t-\lambda_q)g_{q-1}^0,$$

where we have used the simple notation

$$\hat{a}_{N_k,N_k-2}(t)=\beta_k, \quad \alpha_{N_kN_k}=\hat{\alpha}_k.$$

Multiplying (3.34) by $\prod_{\nu=k+1}^{q}(t-\lambda_\nu)$ and summing them over k, we obtain

$$(3.36)\qquad \psi_q\beta_1=-\hat{P}_{n-1}(t)+\sum_{k=1}^{q}(\hat{\alpha}_k-n_k+1)\prod_{\nu=k+1}^{q}(t-\lambda_\nu)\psi_{k-1}$$
$$-\sum_{k=1}^{q-1}(n_{k+1}-1)\prod_{\nu=k+1}^{q}(t-\lambda_\nu)g_k^0-\sum_{k=1}^{q}\prod_{\nu=k}^{q}(t-\lambda_\nu)g_{k-1}^0 .$$

Putting $t=\lambda_\nu$ in (3.36), we then have

$$\hat{P}_{n-1}(\lambda_\nu)=(\hat{\alpha}_\nu-n_\nu+1)\psi_q'(\lambda_\nu)-\sum_{k=\nu}^{q-1}(n_{k+1}-1)n_\nu\psi_q'(\lambda_\nu)-\sum_{k=\nu+1}^{q}n_\nu\psi_q'(\lambda_\nu)$$
$$=(\hat{\alpha}_\nu-n_\nu+1-n_\nu\sum_{k=\nu}^{q-1}n_{k+1})\psi_q'(\lambda_\nu),$$

from which it follows that

$$\hat{\alpha}_\nu=\frac{\hat{P}_{n-1}(\lambda_\nu)}{\psi_q'(\lambda_\nu)}+n_\nu-1+n_\nu(\sum_{k=\nu+1}^{q}n_k) \quad (\nu=1, 2, \cdots, q).$$

Let ρ_j^ν $(j=1, 2, \cdots, n_\nu)$ be eigenvalues of A_ν. Then we have

$$\begin{aligned}\sum_{j=1}^{n_\nu}\rho_j^\nu&=\hat{\alpha}_\nu+(1+2+\cdots+(n_\nu-2))\\&=\frac{\ddot{P}_{n-1}(\lambda_\nu)}{\psi_q'(\lambda_\nu)}+\frac{n_\nu(n_\nu-1)}{2}+n_\nu(\sum_{k=\nu+1}^{q}n_k)\end{aligned}$$

and hence

$$\begin{aligned}\sum_{\nu=1}^{q}\sum_{j=1}^{n_\nu}\rho_j^\nu&=\sum_{\nu=1}^{q}\frac{\hat{P}_{n-1}(\lambda_\nu)}{\psi_q'(\lambda_\nu)}+\frac{1}{2}\sum_{\nu=1}^{n}(n_\nu^2-n_\nu)+\frac{1}{2}(n^2-\sum_{\nu=1}^{q}n_\nu^2)\\&=P_{n-1}^0+\frac{n(n-1)}{2}\\&=-\sum_{k=1}^{n}\mu_k.\end{aligned}$$

Thus we have proved the trace relation for the hypergeometric system.

EXAMPLE 1. A typical equation of the distinct case is the Jordan-Pochhammer equation

$$\phi_n y^{(n)}=\sum_{l=1}^{n}(-1)^{l-1}\{\binom{\rho+l-1}{l}\phi_n^{(l)}(t)+\binom{\rho+l-1}{l-1}\psi_{n-1}^{(l-1)}(t)\}y^{(n-l)},$$

where

$$\begin{cases}\phi_n(t)=\prod_{j=1}^{n}(t-\lambda_j) \quad (\lambda_i\neq\lambda_k;\ i\neq k),\\ \psi_{n-1}(t)/\phi_n(t)=\sum_{j=1}^{n}a_j/(t-\lambda_j),\end{cases}$$

which has the Riemann scheme

$$P\left\{\begin{matrix}\lambda_1 & \cdots & \lambda_j & \cdots & \lambda_n & \infty & \\ 0 & \cdots & 0 & \cdots & 0 & -(\rho+1) & \\ 1 & \cdots & 1 & \cdots & 1 & -(\rho+2) & \\ \vdots & & \vdots & & \vdots & \vdots & t\\ n-2 & & n-2 & \cdots & n-2 & -(\rho+n-1) & \\ \rho+n-1+a_1 & \cdots & \rho+n-1+a_j & \cdots & \rho+n-1+a_n & -(\rho+a_1+\cdots+a_n) & \end{matrix}\right\}.$$

When $n=2$, this is just Gauß equation ($\lambda_1=0$, $\lambda_2=1$, $\rho=-(\alpha+1)$, $a_1=\alpha+1-\gamma$, $a_2=\gamma-\beta$).

EXAMPLE 2. A typical equation of the multiple case is the generalized hypergeometric equation of the Fuchsian type

$$t^{n-1}(t-1)y^{(n)}=\sum_{l=0}^{n-1}(c_l+b_lt)t^{l-1}y^{(l)} \qquad (c_0=0),$$

whose Riemann scheme becomes

$$P\begin{Bmatrix} 0 & 1 & \infty & \\ 0 & 0 & \mu_1 & \\ \rho_1 & 1 & \mu_2 & \\ \rho_2 & 2 & \mu_3 & t \\ \vdots & \vdots & \vdots & \\ \rho_{n-2} & n-2 & \mu_{n-1} & \\ \rho_{n-1} & \rho_n & \mu_n & \end{Bmatrix},$$

where $\rho_1, \rho_2, \cdots, \rho_{n-1}$ are roots of the equation

$$[\rho-1]_{n-1}+\sum_{l=0}^{n-1}c_l[\rho-1]_{l-1}=0,$$

$$\rho_n=c_{n-1}+b_{n-1}+n-1,$$

and $\mu_1, \mu_2, \cdots, \mu_n$ are roots of the equation

$$[-\mu]_n=\sum_{l=0}^{n-1}b_l[-\mu]_l.$$

Obviously, for $n=2$, the above is Gauß equation.

In this case, we put $\phi_j(t)=t^j$ ($j=1, 2, \cdots, n-1$) and then we immediately see that $a_{jk}(t)=\hat{a}_{jk}t^k$. From our method of reduction, we can determine the last row $(\alpha_{n1}, \alpha_{n2}, \cdots, \alpha_{nn})$ as follows:

$$\begin{cases} \alpha_{n,n}=b_{n-1}+c_{n-1}, \\ \alpha_{n,n-k+1}+\sum_{l=0}^{k-2}\alpha_{n,n-l}\hat{a}_{n-l,n-k}=b_{n-k}+c_{n-k} \quad (k=2, 3, \cdots, n-1, n), \end{cases}$$

together with

$$\begin{cases} \hat{a}_{n,n-2} = c_{n-1} - (n-1), \\ \hat{a}_{n,n-k-1} = c_{n-k} - (n-k)\hat{a}_{n,n-k} \quad (k=2, 3, \cdots, n-1). \end{cases}$$

In consequence, we obtain $B = diag(0, \cdots, 0, 1)$ and

$$A = \begin{bmatrix} 0 & 1 & & & & & \\ & 1 & 1 & & & \mathbf{0} & \\ & \mathbf{0} & 2 & \ddots & & & \\ & & & \ddots & 1 & & \\ & & & & n-3 & 1 & \\ \beta_1 & \beta_2 & & \cdots & \beta_{n-2} & \beta_{n-1} & 1 \\ \alpha_{n,1} & \alpha_{n,2} & & \cdots & & \alpha_{n,n-1} & \alpha_{n,n} \end{bmatrix},$$

where $\beta_k = -\hat{a}_{n,k-1}$ $(k=1, 2, \cdots, n-2)$ and $\beta_{n-1} = -\hat{a}_{n,n-2} + n - 2$.

REMARK. Let B be of a general form (3.3). Then it is easy to see that B is invariant under the transformation $T^{-1}BT$, where

(*i*) T is a non-singular diagonal matrix,

and

(*ii*) T is a block-diagonal matrix of the form

$$T = \begin{bmatrix} T_1 & & & 0 \\ & T_2 & & \\ & & \ddots & \\ 0 & & & T_q \end{bmatrix}, \qquad T_k \in GL(n_k, C).$$

From this fact, applying the transformation $X = TY$, we can always assign any values to $(n-1)$ non-zero elements of A and moreover we have

$$(t - B)\frac{dY}{dt} = \begin{bmatrix} J_1 & A_{12} & \cdots & A_{1q} \\ A_{21} & J_2 & \cdots & A_{2q} \\ \vdots & \vdots & \ddots & \vdots \\ A_{q1} & A_{q2} & \cdots & J_q \end{bmatrix} Y,$$

where the diagonal blocks J_k are Jordan canonical matrices. So, if necessary, we can rewrite the hypergeometric system derived so far in the above form.

4. Appell's system of hypergeometric equations

In case of several complex variables, there are some expressions of partial (total) differential equations, which is certainly Fuchsian, however, to oue knowledge, we can not yet find a general and explicit expression of Fuchiaan differential equations of several complex variables like that in case of one complex variable. As special functions of several complex variables, we only know Appell and Horn's hypergeometric functions of two complex variables **F**, **G**, **H**, etc., and their extended functions defined by Lauricella, etc. In [3], the authors give 34 partial differential equations of two complex variables x and y, all of which are of the form:

$$\begin{cases} \phi r+\hat{\psi}t+A_1s+B_1p+C_1q+D_1z=0, \\ \psi t+\hat{\phi}r+A_2s+B_2p+C_2q+D_2z=0, \end{cases}$$

where $p=\dfrac{\partial z}{\partial x}$, $q=\dfrac{\partial z}{\partial y}$, $r=\dfrac{\partial^2 z}{\partial x^2}$, $s=\dfrac{\partial^2 z}{\partial x\partial y}$, $t=\dfrac{\partial^2 z}{\partial y^2}$,

ϕ, $\hat{\phi}$, ψ, $\hat{\psi}$, A_1, A_2 are polynomials of degree at most 2 in x, y, B_1, B_2, C_1, C_2 are polynomials of degree at most 1 in x, y and D_1, D_2 are constants.

It is plausible that under the completely integrable condition, the above partial differential equation reduces to the hypergeometric equation (1.2) on the section $y=$const. or $x=$const. To see this, one needs a very heavy computation. So we here take up some examples.

By solving the above equation with respect to r and t, we can rewrite it in the form

$$\text{(4.1)} \qquad \begin{cases} \phi r=A_1s+B_1p+C_1q+D_1z, \\ \psi t=A_2s+B_2p+C_2q+D_2z. \end{cases}$$

First we consider the reduction of (4.1) into a system of total differential equations. There occur two cases,

$$\text{(i) } \phi\psi-A_1A_2=0, \quad \text{(ii) } \phi\psi-A_1A_2\neq 0.$$

In case (i), under the completely integrable condition, the solution space becomes a three dimensional vector space. A typical equation is Appell's $\mathbf{F}_1$, which is reduced to the system of total differential equations

$$dZ=\left(A\frac{dx}{x}+B\frac{dy}{y}+C\frac{dx}{x-1}+D\frac{d(y-1)}{y-1}+E\frac{d(x-y)}{x-y}\right)Z,$$

where A, B, C, D and E are 3 by 3 constant matrices. (See [5].) From this we can easily obtain the hypergeometric equation on the y-section

$$(4.2)\qquad \begin{aligned} &x(x-1)(x-y)z''' \\ &+\{(\gamma-\beta'+1)(x-1)(x-y)+(\alpha+\beta-\gamma+2)x(x-y)+(\beta+\beta'+1)x(x-1)\}z'' \\ &+(\beta+1)\{(2\alpha+2+\beta)x-(\alpha-\beta'+1)y-\gamma\}z'+\alpha\beta(\beta+1)z=0, \end{aligned}$$

which has the following Riemann scheme:

$$(4.3)\qquad P\left\{\begin{matrix} 0 & 1 & y & \infty & \\ 0 & 0 & 0 & \alpha & \\ 1 & 1 & 1 & \beta & x \\ \beta'-\gamma+1 & \gamma-\alpha-\beta & 1-\beta-\beta' & \beta+1 & \end{matrix}\right\}$$

Hence, (4.2) is just the Jordan-Pochhammer equation.

In this section, we shall treat of the case (ii). We moreover have to divide (ii) into four cases:

$$(\mathrm{ii})_1\ A_1\neq 0,\ A_2\neq 0, \qquad (\mathrm{ii})_2\ A_1=0,\ A_2\neq 0,$$

$$(\mathrm{ii})_3\ A_1\neq 0,\ A_2=0, \qquad (\mathrm{ii})_4\ A_1=A_2=0,$$

where both ϕ and ψ are assumed not to be zero.

We explain one method of reduction only in case $(\mathrm{ii})_1$ and apply it to $\mathbf{F}_2$, $\mathbf{F}_3$, $\mathbf{H}_2$ and $\mathbf{F}_4$.

Differentiating the first equation of (4.1) with respect to y and the second one with respect to x, respectively, we have

$$\begin{cases} \phi\dfrac{\partial s}{\partial x}-A_1\dfrac{\partial s}{\partial y}=(\dfrac{\partial A_1}{\partial y}+B_1)s+C_1t-\dfrac{\partial\phi}{\partial y}r+\dfrac{\partial B_1}{\partial y}p+(\dfrac{\partial C_1}{\partial y}+D_1)q+(\dfrac{\partial D_1}{\partial y})z, \\ \psi\dfrac{\partial s}{\partial y}-A_2\dfrac{\partial s}{\partial x}=(\dfrac{\partial A_2}{\partial x}+C_2)s+B_2r-\dfrac{\partial\psi}{\partial x}t+(\dfrac{\partial B_2}{\partial x}+D_2)p+(\dfrac{\partial C_2}{\partial x})q+(\dfrac{\partial D_2}{\partial x})z. \end{cases}$$

Then we solve the above equations by $\dfrac{\partial s}{\partial x}$ and $\dfrac{\partial s}{\partial y}$, and express them in terms of s, p, q and z. In fact, putting

$$\triangle=\phi\psi-A_1A_2, \tag{4.4}$$

we have

$$\begin{aligned} \triangle\frac{\partial s}{\partial x}=&\{(\frac{\partial A_1}{\partial y}+B_1)\psi+(\frac{\partial A_2}{\partial x}+C_2)A_1+(C_1+A_1\psi\frac{\partial}{\partial x}(\frac{1}{\psi}))A_2+(\frac{A_1B_2}{\phi}+\psi\phi\frac{\partial}{\partial y}(\frac{1}{\phi}))A_1\}s \\ &+\{\frac{\partial B_1}{\partial y}\psi+(\frac{\partial B_2}{\partial x}+D_2)A_1+(C_1+A_1\psi\frac{\partial}{\partial x}(\frac{1}{\psi}))B_2+(\frac{A_1B_2}{\phi}+\psi\phi\frac{\partial}{\partial y}(\frac{1}{\phi})B_1\}p \\ &+\{(\frac{\partial C_2}{\partial y}+D_1)\psi+\frac{\partial C_2}{\partial x}A_1+(C_1+A_1\psi\frac{\partial}{\partial x}(\frac{1}{\psi}))C_2+(\frac{A_1B_2}{\phi}+\psi\phi\frac{\partial}{\partial y}(\frac{1}{\phi}))C_1\}q \\ &+\{\frac{\partial D_1}{\partial y}\psi+\frac{\partial D_2}{\partial x}A_1+(C_1+A_1\psi\frac{\partial}{\partial x}(\frac{1}{\psi}))D_2+(\frac{A_1B_2}{\phi}+\psi\phi\frac{\partial}{\partial y}(\frac{1}{\phi}))D_1\}z \\ =&a_1s+b_1p+c_1q+d_1z, \end{aligned} \tag{4.5}$$

$$\begin{aligned} \triangle\frac{\partial s}{\partial y}=&\{(\frac{\partial A_1}{\partial y}+B_1)A_2+(\frac{\partial A_2}{\partial x}+C_2)\phi+(\frac{A_2C_1}{\psi}+\phi\psi\frac{\partial}{\partial x}(\frac{1}{\psi}))A_2+(B_2+A_2\phi\frac{\partial}{\partial y}(\frac{1}{\phi}))A_1\}s \\ &+\{\frac{\partial B_1}{\partial y}A_2+((\frac{\partial B_2}{\partial x}+D_2)\phi+(\frac{A_2C_1}{\psi}+\phi\psi\frac{\partial}{\partial x}(\frac{1}{\psi}))B_2+(B_2+A_2\phi\frac{\partial}{\partial y}(\frac{1}{\phi}))B_1\}p \\ &+\{(\frac{\partial C_1}{\partial y}+D_1)A_2+\frac{\partial C_2}{\partial x}+(\frac{A_2C_1}{\psi}+\phi\psi\frac{\partial}{\partial x}(\frac{1}{\psi}))C_2+(B_2+A_2\phi\frac{\partial}{\partial y}(\frac{1}{\phi}))C_1\}q \\ &+\{\frac{\partial D_1}{\partial y}A_2+\frac{\partial D_2}{\partial x}\phi+(\frac{A_2C_1}{\psi}+\phi\psi\frac{\partial}{\partial x}(\frac{1}{\psi}))D_2+(B_2+A_2\phi\frac{\partial}{\partial y}(\frac{1}{\phi}))D_1\}z \\ =&a_2s+b_2p+c_2q+d_2z. \end{aligned} \tag{4.6}$$

Now, in order to obtain a system of total differential equations, we put

$$(4.7)\qquad \begin{cases} z_1 = z, \\ z_2 = \phi p + \phi_1 z, \\ z_3 = \psi q + \psi_1 z, \\ z_4 = \triangle s + \xi p + \zeta q + \eta z. \end{cases}$$

The functions ϕ_1, ψ_1, ξ, ζ and η are determined as follows:

$$\begin{aligned} dz_2 &= \phi dp + p d\phi + \phi_1 dz + z d\phi_1 \\ &= (\phi r + \phi_1 p)dx + (\phi s + \phi_1 q)dy + p d\phi + z d\phi_1 \\ &= \frac{1}{\triangle}(A_1 dx + \phi dy) z_4 \\ &+ \{(B_1 + \phi_1 - \frac{A_1}{\triangle}\xi)\frac{z_2}{\phi} + (D_1 - \frac{A_1}{\triangle}\eta - \frac{\phi_1}{\phi}(B_1 + \phi_1 - \frac{A_1}{\triangle}\xi))z_1 + \underline{(C_1 - \frac{A_1}{\triangle}\zeta)}q\}dx \\ &+ \{-\frac{\xi}{\triangle}z_2 + \underline{(\phi_1 - \frac{\phi\zeta}{\triangle})}q + \underline{(\frac{\xi}{\triangle}\phi_1 - \frac{\phi}{\triangle}\eta)}z_1\}dy + p d\phi + z d\phi_1. \end{aligned}$$

$$\begin{aligned} dz_3 &= \frac{1}{\triangle}(\psi dx + A_2 dy) z_4 \\ &+ \{-\frac{\zeta}{\triangle}z_3 + \underline{(\psi_1 - \frac{\psi}{\triangle}\xi)}p + \underline{(\frac{\zeta}{\triangle}\psi_1 - \frac{\psi}{\triangle}\eta)}z_1\}dx \\ &+ (C_2 + \psi_1 - \frac{A_2}{\triangle}\zeta)\frac{z_3}{\psi} + (D_2 - \frac{A_2}{\triangle}\eta - \frac{\psi_1}{\psi}(C_2 + \psi_1 - \frac{A_2}{\triangle}\zeta))z_1 + \underline{(B_2 - \frac{A_2}{\triangle}\xi)}p\}dy \\ &+ q d\psi + z d\psi_1. \end{aligned}$$

In the above, we put

$$(4.8)\qquad \begin{cases} C_1 - \frac{A_1}{\triangle}\zeta = 0, \quad \text{i.e.,} \quad \zeta = \frac{C_1}{A_1}\triangle, \\ B_2 - \frac{A_2}{\triangle}\xi = 0, \quad \text{i.e.,} \quad \xi = \frac{B_2}{A_2}\triangle \end{cases}$$

and moreover,

(4.9)
$$\begin{cases} \phi_1 = \dfrac{\phi\zeta}{\triangle} = \dfrac{C_1}{A_1}\phi, \\ \psi_1 = \dfrac{\psi\xi}{\triangle} = \dfrac{B_2}{A_2}\psi, \\ \eta = \dfrac{\phi_1}{\phi}\xi = \dfrac{\psi_1}{\psi}\zeta = \dfrac{\xi\zeta}{\triangle} = (\dfrac{C_1}{A_1})(\dfrac{B_2}{A_2})\triangle. \end{cases}$$

Then, putting

(4.10)
$$X = \frac{C_1}{A_1}, \quad Y = \frac{B_2}{A_2},$$

we first obtain

(4.11)
$$dz_1 = pdx + qdy = (\frac{z_2 - \phi_1 z_1}{\phi})dx + \frac{(z_3 - \psi_1 z_1)}{\psi}dy$$
$$= (-Xdx - Ydy)z_1 + (\frac{dx}{\phi})z_2 + (\frac{dy}{\psi})z_3.$$

And, since

$$B_1 + \phi_1 - \frac{A_1}{\triangle}\xi = B_1 + X\phi - A_1 Y,$$

$$D_1 - \frac{A_1}{\triangle}\eta - \frac{\phi_1}{\phi}(B_1 + \phi_1 - \frac{A_1}{\triangle}\xi) = D_1 - B_1 X - \phi X^2,$$

$$-\frac{\phi_1}{\phi}d\phi + d\phi_1 = -Xd\phi + d(X\phi) = \phi dX,$$

we obtain

(4.12)
$$dz_2 = \{(D_1 - B_1 X - \phi X^2)dx + \phi dX\}z_1$$
$$+ \{(B_1 + X\phi - A_1 Y)\frac{dx}{\phi} - Ydy + \frac{d\phi}{\phi}\}z_2$$
$$+ \frac{1}{\triangle}(A_1 dx + \phi dy)z_4.$$

Similarly, we obtain

(4.13)
$$dz_3 = \{(D_2 - C_2 Y - \psi Y^2)dy + \psi dY\}z_1$$
$$+ \{(C_2 + Y\psi - A_2 X)\frac{dy}{\psi} - Xdx + \frac{d\psi}{\psi}\}z_3$$
$$+ \frac{1}{\triangle}(\psi dx + A_2 dy)z_4.$$

Now, substituting (4.1), (4.5) and (4.6) into

(4.14)
$$dz_4=\triangle ds+d\triangle s+\xi dp+pd\xi+\zeta dq+qd\zeta+\eta dz+zd\eta$$
$$=\triangle(\frac{\partial s}{\partial x}dx+\frac{\partial s}{\partial y}dy)+\xi rdx+\zeta tdy$$
$$+(d\triangle+\xi dy+\zeta dx)s+(d\xi+\eta dx)p+(d\zeta+\eta dy)q+d\eta z$$
$$=e_1s+e_2p+e_3q+e_4z$$
$$=\frac{e_1}{\triangle}z_4+\frac{1}{\psi}(e_3-\zeta\frac{e_1}{\triangle})z_3+\frac{1}{\phi}(e_2-\xi\frac{e_1}{\triangle})z_2$$
$$+\{e_4-\frac{e_1}{\triangle}\eta-\frac{\phi_1}{\phi}(e_2-\xi\frac{e_1}{\triangle})-\frac{\psi_1}{\psi}(e_3-\zeta\frac{e_1}{\triangle})\}z_1,$$

we have to calculate the coefficients of the above last formula. Taking account of the fact that

$$a_1+\zeta+\frac{\xi}{\phi}A_1=a_1+(\frac{C_1}{A_1}+\frac{A_1}{\phi}(\frac{B_2}{A_2}))\triangle$$
$$=(\frac{\partial A_1}{\partial y}+B_1)\psi+(\frac{\partial A_2}{\partial x}+C_2)A_1+A_1A_2\psi\frac{\partial}{\partial x}(\frac{1}{\psi})+A_1\psi\phi\frac{\partial}{\partial y}(\frac{1}{\phi})$$
$$+(A_2C_1+\frac{A_1^2B_2}{\phi})+(\frac{C_1}{A_1}+\frac{A_1}{\phi}(\frac{B_2}{A_2}))(\phi\psi-A_1A_2)$$
$$=(\frac{\partial A_1}{\partial y}+B_1+A_1\phi\frac{\partial}{\partial y}(\frac{1}{\phi}))\psi+(\frac{\partial A_2}{\partial x}+C_2+A_2\psi\frac{\partial}{\partial x}(\frac{1}{\psi}))A_1+\phi\psi(X+\frac{A_1}{\phi}Y),$$
$$a_2+\xi+\frac{\zeta}{\psi}A_2=(\frac{\partial A_1}{\partial y}+B_1+A_1\phi\frac{\partial}{\partial y}(\frac{1}{\phi}))A_2+(\frac{\partial A_2}{\partial x}+C_2+A_2\psi\frac{\partial}{\partial x}(\frac{1}{\psi}))\phi+\phi\psi(Y+\frac{A_2}{\psi}X),$$

we obtain

(4.15)
$$e_1=d\triangle+(a_1+\zeta+\frac{\xi}{\phi}A_1)dx+(a_2+\xi+\frac{\zeta}{\psi}A_2)dy$$
$$=d\triangle+(\frac{\partial A_2}{\partial x}+A_2\psi\frac{\partial}{\partial x}(\frac{1}{\psi})+C_2+Y\psi)\lambda+(\frac{\partial A_1}{\partial y}+A_1\phi\frac{\partial}{\partial y}(\frac{1}{\phi})+B_1+X\phi)\mu,$$

where we have put

(4.16)
$$\lambda=A_1dx+\phi dy,\quad \mu=\psi dx+A_2dy.$$

Similarly, we can obtain

(4.17) $$e_2=d\xi+(b_1+\frac{\xi}{\phi}B_1+\eta)dx+(b_2+\frac{\zeta}{\psi}B_2)dy$$

$$=(\frac{\partial B_2}{\partial x}+B_2\psi\frac{\partial}{\partial x}(\frac{1}{\psi})+D_2)\lambda+(\frac{\partial B_1}{\partial y}+B_1Y+B_1\phi\frac{\partial}{\partial y}(\frac{1}{\psi})+XY\phi)\mu+d(Y\triangle),$$

(4.18) $$e_3=d\zeta+(c_1+\frac{\xi}{\phi}C_1)dx+(c_2+\frac{\zeta}{\psi}C_2+\eta)dy$$

$$=(\frac{\partial C_2}{\partial x}+C_2X+C_2\psi\frac{\partial}{\partial x}(\frac{1}{\psi})+XY\psi)\lambda+(\frac{\partial C_1}{\partial y}+C_1\phi\frac{\partial}{\partial y}(\frac{1}{\phi})+D_1)\mu+d(X\triangle),$$

(4.19) $$e_4=d\eta+(d_1+\frac{\xi}{\phi}D_1)dx+(d_2+\frac{\zeta}{\psi}D_2)dy$$

$$=(\frac{\partial D_2}{\partial x}+D_2X+D_2\psi\frac{\partial}{\partial x}(\frac{1}{\psi}))\lambda+(\frac{\partial D_1}{\partial y}+D_1Y+D_1\phi\frac{\partial}{\partial y}(\frac{1}{\phi}))\mu+d(XY\triangle).$$

From these, the coefficients of (4.14) can be calculated as follows:

$$\frac{e_1}{\triangle}=\frac{d\triangle}{\triangle}+(\frac{\partial A_2}{\partial x}+A_2\psi\frac{\partial}{\partial x}(\frac{1}{\psi})+C_2+Y\psi)\frac{\lambda}{\triangle}+(\frac{\partial A_1}{\partial y}+A_1\phi\frac{\partial}{\partial y}(\frac{1}{\phi})+B_1+X\phi)\frac{\mu}{\triangle},$$

$$\frac{1}{\psi}(e_3-Xe_1)=\frac{1}{\psi}\{\frac{\partial C_2}{\partial x}-X\frac{\partial A_2}{\partial x}+(C_2-A_2X)\psi\frac{\partial}{\partial x}(\frac{1}{\psi})\}\lambda$$

$$+\frac{1}{\psi}(A_1\frac{\partial x}{\partial y}+D_1-B_1X-X^2\phi)\mu+\frac{\triangle}{\psi}dX,$$

$$\frac{1}{\phi}(e_2-Ye_1)=\frac{1}{\phi}(A_2\frac{\partial Y}{\partial x}+D_2-C_2Y-Y^2\psi)\lambda$$

$$+\frac{1}{\phi}\{\frac{\partial B_1}{\partial y}-Y\frac{\partial A_1}{\partial y}+(B_1-A_1Y)\phi\frac{\partial}{\partial y}(\frac{1}{\phi})\}\mu+\frac{\triangle}{\phi}dY,$$

$$e_4-XYe_1-X(e_2-Ye_1)-Y(e_3-Xe_1)$$

$$=\{\frac{\partial D_2}{\partial x}-X\frac{\partial B_2}{\partial x}-Y\frac{\partial C_2}{\partial x}+XY\frac{\partial A_2}{\partial x}+(D_2-C_2Y)\psi\frac{\partial}{\partial x}(\frac{1}{\psi})\}\lambda$$

$$+\{\frac{\partial D_1}{\partial y}-X\frac{\partial B_1}{\partial y}-Y\frac{\partial C_1}{\partial y}+XY\frac{\partial A_1}{\partial y}+(D_1-B_1X)\phi\frac{\partial}{\partial y}(\frac{1}{\phi})\}\mu$$

$$+d(XY\triangle)-Xd(Y\triangle)-Yd(X\triangle)+XYd\triangle$$

$$=\{\frac{\partial D_2}{\partial x}-XA_2\frac{\partial Y}{\partial x}-Y\frac{\partial C_2}{\partial x}+(D_2-C_2Y)\psi\frac{\partial}{\partial x}(\frac{1}{\psi})\}\lambda$$

$$+\{\frac{\partial D_1}{\partial y}-YA_1\frac{\partial X}{\partial y}-X\frac{\partial B_1}{\partial y}+(D_1-B_1X)\phi\frac{\partial}{\partial y}(\frac{1}{\phi})\}\mu.$$

We summarize the above results in the following: Let us put

$$X=\frac{C_1}{A_1},\qquad Y=\frac{B_2}{A_2},$$

$$\lambda=A_1dx+\phi dy,\quad \mu=\psi dx+A_2dy.$$

When we write the reduced system of total differential equations in the form

$$dz_j=\omega_{j1}z_1+\omega_{j2}z_2+\omega_{j3}z_3+\omega_{j4}z_4\ (j=1,2,3,4),\tag{4.20}$$

we have

$$\begin{cases}\omega_{11}=-Xdx-Ydy,\\ \omega_{12}=\dfrac{dx}{\phi},\\ \omega_{13}=\dfrac{dy}{\psi},\\ \omega_{14}=0,\end{cases}\tag{4.21}$$

$$\begin{cases}\omega_{21}=(D_1-B_1X-\phi X^2)dx+\phi dX,\\ \omega_{22}=(\dfrac{B_1}{\phi}+X)dx-\dfrac{Y}{\phi}\lambda+\dfrac{d\phi}{\phi}\\ \omega_{23}=0,\\ \omega_{24}=\dfrac{\lambda}{\triangle},\end{cases}\tag{4.22}$$

$$\begin{cases}\omega_{31}=(D_2-C_2Y-\psi Y^2)dy+\psi dY,\\ \omega_{32}=0,\\ \omega_{33}=(\dfrac{C_2}{\psi}+Y)dy-\dfrac{X}{\psi}\mu+\dfrac{d\psi}{\psi},\\ \omega_{34}=\dfrac{\mu}{\triangle},\end{cases}\tag{4.23}$$

$$\begin{cases}\omega_{41}=\{\psi\dfrac{\partial}{\partial x}(\dfrac{D_2}{\psi})-Y\psi\dfrac{\partial}{\partial x}(\dfrac{C_2}{\psi})-XA_2\dfrac{\partial Y}{\partial x}\}\lambda+\{\phi\dfrac{\partial}{\partial y}(\dfrac{D_1}{\phi})-X\phi\dfrac{\partial}{\partial y}(\dfrac{B_1}{\phi})-YA_1\dfrac{\partial X}{\partial y}\}\mu,\\ \omega_{42}=\dfrac{1}{\phi}\{A_2\dfrac{\partial Y}{\partial x}+D_2-C_2Y-\psi Y^2\}\lambda+\{\dfrac{\partial}{\partial y}(\dfrac{B_1}{\phi})-Y\dfrac{\partial}{\partial y}(\dfrac{A_1}{\phi})\}\mu+\dfrac{\triangle}{\phi}dY,\end{cases}\tag{4.24}$$

$$\begin{cases} \omega_{43}=\{\frac{\partial}{\partial x}(\frac{C_2}{\psi})-X\frac{\partial}{\partial x}(\frac{A_2}{\psi})\}\lambda+\frac{1}{\psi}\{A_1\frac{\partial X}{\partial y}+D_1-B_1X-\phi X^2\}\mu+\frac{\triangle}{\psi}dX, \\ \omega_{44}=\frac{d\triangle}{\triangle}+\frac{1}{\triangle}\{\psi\frac{\partial}{\partial x}(\frac{A_2}{\psi})+C_2+\psi Y\}\lambda+\frac{1}{\triangle}\{\phi\frac{\partial}{\partial y}(\frac{A_1}{\phi})+B_1+\phi X\}\mu. \end{cases}$$

If we denote the matrix of differential 1-form (ω_{ij}; $i, j=1, 2, 3, 4$) by Ω then the completely integrable condition becomes

$$d\Omega=\Omega \wedge \Omega, \tag{4.25}$$

where $\wedge$ denotes an exterior product. Under this condition, the system of total differential equations (4.20) forms a 4-dimensional solution space.

Now, from (4.20-4) *we can immediately obtain a differential equation on the y-section, which is given by the following:*

$$\frac{dz_j}{dx}=\hat{\omega}_{j1}z_1+\hat{\omega}_{j2}z_2+\hat{\omega}_{j3}z_3+\hat{\omega}_{j4}z_4 \quad (j=1, 2, 3, 4), \tag{4.26}$$

where

$$\begin{cases} \hat{\omega}_{11}=-X, \\ \hat{\omega}_{12}=\frac{1}{\phi}, \\ \hat{\omega}_{13}=\hat{\omega}_{14}=0, \end{cases} \tag{4.27}$$

$$\begin{cases} \hat{\omega}_{21}=(D_1-B_1X-\phi X^2)+\phi\frac{\partial X}{\partial x}, \\ \hat{\omega}_{22}=(\frac{B_1}{\phi}+X-\frac{Y}{\phi}A_1)+\frac{1}{\phi}\frac{\partial\phi}{\partial x}, \\ \hat{\omega}_{23}=0, \\ \hat{\omega}_{24}=\frac{A_1}{\triangle}, \end{cases} \tag{4.28}$$

$$\begin{cases} \hat{\omega}_{31}=\psi\frac{\partial Y}{\partial x}, \\ \hat{\omega}_{32}=0, \\ \hat{\omega}_{33}=-X+\frac{1}{\psi}\frac{\partial\psi}{\partial x}, \\ \hat{\omega}_{34}=\frac{\psi}{\triangle}, \end{cases} \tag{4.29}$$

$$(4.30)\begin{cases} \hat{\omega}_{41}=A_1\{\psi\frac{\partial}{\partial x}(\frac{D_2}{\psi})-Y\psi\frac{\partial}{\partial x}(\frac{C_2}{\psi})-XA_2\frac{\partial Y}{\partial x}\}+\psi\{\phi\frac{\partial}{\partial y}(\frac{D_1}{\phi})-X\phi\frac{\partial}{\partial y}(\frac{B_1}{\phi})-YA_1\frac{\partial X}{\partial y}\}, \\ \hat{\omega}_{42}=\frac{A_1}{\phi}\{A_2\frac{\partial Y}{\partial x}+D_2-C_2Y-\psi Y^2\}+\psi\{\frac{\partial}{\partial y}(\frac{B_1}{\phi})-Y\frac{\partial}{\partial y}(\frac{A_1}{\phi})\}+\frac{\triangle}{\phi}\frac{\partial Y}{\partial x}, \\ \hat{\omega}_{43}=A_1\{\frac{\partial}{\partial x}(\frac{C_2}{\psi})-X\frac{\partial}{\partial x}(\frac{A_2}{\psi})\}+\{A_1\frac{\partial X}{\partial y}+D_1-B_1X-\phi X^2\}+\frac{\triangle}{\phi}\frac{\partial X}{\partial x}, \\ \hat{\omega}_{44}=\frac{1}{\triangle}\frac{\partial\triangle}{\partial x}+\frac{A_1}{\triangle}\{\psi\frac{\partial}{\partial x}(\frac{A_2}{\psi})+C_2+\psi Y\}+\frac{\psi}{\triangle}\{\phi\frac{\partial}{\partial y}(\frac{A_1}{\phi})+B_1+\phi X\}. \end{cases}$$

Now, applying the above result to some Appell and Horn's hypergeometric equations, we shall seek their reduced system of total differential equation.

(i) $F_2(\alpha, \beta, \beta', \gamma, \gamma';\ x, y)$.

$$\begin{cases} x(1-x)r=xys+\{(\alpha+\beta+1)x-\gamma\}p+\beta yq+\alpha\beta z, \\ y(1-y)t=xys+\beta' xp+\{(\alpha+\beta'+1)y-\gamma'\}q+\alpha\beta' z \end{cases}$$

We have

$$\begin{cases} \phi=x(1-x), \quad \psi=y(1-y), \quad \triangle=xy(1-x-y), \\ X=\frac{\beta}{x}, \qquad Y=\frac{\beta'}{y}. \end{cases}$$

Substituting these into (4.21-4), we immediately obtain

$$\begin{cases} \omega_{11}=-\beta\frac{dx}{x}-\beta'\frac{dy}{y}, \\ \omega_{12}=\frac{dx}{x}-\frac{d(1-x)}{1-x}, \\ \omega_{13}=\frac{dy}{y}-\frac{d(1-y)}{1-y}, \end{cases}$$

$$\begin{cases} \omega_{21}=\beta(\gamma-\beta-1)\frac{dx}{x}, \\ \omega_{22}=(\beta-\gamma+1)\frac{dx}{x}-(\alpha+\beta-\beta'-\gamma)\frac{d(1-x)}{1-x}-\beta'\frac{dy}{y}, \\ \omega_{24}=-\frac{d(1-x-y)}{1-x-y}+\frac{dy}{y}, \end{cases}$$

$$\begin{cases} \omega_{31}=\beta'(\gamma'-\beta'-1)\dfrac{dy}{y}, \\ \omega_{33}=(\beta'-\gamma'+1)\dfrac{dy}{y}-(\alpha+\beta'-\beta-\gamma')\dfrac{d(1-y)}{1-y}-\beta\dfrac{dx}{x}, \\ \omega_{34}=\dfrac{dx}{x}-\dfrac{d(1-x-y)}{1-x-y}, \end{cases}$$

$$\begin{cases} \omega_{41}=0, \\ \omega_{42}=\beta'(\beta'-\gamma'+1)\dfrac{d(1-x)}{1-x}+\beta'(\gamma'-\beta'-1)\dfrac{dy}{y}, \\ \omega_{43}=\beta(\beta-\gamma+1)\dfrac{d(1-y)}{1-y}+\beta(\gamma-\beta-1)\dfrac{dx}{x}, \\ \omega_{44}=(\beta-\gamma+1)\dfrac{dx}{x}+(\beta'-\gamma'+1)\dfrac{dy}{y}-(\alpha+\beta+\beta'-\gamma-\gamma'+1)\dfrac{d(1-x-y)}{1-x-y}. \end{cases}$$

Hence, according to our reduction, F_2 is a typical equation like F_1, which has a simple reduced form:

$$dZ=(A\frac{dx}{x}+B\frac{dy}{y}+C\frac{d(x-1)}{x-1}+D\frac{d(y-1)}{y-1}+E\frac{d(x+y-1)}{x+y-1})Z,$$

where

$$A=\begin{bmatrix} -\beta & 1 & 0 & 0 \\ \beta(\gamma-\beta-1) & \beta-\gamma+1 & 0 & 0 \\ 0 & 0 & -\beta & 1 \\ 0 & 0 & \beta(\gamma-\beta-1) & \beta-\gamma+1 \end{bmatrix},$$

$$B=\begin{bmatrix} -\beta' & 0 & 1 & 0 \\ 0 & -\beta' & 0 & 1 \\ \beta'(\gamma'-\beta'-1) & 0 & \beta'-\gamma'+1 & 0 \\ 0 & \beta'(\gamma'-\beta'-1) & 0 & \beta'-\gamma'+1 \end{bmatrix},$$

$$C=\begin{bmatrix} 0 & -1 & 0 & 0 \\ 0 & \gamma+\beta'-\alpha-\beta & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \beta'(\beta'-\gamma'+1) & 0 & 0 \end{bmatrix},$$

$$D=\begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \gamma'+\beta-\alpha-\beta' & 0 \\ 0 & 0 & \beta(\beta-\gamma+1) & 0 \end{bmatrix},$$

$$E=\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & \gamma+\gamma'-\alpha-\beta-\beta'-1 \end{bmatrix}.$$

Using (4.27-30), we have a system of differential equations on the y-section.

$$\begin{cases} \dfrac{dz_1}{dx}=(-\dfrac{\beta}{x})z_1+\dfrac{1}{x(1-x)}z_2, \\ \dfrac{dz_2}{dx}=\dfrac{\beta(\gamma-\beta-1)}{x}z_1+\dfrac{(\alpha-\beta'-1)x+\beta-\gamma+1}{x(1-x)}z_2+\dfrac{1}{1-x-y}z_4, \\ \dfrac{dz_3}{dx}=(-\dfrac{\beta}{x})z_3+\dfrac{1-y}{(1-x-y)x}z_4, \\ \dfrac{dz_4}{dx}=\dfrac{\beta'(\gamma'-\beta'-1)}{1-x}z_2+\dfrac{\beta(\gamma-\beta-1)}{x}z_3+(\dfrac{\beta-\gamma+1}{x}+\dfrac{\alpha+\beta+\beta'-\gamma-\gamma'+1}{1-x-y})z_4 \end{cases}$$

and a differential equation for $z_1=z$ is just the following hypergeometric equation

$$\begin{aligned} & x^2(1-x)(1-x-y)z^{(4)} \\ & +[\{2(\gamma+1)-(\alpha+\beta-\beta'+\gamma+5)x\}(1-x-y)-(\alpha+\beta+\beta'-\gamma-\gamma'+3)x(1-x)]xz''' \\ & +[\{\gamma(\gamma+1)-((\beta+2)(\alpha-\beta'+2)+\gamma(\alpha+\beta-\beta'+3))x\}(1-x-y) \\ & \quad -\{(\beta+1)(\alpha+\beta'-\gamma'+1)+(\gamma+1)(\alpha+\beta+\beta'-\gamma-\gamma'+3)\}x \\ & \quad +\{(\alpha+\beta-\beta'+3)(\alpha+\beta+\beta'-\gamma-\gamma'+3)+(\beta+1)(\alpha+\beta'-\gamma'+1)+\beta'(\beta'-\gamma'+1)\}x^2]z'' \\ & +(\beta+1)[\{2\alpha+2\beta'-2\gamma'-\gamma+4)(\alpha-\beta'+1)+\beta(2\alpha-\gamma'+2)+2\beta'(\beta'-\gamma'+1)\}x \\ & \quad -\gamma(\alpha+\beta'-\gamma'+1)-\gamma(\alpha-\beta'+1)(1-x-y)]z' \\ & +\beta(\beta+1)\alpha(\alpha-\gamma'+1)z=0\ , \end{aligned}$$

which has the Riemann scheme

$$P\left\{\begin{array}{ccccc} x=0 & x-1=0 & 1-x-y=0 & x=\infty & \\ 0 & 0 & 0 & \beta & \\ 1 & 1 & 1 & \beta+1 & x \\ 1-\gamma & 2 & 2 & \alpha & \\ 2-\gamma & \gamma+\beta'-\beta-\alpha & \gamma+\gamma'-\alpha-\beta-\beta' & \alpha+1-\gamma' & \end{array}\right\}.$$

(ii) $\underline{\mathbf{F}_3(\alpha,\alpha',\beta,\beta',\gamma;\ x,y)}$.

$$\begin{cases} x(1-x)r=-ys+\{(\alpha+\beta+1)x-\gamma\}p+\alpha\beta z, \\ y(1-y)t=-xs+\{(\alpha'+\beta'+1)y-\gamma\}q+\alpha'\beta' z. \end{cases}$$

In this case, $\phi=x(1-x)$, $\psi=y(1-y)$, $\triangle=xy(xy-x-y)$, $X=0$ and $Y=0$. We then obtain

$$\begin{cases} \omega_{11}=0, \\ \omega_{12}=\dfrac{dx}{x}-\dfrac{d(1-x)}{1-x}, \\ \omega_{13}=\dfrac{dy}{y}-\dfrac{d(1-y)}{1-y}, \end{cases}$$

$$\begin{cases} \omega_{21}=\alpha\beta dx, \\ \omega_{22}=(1-\gamma)\dfrac{dx}{x}-(\alpha+\beta-\gamma)\dfrac{d(1-x)}{1-x}, \\ \omega_{24}=\dfrac{dx}{xy}-\dfrac{d(xy-x-y)}{y(xy-x-y)}, \end{cases}$$

$$\begin{cases} \omega_{31}=\alpha'\beta' dy, \\ \omega_{32}=(1-\gamma)\dfrac{dy}{y}-(\alpha'+\beta'-\gamma)\dfrac{d(1-y)}{1-y}, \\ \omega_{34}=\dfrac{dy}{xy}-\dfrac{d(xy-x-y)}{x(xy-x-y)}, \end{cases}$$

$$\begin{cases} \omega_{41}=0, \\ \omega_{42}=(-\alpha'\beta' y)\dfrac{dx}{x}+(\alpha'\beta' y)\dfrac{d(1-x)}{1-x}+\alpha'\beta' dy, \end{cases}$$

$$\left\{\begin{array}{l} \omega_{43}=\alpha\beta dx-(\alpha\beta x)\dfrac{dy}{y}+(\alpha\beta x)\dfrac{d(1-y)}{1-y}, \\ \omega_{44}=\dfrac{(\alpha'+\beta'+2)y-\gamma-1}{y}\dfrac{dx}{x}+\dfrac{(\alpha+\beta+2)x-\gamma-1}{x}\dfrac{dy}{y} \end{array}\right.$$

$$+\{-(\alpha+\alpha'+\beta+\beta'+1)+\frac{\gamma+1}{x}+\frac{\gamma+1}{y}\}\frac{d(xy-x-y)}{xy-x-y},$$

F_3 has no longer a simple form like F_2. A system of differential equations on the y-section is given as follows:

$$\left\{\begin{array}{l} \dfrac{dz_1}{dx}=\dfrac{z_2}{x(1-x)}, \\ \dfrac{dz_2}{dx}=\alpha\beta z_1+\dfrac{(\alpha+\beta-1)x+1-\gamma}{x(1-x)}z_2-\dfrac{z_4}{x(xy-x-y)}, \\ \dfrac{dz_3}{dx}=\dfrac{(1-y)}{x(xy-x-y)}z_4, \\ \dfrac{dz_4}{dx}=-\dfrac{\alpha'\beta'y}{x(1-x)}z_2+\alpha\beta z_3+\dfrac{(\alpha+\beta-1)x(1-y)+(\gamma-\alpha'-\beta'-1)y}{x(xy-x-y)}z_4, \end{array}\right.$$

and the hypergeometric equation for $z_1=z$ is given by

$$\begin{aligned} &x^2(1-x)(xy-x-y)z^{(4)} \\ &+[\{\gamma+2-(\alpha+\beta+5)x\}(xy-x-y)-(1-x)\{(\alpha+\beta+3)x(1-y)+(\gamma-\alpha'-\beta'+1)y\}]xz''' \\ &+[\{(\alpha+\beta+3)x-\gamma-1\}\{(\alpha+\beta+3)(1-y)x+(\gamma-\alpha'-\beta'+1)y\} \\ &\quad-(\alpha+2)(\beta+2)x(xy-x-y)-(\alpha+1)(\beta+1)(1-y)x(1-x)-\alpha'\beta'y]z'' \\ &+(\alpha+1)(\beta+1)[\{(\alpha+\beta+1)x-\gamma\}(1-y)+(\alpha+\beta+3)(1-y)x+(\gamma+1-\alpha'-\beta')y]z' \\ &+\alpha\beta(\alpha+1)(\beta+1)(1-y)z=0 \end{aligned}$$

with the Riemann scheme

$$P\left\{\begin{array}{ccccc} x=0 & x=1 & xy-x-y=0 & x=\infty & \\ 0 & 0 & 0 & \alpha & \\ 1 & 1 & 1 & \alpha+1 & x \\ \alpha'-\gamma+1 & 2 & 2 & \beta & \\ \beta'-\gamma+1 & \gamma-\alpha-\beta & \gamma-\alpha-\alpha'-\beta-\beta'+1 & \beta+1 & \end{array}\right\}$$

(iii) **$\mathbf{H}_2(\alpha, \beta, \gamma, \delta, \varepsilon;\ x,y)$.**

$$\begin{cases} x(x-1)r=xys-\{(\alpha+\beta+1)x-\varepsilon\}p+\beta yq-\alpha\beta z, \\ y(y+1)t=xs-\{(\gamma+\delta+1)y+1-\alpha\}q-\gamma\delta z. \end{cases}$$

$$\phi=x(x-1),\ \psi=y(y+1),\ \triangle=xy(xy-y-1),\ X=\frac{\beta}{x},\ Y=0.$$

$$\begin{cases} \omega_{11}=-\beta\dfrac{dx}{x}, \\ \omega_{12}=-\dfrac{dx}{x}+\dfrac{d(x-1)}{x-1}, \\ \omega_{13}=\dfrac{dy}{y}-\dfrac{d(y+1)}{y+1}, \end{cases}$$

$$\begin{cases} \omega_{21}=\beta(\beta-\varepsilon+1)\dfrac{dx}{x}, \\ \omega_{22}=(\beta+1-\varepsilon)\dfrac{dx}{x}+(\varepsilon-\alpha-\beta)\dfrac{d(x-1)}{x-1}, \\ \omega_{24}=\dfrac{d(xy-y-1)}{y(xy-y-1)}, \end{cases}$$

$$\begin{cases} \omega_{31}=(-\gamma\delta)dy, \\ \omega_{33}=(\alpha-\beta)\dfrac{dy}{y}+(\beta-\alpha-\gamma-\delta+1)\dfrac{d(y+1)}{y+1}-\beta\dfrac{dx}{x}, \\ \omega_{34}=\dfrac{y+1}{xy-y-1}\dfrac{dx}{x}+\dfrac{1}{xy-y-1}\dfrac{dy}{y}, \end{cases}$$

$$\begin{cases} \omega_{41}=0, \\ \omega_{42}=(-\gamma\delta)y\dfrac{d(x-1)}{x-1}-(\gamma\delta)dy, \\ \omega_{43}=\beta(\beta-\varepsilon+1)\dfrac{dx}{x}+\dfrac{\beta(x+\beta-\varepsilon)}{y-1}\dfrac{dy}{y}, \\ \omega_{44}=\dfrac{\alpha-(\gamma+\delta)y}{y}\dfrac{d(xy-y-1)}{xy-y-1}+\{1+\dfrac{((\varepsilon-\beta)-\alpha x)(y+1)}{xy-y-1}\cdot\dfrac{dx}{x}+\{1+\dfrac{((\varepsilon-\beta)-\alpha x)}{xy-y-1}\}\dfrac{dy}{y}. \end{cases}$$

Putting $\theta=\alpha+\beta+\gamma+\delta-\varepsilon$, we can write the hypergeometric equation for $z_1=z$ in the form

$$x^2(x-1)(xy-y-1)z^{(4)}$$
$$+[(xy-y-1)\{(\alpha+\beta+\varepsilon+5)x-2(\varepsilon+1)\}+(\theta+2)yx(x-1)]xz'''$$
$$+[(xy-y-1)\{((\alpha+2)(\beta+2)+\varepsilon(\alpha+\beta+3))x-\varepsilon(\varepsilon+1)\}$$
$$+yx\{(\theta+2)((\alpha+\beta+3)x-(\varepsilon+1))+(\beta+1)(\theta+\varepsilon-\beta)(x-1)\}]z''$$
$$+[(xy-y-1)\varepsilon(\alpha+1)(\beta+1)+(\theta+2)(\alpha+1)(\beta+1)yx+(\beta+1)(\theta+\varepsilon-\beta)((\alpha+\beta+1)x-\varepsilon)]z'$$
$$+\alpha\beta(\beta+1)(\theta+\varepsilon-\beta)yz=0,$$

which has the Riemann scheme

$$P\begin{Bmatrix} x=0 & x=1 & xy-y-1=0 & x=\infty & \\ 0 & 0 & 0 & \alpha & \\ 1 & 1 & 1 & \beta & x \\ 1-\varepsilon & 2 & 2 & \beta+1 & \\ 2-\varepsilon & \varepsilon-\alpha-\beta & 1-\theta & \theta+\varepsilon-\beta & \end{Bmatrix}$$

(iv) $\underline{\mathbf{F}_4(\alpha,\beta,\gamma,\gamma';\ x,y)}$.

$$\begin{cases} x(1-x)r-y^2t-2xys+\{\gamma-(\alpha+\beta+1)x\}p-(\alpha+\beta+1)yq-\alpha\beta z=0, \\ y(1-y)t-x^2r-2xys+\{\gamma'-(\alpha+\beta+1)y\}q-(\alpha+\beta+1)xp-\alpha\beta z=0. \end{cases}$$

We first rewrite the above in the form (4.1):

$$\begin{cases} x(1-x-y)r=2xys+\{(\alpha+\beta+1)x-\gamma(1-y)\}p+(\alpha+\beta-\gamma'+1)yq+\alpha\beta z, \\ y(1-x-y)t=2xys+(\alpha+\beta-\gamma+1)xp+\{(\alpha+\beta+1)y-\gamma'(1-x)\}q+\alpha\beta z. \end{cases}$$

In this case $\phi=x(1-x-y)$ and $\psi=y(1-x-y)$, polynomials in x and y, which is different from those described above. For simplicity, we put

$$f=1-x-y,\quad \theta=\alpha+\beta-\gamma+1,\quad \theta'=\alpha+\beta-\gamma'+1$$

and then we have

$$\begin{cases} \phi=xf,\ A_1=2xy,\quad B_1=\theta x-\gamma f,\quad C_1=\theta' y,\quad D_1=\alpha\beta, \\ \psi=yf,\ A_2=2xy,\quad B_2=\theta x,\quad C_2=\theta' y-\gamma' f,\quad D_2=\alpha\beta \end{cases}$$

with

$$\triangle = xy(f^2-4xy), \quad X=\frac{\theta'}{2x}, \quad Y=\frac{\theta}{2y}.$$

Substituting these into (4.21-4), we have

$$\begin{cases} \omega_{11} = -\dfrac{\theta'}{2}\dfrac{dx}{x} - \dfrac{\theta}{2}\dfrac{dy}{y}, \\ \omega_{12} = \dfrac{dx}{fx}, \\ \omega_{13} = \dfrac{dy}{fy}, \end{cases}$$

$$\begin{cases} \omega_{21} = \{(\alpha\beta - \dfrac{\theta\theta'}{2}) + \dfrac{\theta'}{2}(\gamma - 1 - \dfrac{\theta'}{2})\dfrac{f}{x}\}dx, \\ \omega_{22} = (\dfrac{\theta'}{2} - \gamma + 1)\dfrac{dx}{x} - \dfrac{\theta}{2}\dfrac{dy}{y} + \dfrac{df}{f}, \\ \omega_{24} = \dfrac{2y\,dx + f\,dy}{y(f^2-4xy)}, \end{cases}$$

$$\begin{cases} \omega_{31} = \{(\alpha\beta - \dfrac{\theta\theta'}{2}) + \dfrac{\theta}{2}(\gamma - 1 - \dfrac{\theta}{2})\dfrac{f}{y}\}dy, \\ \omega_{33} = (\dfrac{\theta}{2} - \gamma' + 1)\dfrac{dy}{y} - \dfrac{\theta'}{2}\dfrac{dx}{x} + \dfrac{df}{f}, \\ \omega_{34} = \dfrac{f\,dx + 2x\,dy}{x(f^2-4xy)}, \end{cases}$$

$$\begin{cases} \omega_{41} = (\alpha\beta - \dfrac{\theta\theta'}{2})\{y\,dx + x\,dy - 2xy\dfrac{df}{f}\}, \\ \omega_{42} = (\alpha\beta - \dfrac{\theta\theta'}{2})\{\dfrac{2y}{f}dx + dy\} + \theta(\gamma' - 1 - \dfrac{\theta}{2})\{dx + \dfrac{f}{2y}dy\}, \\ \omega_{43} = (\alpha\beta - \dfrac{\theta\theta'}{2})\{dx + \dfrac{2x}{f}dy\} + \theta'(\gamma - 1 - \dfrac{\theta'}{2})\{\dfrac{f}{2x}dx + dy\}, \\ \omega_{44} = \dfrac{1}{(f^2-4xy)}\{2(\theta'+1)y + 2(\theta-\gamma'+1)f + (\dfrac{\theta'}{2} - \gamma + 1)\dfrac{f^2}{x} + \dfrac{4xy}{f}\}dx \end{cases}$$

$$+\frac{1}{(f^2-4xy)}\{2(\theta+1)x + 2(\theta'-\gamma+1)f + (\frac{\theta}{2} - \gamma' + 1)\frac{f^2}{y} + \frac{4xy}{f}\}dy + \frac{d(f^2-4xy)}{f^2-4xy}.$$

In order to seek a differential equation on the y-section, we moreover put

$$\delta=\alpha\beta-\frac{\theta\theta'}{2},\quad \varepsilon=\frac{\theta}{2}(\gamma'-1-\frac{\theta}{2}),\quad \varepsilon'=\frac{\theta'}{2}(\gamma-1-\frac{\theta'}{2})$$

and then obtain the system

$$\begin{cases}\dfrac{dz_1}{dx}=(-\dfrac{\theta'}{2x})z_1+(\dfrac{1}{fx})z_2,\\[2mm] \dfrac{dz_2}{dx}=(\delta+\varepsilon'\dfrac{f}{x})z_1-(\dfrac{2\varepsilon'}{\theta'}\dfrac{1}{x}+\dfrac{1}{f})z_2+\dfrac{2}{f^2-4xy}z_4,\\[2mm] \dfrac{dz_3}{dx}=-(\dfrac{\theta'}{2x}+\dfrac{1}{f})z_3+\dfrac{f}{x(f^2-4xy)}z_4,\\[2mm] \dfrac{dz_4}{dx}=\delta(y+\dfrac{2xy}{f})z_1+2\{\delta(\dfrac{y}{f})+\varepsilon\}z_2+(\delta+\varepsilon'\dfrac{f}{x})z_3\end{cases}$$

$$+\frac{2}{f^2-4xy}\{(\theta'-1)y+(\theta-\gamma')f+\frac{2xy}{f}-\frac{\varepsilon'}{\theta'}\frac{f^2}{x}\}z_4.$$

The differential equation for $z_1=z$ is given by

$$\begin{aligned}&x^2(\delta x+\varepsilon'f)(f^2-4xy)z^{(4)}\\ &+[(\delta x+\varepsilon'f)\{2(\gamma+1)(f^2-4xy)+(2\gamma'-2\theta-5)x(1-x+y)\}+(\varepsilon'-\delta)x(f^2-4xy)]xz'''\\ &+[x(\delta x+\varepsilon'f)\{(2\gamma^2-(2\theta'+5)\gamma-\theta'-3-\delta)f-2(\gamma+1)(\theta'+3)y-(4\varepsilon+2\gamma-2\theta'-3)x\}\\ &\quad+f(\delta x+\varepsilon'f)\{\gamma(\gamma-1-\frac{\theta'}{2})f-(\delta x+\varepsilon'f)\}\\ &\quad+\{(\theta'+2)(\delta x+\varepsilon'f)+2(\varepsilon'-\delta)x\}\{(\gamma-\frac{\theta'}{4})(f^2-4xy)+(\gamma-\theta'-\frac{3}{2})xf+(2\gamma-2\theta'-3)xy\}]z''\\ &+[(\delta x+\varepsilon'f)\{\{(\gamma+\delta)(2\theta'-2\gamma+3)-2\varepsilon(\theta'+2)\}x-(\gamma-1-\frac{\theta'}{2})(2\gamma+\delta)f\}\\ &\quad-\{(\theta'+2)(\delta x+\varepsilon'f)+2(\varepsilon'-\delta)x\}\{(\gamma(\theta'-\gamma+\frac{3}{2})+\frac{\delta}{2})f+\gamma(\theta'+1)y+2\varepsilon x\}\\ &\quad+(\varepsilon'-\delta)(\gamma-1-\frac{\theta'}{2})\gamma f^2-\gamma\delta(\delta x+\varepsilon'f)f]z'\\ &+[(\delta x+\varepsilon'f)(\gamma-1-\frac{\theta'}{2}+\delta)\delta+\{(\theta'+2)(\delta x+\varepsilon'f)+2(\varepsilon'-\delta)x\}\{\delta(\theta'-\gamma+1)-\varepsilon\theta'\}\\ &\quad-(\varepsilon'-\delta)\delta(\gamma-1-\frac{\theta'}{2})f]z=0.\end{aligned}$$

This is a Fuchsian equation. In order to obtain the hypergeometric equation, we have only to differentiate it once more. The Riemann scheme is

$$P\left\{\begin{matrix} \delta x+\varepsilon' f=0 & x=0 & x=y^{+} & x=y^{-} & x=\infty & \\ 0 & 0 & 0 & 0 & \alpha & \\ 1 & 1 & 1 & 1 & \beta & x \\ 2 & 1-\gamma & 2 & 2 & \alpha-\gamma'+1 & \\ 4 & 2-\gamma & \gamma+\gamma'-\alpha-\beta-\frac{1}{2} & \gamma+\gamma'-\alpha-\beta-\frac{1}{2} & \beta-\gamma'+1 & \end{matrix}\right\},$$

where $y^{\pm}=y+1\pm 2\sqrt{y}$. There appeared an apparent singularity $\delta x+\varepsilon' f=0$.

When one attempts to analyze completely integrable partial differential (Fuchsian) equations in the large, it is useful and often plays an important role to investigate their differential equations on sections. For that purpose, we have explained one method of reduction of partial differential equations to total differential equations.

5. An algebraic manipulation for the reduction

In this section we shall explain a program of an algebraic manipulation for the algorithm described in §2 – §3. We used: Computer-VAX11/785, OS-UNIX4.3BSD, Language−REDUCE3.2.

We input the order N of the hypergeometric equation, its finite regular singularities $L(J)=\lambda_j$ and the coefficients $P(J)=P_j(t)$. For example, consider the third order generalized hypergeometric equation

(5.1) $$t^2(t-1)y''' = t(c_2+b_2t)y''+(c_1+b_1t)y'+b_0y$$

$$= t(C2+B2\,t)y''+(C1+B1\,t)y'+B0y.$$

Then we input N=3, L(1)=L(2)=0, L(3)=1, $P(3)=t^2(t-1)$, $P(2)=t(C2+B2\,t)$, $P(1)=(C1+B1\,t)$, P(0)=B0 as follows:

```
COMMENT:  N    IS  ORDER  OF  DIFFERENTIAL  EQUATION.
          L(N) IS  REGULAR  SINGULAR  POINT.
          P(N) IS  COEFFICIENT  OF  DIFFERENTIAL  EQUATION;
LET  N=3;
OPERATOR  P,L;
OFF  EXP;
```

```
L(1):=0$ L(2):=0$ L(3):=1$

P(3):=T**2*(T-1)$
P(2):=(C2+B2*T)*T$
P(1):=(C1+B1*T)$
P(0):=B0$

END;
```

The following is our main program, where PHI(K) denotes a function calculating $(t-L(1))(t-L(2))\cdots(t-L(K))$, the operator A denotes elements of the repuired matrix A, i.e., $A(J,K)=\alpha_{j,k}$, the operator AT denotes the polynomials of the transformation (2.4), i.e., $AT(J,K)=a_{j,k}(t)$ and the operator DAT denotes the differentiation of the polynomials $a_{j,k}(t)$ with respect to t. In this program, for the sake of saving of memories, all terms which become useless are eliminated one after another.

```
COMMENT: PHI(K) IS A USED FUNCTION IN A MAIN PROGRAM.;

PROCEDURE PHI(K);
   BEGIN
      SCALAR PRO;
      PRO:=1;
      IF K=0 THEN RETURN PRO$
      PRO:=(FOR J:=1:K PRODUCT T-L(J));
      RETURN PRO$
   END;

COMMENT:THIS IS A REDUCTION PROGRAM FOR KOHNO'S TYPE.

                                        1986-05-23 FRIDAY.
                                        MADE BY TETSUYA.SUZUKI.
                                        AT YAMANASHI  UNIVERSITY.;

COMMENT:   A(N, N) IS A REQUIRED MATRIX.
           AT(N, N) IS A POLYNOMIAL MATRIX WITH A(N, N).
```

```
        DAT(N, N) IS A DIFFERENTIATED AT(N, N).;

OPERATOR A,AT,DAT;

DEPEND AT,T;
DEPEND DAT,T;

FOR ALL J,K SUCH THAT NUMBERP LET DAT(J,K)=DF(AT(J,K),T);
FOR ALL J SUCH THAT NUMBERP LET DAT(J,J-1)=DF(PHI(J-1),T);
FOR ALL J SUCH THAT NUMBERP LET DAT(J,0)=0;

FOR J:=0:N-1  DO AT(N+1,J):=-P(J)$
FOR J:=0:N-1 DO CLEAR P(J);

FOR K:=1:N DO
  BEGIN
    FOR J:=N STEP -1 UNTIL 1 DO
      BEGIN
        SIG:=0;
        SIG:=SIG+(FOR I:=0:K-2 SUM A(J,J-I) *AT(J-I,J-K));
        IF J-K+1<1 THEN RETURN;
        A(J,J-K+1):=0;
        A(J,J-K+1):=A(J,J-K+1)-SIG;
        A(J,J-K+1):=A(J,J-K+1)-AT(J+1,J-K);
        A(J,J-K+1):=A(J,J-K+1)+(T-L(J)) *DAT(J,J-K);
        A(J,J-K+1):=A(J,J-K+1)/PHI(J-K);
        A(J,J-K+1):=SUB(T=L(J),A(J,J-K+1));
        IF J-K-1<0 THEN RETURN;
        AT(J,J-K-1):=0;
        AT(J,J-K-1):=AT(J,J-K-1)+SIG;
        AT(J,J-K-1):=AT(J,J-K-1)+AT(J+1,J-K);
        AT(J,J-K-1):=AT(J,J-K-1)+A(J,J-K+1) *PHI(J-K);
        AT(J,J-K-1):=AT(J,J-K-1)/(T-L(J));
        AT(J,J-K-1):=AT(J,J-K-1)-DAT(J,J-K);
      END;
    FOR I:=0:K-2 DO CLEAR AT(N-I,N-K);
    FOR I:=1:K DO
      BEGIN
```

```
            WRITE''A('',K,'','',I,'') :='',A(K,I);
            CLEAR A(K,I);
        END;
    END;
END;
```

In the above case (5.1), we obtain the output

A(1,1):=0, $\alpha_{11}=0$,
A(2,1):=C2−C1−2, $\alpha_{21}=c_2-c_1-2$,
A(2,2):=−(C2−3), $\alpha_{22}=-c_2+3$,
A(3,1):=(C2+B2) *(C2−C1−2)+B0, $\alpha_{31}=b_0-(b_2+c_2)(c_1-c_2+2)$,
A(3,2):=−((C2+B2) *(C2−2)−C1−B1), $\alpha_{32}=b_1+c_1-(b_2+c_2)(c_2-2)$,
A(3,3):=C2+B2, $\alpha_{33}=b_2+c_2$,

with the polynomials $a_{jk}(t)$:

AT(3,1):=(C2−2) *T, $a_{31}(t)=(c_2-2)t$,
AT(3,0):=−(C2−C1−2), $a_{30}=c_1-c_2+2$,
AT(2,0):=0, $a_{20}=0$.

In this case, CPU TIME is 4,573MS.

Owing to limited space, we only show two other examples.

(i) A fourth order generalized hypergeometric equation.

$$t^3(t-1)y^{(4)}=t^2(\mathrm{C3}+\mathrm{B3}\,t)y'''+t(\mathrm{C2}+\mathrm{B2}\,t)y''+(\mathrm{C1}+\mathrm{B1}\,t)y'+\mathrm{B0}\,y,$$

The output is as follows:

```
A(1,1):=0
A(2,1):=0
A(2,2):=1
A(3,1):=-(2 *C3-C2+C1-6)
A(3,2):=2 *C3-C2-6
```

```
A(3,3):=-(C3-5)
A(4,1):=-((2 *C3-C2+C1-6) *(C3+B3)-B0)
A(4,2):=(2 *C3-C2-6) *(C3+B3)+C1+B1
A(4,3):=-((C3+B3) *(C3-3)-C2-B2)
A(4,4):=C3+B3

AT(4,2):=(C3-3) *T²
AT(4,1):=-(2 *C3-C2-6) *T
AT(3,1):=0
AT(4,0):=2 *C3-C2+C1-6
AT(3,0):=0
AT(2,0):=0
                              CPU TIME 7, 752 MS
```

(ii) A third order Jordan-Pochhammer equation

$$\phi(t)y'''=P_2(t)y''+P_1(t)y'+P_0(t)y,$$

where

$$\phi(t)=(t-L1)(t-L2)(t-L3),$$

$$\psi(t)=B1(t-L2)(t-L3)+B2(t-L1)(t-L3)+B3(t-L1)(t-L2),$$

$$P_{n-l}(t)=(-1)^{l-1}\left\{\binom{Q+l-1}{l}\phi^{(l)}(t)+\binom{Q+l-1}{l-1}\psi^{(l-1)}(t)\right\}\quad(l=1,2,3).$$

```
A(1,1):=Q+B1+2
A(2,1):=(((B2+B3-2) *L1-(B2-1) *L3-(B3-1) *L2) *B1)/ (L1-L3)
A(2,2):=Q+B2+1
A(3,1):=((((B3-1) *B1-B3²+3*B3-2) *L2+((B3-1) *B2+B3²
        -3 *B3+2) *L1-(B1+B2) *(B3-1) *L3) *B1)/(L1-L3)
A(3,2):=(-((B1+B2) *L3-L1 *B2-L2 *B1) *(B3-1))/(L1-L3)
A(3,3):=Q+B3

AT(3,1):=T *(-2 *Q-B1-B2-2)+Q *L1+Q *L2+L1 *B2+L1+L2
        *B1+L2
```

```
AT(3,0):=(Q^2*L1-Q^2*L3+Q*L1*B1+Q*L1*B2+3*Q*L1-Q*L3*
         B1-Q*L3*B2-3*Q*L3-L1*B1*B3+3*L1*B1+2*L1*B2
         +2*L1+L2*B1*B3-L2*B1-2*L3*B1-2*L3*B2-2*L3)
         /(L1-L3)
AT(2,0):=-Q-B1-2
                                        CPU TIME 6,273 MS
```

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