REMARKS ON SPACES WITH NON-POSITIVE CURVATURE.

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H. Busemann [1], [2], [3] had dealt with the metric spaces called **G**-spaces or **E**-spaces by him. In his work [3], he defined **G**-spaces with non-positive curvature. His definition was a metric condition. But in differentiable Riemannian manifolds of class C^r ($r \geqq 4$), this is equivalent to the condition of non-positive curvature in the usual sense. Among his works, it is to be in particular noticed that he systematically studied the relation between a ray and its coray and the theory of asymptotes. He [2], [3] obtained many interesting results in **G**-spaces or in **E**-spaces. On the other hand, F. P. Pederson [4] defined **G**-spaces with non-positive curvature under the more weak condition than the H. Busemann's and showed that H. Busemann's results hold again in this case. The purpose of this note is to show that F. P. Pederson's definition can be replaced by the more convenient form, which will be stated in §1.

1. ***Basic concepts and some theorems.*** To define an **E**-space (or a **G**-space), H. Busemann considered the following axioms: **A.** $\mathfrak{G}$ is a usual metric space with distance ab for any two points $a, b \in \mathfrak{G}$. **B.** $\mathfrak{G}$ ' is finitely compact. **C.** $\mathfrak{G}$ is convex metric. **D.** Every point x has a neighborhood $S(x, \alpha_x) = \{y \mid xy < \alpha_x, yx < \alpha_x\}$ $(\alpha_x > 0)$ such that for any positive number ε and any two points $a, b \in S(x, \alpha_x)$, there exists real numbers $0 < \delta_k(a, b) \leqq \varepsilon$ $(k=1,2)$ for which the points a_1, b_1 satisfying the conditions $a_1 a + ab = a_1 b$, $aa_1 = \delta_1$; $ab + bb_1 = ab_1$, $bb_1 = \delta_2$ are uniquely determined.

When the above four axioms **A, B, C, D** are satisfied, $\mathfrak{G}$ is said an **E**-psace. Moreover, if the metric is symmetric, $\mathfrak{G}$ is said a **G**-space.

Following the usual method, we can define a continuous curve in an **E**-space (or in a **G**-space) and its length. For any two points p, q there exists a segment from p to q (or from q to p) and its length is equal to the distance pq (or qp). This fact easily follows from the axioms **A, B, C.** Under the axiom **D,** the prolongation of segments is locally possible and unique. We come from these facts to the concept of extremals.

G. Birkhoff [5] also gave another definition of **G**-spaces, which is apparently different from, but is equivalent to the H. Busemann's.

At first we shall formulate the following definition:

(1.1). ***Definition.*** *A continuous funtion $f(t)$ will be called semi-convex in the interval $[\alpha, \beta]$, if $f(t) \leqq max(f(t_1), f(t_2))$ for any t_1, t_2 with $\alpha \leqq t_1 < t < t_2 \leqq \beta$. If* *the equality never hold, then we shall call $f(t)$ strictly semi-convex.*

In addition to this, if we assume that whenever $f(t) = max\ (f(t_1), f(t_2))$ for a value t with $t_1 < t < t_2$, then $f(t) = const.$ in $[t_1, t_2]$, then this definition coincides with F. P. Pederson's one.

The following properties of semi-convex functions are obvious from the definition.

(**1.2**) *If a continuons fuaction $f(t)$ is semi-convex in $[[\alpha, \beta]$, and has the minimum at the left or right end point of $[\alpha, \beta]$, then $f(t)$ is monotone. If $f(t_0)$ is a minimum at an inner point t_0 of $[\alpha, \beta]$, then $f(t)$ is monotone decreasing in $[\alpha, t_0]$ and monotone increasing in $[t_0, \beta]$. If $f(t)$ is strictly semi-convex, then $f(t)$ has the minimum value at only one point t_0. According as $t_0 = \alpha$, $\alpha < t_0 < \beta$, or $t_0 = \beta$, $f(t)$ is strictly increasing, strictly decreasing in $[\alpha, t_0]$ and strictly increasing in $[t_0, \beta]$, or strictly decreasing.*

H. Busemann showed that there exists a parametric representation $x(t)$, $-\infty < t < +\infty$, for an extremal $\mathfrak{X}$ such that
$\mathfrak{X}$: $x(t_1)x(t_2) = | t_2 - t_1 |$ for sufficiently near t_1, t_2, where t denotes the arc-length of $\mathfrak{X}$. By making use of this, we shall define **G**-spaces with non-positive curvature.

(*1.3*) ***Definition.*** *In a G-space $\mathfrak{G}$, let $x(t)$, $\alpha_1 \leqq t \leqq \alpha_2$ $(\alpha_1 < \alpha_2)$; $y(s)$, $\beta_1 \leqq s \leqq \beta_2$ $(\beta_1 < \beta_2)$, be parametric representations of two extremal subarcs. Under the correspondence $s = ct + d$ $(c \neq 0)$ between t and s such that $\beta_i = c\alpha_i + d$ $(i = 1, 2)$, if the segment $T(x(t), y(ct+d))$ is unique for every t in $[\alpha_1, \alpha_2]$, and $f(t) = x(t)\,y(ct+d)$ is (strictly) semi-convex, then we say that $\mathfrak{G}$ has (negative) non-positive curvature.*

Let $\bar{\alpha}_x$ be the least upper bound of those α for which the segment $T(a, b)$ with end points a, b in $S(x, \alpha)$ is unique. H. Busemann's definition of **G**-spaces with non-positive curvature is as follows: The **G**-space is said to have non-positive curvature if every point x of $\mathfrak{G}$ has a nieghborhood $S(x, \beta_x)$ $(0 < \beta_x \leqq \bar{\alpha}_x)$ such that three points a, b, c in $S(x, \beta_x)$ satisfy the relation $2m(a, b)\, m(a, c) \leqq bc$, where $m(a, c)$, $m(b, c)$ are the middle points of a, b; a, c respectively. If, under the same conditions, $2m(a, b)\, m(a, c) = bc$, then $\mathfrak{G}$ is said to have curvature 0.

In the definiton (1.3), if $f(t)$ is convex, then $\mathfrak{G}$ has non-positive curature in H. Busemann's sene and vice versa [3]. The **G**-space $\mathfrak{G}$ with non-positive curvature in the present sense locally satisfies the condition (1.3), but the converse is not necessarily true. This easily follows from the fact that even if $f(t)$ is locally semi-convex, it is not necessarily semi-convex. From this, we can see that the condition of non-positive curvature in the present sense is weaker than H. Busemann's one.

The reason why we adopt the definition (1.3) is due to the following facts (i) and (ii).

(i) *F. P. Pederson defined **G**-spaces with (negative) non-positive curvature as follows: if every point x of a **G**-space has a neigehborhood $S(x, \gamma_x)$, $0 < \gamma_x \leqq \bar{\alpha}_x$, such that for any* *two (non-collinear) segments* T_1 *and* T_2 *in* $S(x, \gamma_x)$, *the distances* $x(t)\,T_2$ *and* $y(t)\,T_1$ *are (strictly) semi-convex, where* $x(t)$, $\alpha_1 \leqq t \leqq \alpha_2$; $y(t)$, $\beta_1 \leqq t \leqq \beta_2$, *are respectively parametric representations of* T_1, T_2.

But the foot of $x(t)$ *on* T_2 *is not necessarily continuous even if* t *varies continuously. Therefore it seems difficult to derive (1.3) as theorem from F. P. Pedersonos's definition. It is important that, when* $\mathfrak{G}$ *is straight, F. P. Pederson's definition implies (1.3), as we can see from the results of H. Busemann* [3].

(ii) *The condition (1.1) of semi-convex functions is weaker than the F. P. Pederson's.*

The definition of non-positive curvature in the present sense coincides with the usual one in differentiable Riemannian manifolds in spite of such weak condition. Next we shall prove this.

(1.4). A Riemannian manifold V_n *of class* C^r $(r \geqq 4)$ *has non-positive curvatre in the usual sense if it has non-psitive curvature in the present sense.*

H. Busemann [3] proved that V_n has non-positive curvature in his sense if and only if it has non-positive curvature in the usual sense. Hence, if V_n has non-positive curvature in the usual sense, then $\mathfrak{G}$ has, of course, non-positive curvature in the present sense. Accordingly we shall prove that if V_n has the positive curvature $K(p, \tau)$ where $p \in V_n$ and τ is the surface formed by all geodesics tangent to a 2-plane element through p, then V_n hasn't non-positive curvature in the present sense.

For this purpose we shall prove

(1.5) If V_n *has the positive curvature* $K(p, \tau)$ *at* p $(\in V_n)$ *for a geodesic surface* τ *through* p *as above, then there is a neighborhood* $S(p, \beta_p)$ $(\beta_p > 0)$ *such that for any two points* $a, b \in S(p, \beta_p) \cap \tau$, *if* p, a, b *are non-collinear, the inequality*

$$ab < 2m(p, a)\, m(p, b) \tag{1.6}$$

holds.

From the well known theorem in differential geometry (see *e.g.* [6]), p has a neighborhood $S(p, \beta_p)$ such that for any two points $a, b \in S(p, \beta_p) \cap \tau$, if p, a, b are non-collinear, we have the following inequality

$$\gamma^2 < \alpha^2 + \beta^2 - 2\alpha\beta \cos \widehat{apb}, \tag{1.7}$$

where α, β, γ denote the geodesic distances pa, pb, ab respectively and $\widehat{apb}$ denotes the angle between the geodesic arcs $T(p, a)$, $T(p, b)$ at p.

When β_p is sufficiently small, we can assume that the Riemannain curvatures $K(m(p, b), \tau[T(m(p, b), p), T(m(p, b), m(p, a))])$, $K(m(p, b), \tau[T(m(p, b), m(p, a)), T(m(p, b), b)])$, $K(m(p, a), \tau[T(m(p, a), p)T, (m(p, a), b)])$, $K(m(p, a), \tau[T(m(p, a), a), T(m(p, a), b)])$ are positive, where $\tau[T(m(p, b), p), T(m(p, b), m(p, a))]$ is the geodesic surface spanned by the two geodesic arcs $T(m(p, b), p)$ and $T(m(p, b), m(p, a))$ at $m(p, b)$, etc..

Putting $\alpha=2\lambda$, $\beta=2\mu$, $bm(p, a)=\nu$, $m(p, a)m(p, b)=\gamma'$, we have by (1,7)

$$\lambda^2<\gamma'^2+\mu^2-2\gamma'\mu\ \cos \widehat{m(p, a)m(p, b)p},$$

$$\nu^2<\gamma'^2+\mu^2-2\gamma'\mu\ \cos \widehat{m(p, a)m(p, b)b}.$$

Hence, we have by adding the above two inequalities

$$\lambda^2+\nu^2<2(\gamma'^2+\mu^2). \tag{1.8}$$

Similarly the inequality $\gamma^2+\beta^2<2(\lambda^2+\nu^2)$ holds. *I. e.*

$$\gamma^2+\beta^2<4(\gamma'^2+\mu^2).$$

Therefore we have $\gamma<2\gamma'$, since $\beta=2\mu$, which proves (1.5).

Proof of (1.4). Let V_n has the positive curvatnre $K(p, \gamma)$ for a geodesic surface γ through p. Consider a triangle pab where $a, b \in S(p, \beta_p/8)\cap\gamma$ and the point e satisfying the conditions $m(p, a)\,e=2m(p, b)\,e=2m(p, a)\,m(p, b)$. Such a point e is uniquely determined because of

$$pm(p, a)<\beta_p/16, \quad pm(p, b)<\beta_p/16, \quad m(p, a)\,m(p, b)<\beta_p/8 \text{ and}$$
$$pe\leqq pm(p, b)+m(p, b)e<\beta_p/4.$$

Let d be a point such that $2be=2ed=bd$. Then

$$de=be\leqq pe+pb<\beta_p/2, \quad i.\ e.$$
$$pd\leqq pe+ed<\beta_p.$$

Thus such a point d is also uniquely determined. On the other hand, we may assume that $S(p, \beta_p)$ is convex [7]. Therefore $T(p, a)$, $T(a, b)$, $T(b, e)$, $T(p, d)$ are contained in $S(p, \beta_p)$. We then have $2m(p, b)\,e>pd$, since $K(b, \gamma[T(b, p), T(b, d)])$ is positve for sufficiently small β_p. From this inequality and (1.6) we conclude $m(p, a)e<pd$, $m(p, a)e>ab$. Thus we have completely proved (1.4).

The following theorem is clear.

(1.9) In a **G**-*space* $\mathfrak{G}$ *with non-positive curvature, if the segments connecting a point* p *to points on an extremal subarc* $x(t)$, $t_1\leqq t\leqq t_2$, *are unique then* $f(t)=px(t)$ *is semi-convex in* $[t_1, t_2]$.

It is to be noticed that $f(t)$ can not necessarily be strictly semi-convex, even if $\mathfrak{G}$ has negative curvature. (1.9) yields

(1.10) $S(p, \bar{\alpha}_p)$ *is convex.*

We shall say that $\mathfrak{G}$ has negative curvature in strong sense, if $\mathfrak{G}$ has negative curvature and is straight and the function $f(t)$ in (1.9) is strictly semi-convex whenever $p \bar{\in} x(t)$, $t_1 \leqq t \leqq t_2$.

(1.11) *Let a G-space $\mathfrak{G}$ has non-positive curvature in H. Busemann's sense, and be straight. If a point $p\ (\in \mathfrak{G})$ has a foot f on a straight line $\mathfrak{l}\ (\bar{\in} p)$, then every point on the half ray $\mathfrak{x}$ which has the initial point f and contains p has only one foot f on $\mathfrak{x}$.*

Proof. Let $x(t), -\infty < x < +\infty$, be a parametric representation of $\mathfrak{l}$. Then $f(t) = px(t)$ is strictly convex [3]. Hence p has only one foot f on $\mathfrak{l}$. On the other hand, it is easily proved that every point on the subarc $T(f, p)$ has a unique foot f on $\mathfrak{l}$ except p. Consider now a point q on $\mathfrak{x}$ such that $q \bar{\in} T(p, f)$ and $q\,m(p, f) < pf$. Let f' be the foot of q on $\mathfrak{l}$. We assume $f \neq f'$. Then $qf > qf'$ because $qx(t) = g(t)$ is strictly convex. From the assumption of (1.11), we have

$$m(q, f)m(f, f') \leqq \frac{1}{2}\, qf' < \frac{1}{2}\, qf$$

Therefore

$$m(q, f)m(f, f') < m(q, f)f.$$

This fact shows us that the foot of $m(q, f)\ (\in T(f, q))$ is different from f, which contradicts the above mentioned fact. This proves (1.11).

It seems difficult to prove (1.11) in G-spaces with non-positve curvature in the present sense. But we can prove the following theorem.

(1.12) *If a G-space $\mathfrak{G}$ has negative curvature in strong sense, then a point $p\ (\in \mathfrak{G})$ has only one foot on a straight line $\mathfrak{l}\ (\bar{\in} p)$, and when p varies continuously, the foot also varies continuously.*

Proof. Let $x(t), -\infty < t < +\infty$, be a parametric representation of $\mathfrak{l}$. Then $px(t) = g(t)$ is strictly semi-convex. Accordingly p has only one foot f on $\mathfrak{l}$. Let $\{p_n\}$ be a sequence of points which converges to a point p and f_n be the foot of p_n on $\mathfrak{l}$ $(n=1, 2)$. It is sufficient to prove that $\{f_n\}$ converges to the point f which is the foot of p. If it were not the case, there would be a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ such that $f_{n_k} \to f'$ as $n_k \to \infty$ and $f' \neq f$. Then $T(p_{n_k}, f_{n_k})$ converges uniformly to $T(p, f')$ and we can conclude that f' is the foot of p, which contradicts that f is the unique foot of p. Therefore $\{f_n\}$ converges to f.

2. ***The universal covering spaces for a G-space with non-positive curvature.*** In this paragraph, we assume that G-spaces have the property of domain invariance. This assumption was adopted by H. Busemann [3], and formulated as follows: "If X is a subset of a G-space $\mathfrak{G}$ and is homeomorphic to an open set U, then X is open in $\mathfrak{G}$".

By making use of this assumption, he extended the well known theorem in Riemannian spaces to that of G-spaces, that is to say,

(2. 1) *If a G-space* $\mathfrak{G}$ *satisfies the property of domain invariance and has non-positive curvature then the universal covering spaces for* $\mathfrak{G}$ *are straight, and have non-positive curvature.*

In the above theorem, if $\mathfrak{G}$ has non-positive curvature in H. Busemann's sense and satisfies the property of domain invariance, then the universal covering spaces for $\mathfrak{G}$ also have the same properties and are straight [3]. Next we shall prove (2. 1) in the present sense.

Proof. Fix a point $p\ (\in \mathfrak{G})$, and put $K=\{x \mid px=\alpha_p/2\}\ (\alpha_p>0)$. For $u\in K$ let $x(u,t)$, $0\leqq t<+\infty$, be the parametric representation of the half extremal issuing from the point $p\ (=x(u,0))$ and containing u. Then (u,t) and $x(u,t)$ are in one to one correspondence for $0\leqq t<\alpha_p$. Fix u_0 and $t_0>0$.
Then

$$\delta=\inf_{0\leqq t\leqq t_0}\alpha_{x(u_0,t)}$$

is positive. Consider a sequence $\{u_n\}\ (\subset K)$ which converges to u_0. $\{x(u_n,t)\}$ converges uniformly to $x(u_0,t)$ for all t in $[0,t_0]$. Therefore there exists a positive integer N such that

$$\bigcup_{0\leqq t\leqq t_0} S(x(u_0,t),\ \delta/4)\supset x(u_n,\ t),\ \ 0\leqq t\leqq t_0,\ \ for\ n\geqq N.$$

We shall now prove that each extremal subarc of $\{x(u_n,t),\ 0\leqq t\leqq t_0\}\ (n\geqq N)$ disjoins the others except p. For, let $x(u_{n_0},t)$, $0<t\leqq t_0$, intersects $x(u_0,t)$, $0\leqq t\leqq t_0$, at $x(u_0,t')$ $(t'\neq 0)$. Then $x(u_{n_0},t)\,x(u_0,t)<\delta/4$. Let t' be the first value of t for which $x(u_0,t)$, $0<t\leqq t_0$, intersects $x(u_{n_0},t)$, $0<t\leqq t_0$. Put $x(u_0,t')=x(u_{n_0},t'')$ and $\bar{t}'=\bar{t}\,t''/t'$ $(0\leqq\bar{t}\leqq t')$.
Then

$$|\bar{t}'-\bar{t}|=|t''-t'|\,\bar{t}/t'\leqq|t''-t'|=x(u_0,\ t')\,x(u_0,t'')<\delta/4.$$

we have from this

$$x(u_{n_0},\bar{t}')x(u_0,\bar{t})\leqq x(u_{n_0},t')x(u_0,\bar{t}')+x(u_0,\bar{t}')x(u_0,\bar{t})<\delta/2.$$

Therefore the $T(x(u_{n_0},\bar{t}'),\ x(u_0,\bar{t}))$ is unique for every pair $(\bar{t},\bar{t}')$, and the functon $f(t)=x(u_{n_0},\ \bar{t}\,t''/t')\,x(u_0,\ \bar{t})$ is semi-convex, but $f(t')=f(0)=0$, which contradicts the definition of the semi-convexity. Similarly we can prove that every $x(u_n,t)$, $0<t\leqq t_0\ (n\geqq N)$ hasn't common points with the others.

On the other hand, we can find that all half extremals $x(u,t)$, $0 \leqq t \leqq \alpha_p$, $(u \in S(u_0, \delta/4) \cup K)$ simply cover a neighborhood $S(u_0, \varepsilon)$ $(0 < \varepsilon < \delta/4)$. For if this were not the case, then there would be a sequence $\{v_n\}$ which converges to u_0 and is not contained in $\{x(u,t) \mid 0 \leqq t \leqq \alpha_p,\ u \in S(u_0, \delta/4) \cap K\}$. Every half extremal passing through v_n and having p as initial point intersects in the first K at only one point u_n, then $\{u_n\}$ converges to u_0 as $\{T(p, v_n)\}$ converges to $T(p, u_0)$. Therefore there exists a positive integer N such that $S(u_0, \delta/4) \cap K \ni u_n$ for $n \geqq N$, which contradicts our assumption.

The half extremal connecting p to $q(\in S(u_0, \varepsilon))$ is unique and expressible by $x(u', t)$, $0 \leqq t < +\infty$, Put $q = x(u', t)$. The point $x(u', t')$ corresponding by the relation $t' = t_0 - \varepsilon/2 + t$ to q makes a set of points U as q varies in $S(u_0, \varepsilon)$. This correspondence is one-to-one and bicontinuous. Therefore U is open by the assumption of domain invariance. We know in the above correspondence that $U \ni x(u_0, t_0)$. Hence we conclude

(2.2) *If a G-space* $\mathfrak{G}$ *has non-positive curvature and satisfies the property of domain invariance, then every point of a half extremal* $\mathfrak{x}$ *has a neighborhood simply covered by a system of half extremals containing* $\mathfrak{x}$ *and having the same initial point as* $\mathfrak{x}$. *All points in such a neighborhood are locally in one-to-one correspondence with pairs* (u, t).

Let $\overline{\mathfrak{G}}$ be the set of pairs (u, t), i. e., the topological product of K and half real line: $0 \leqq t < +\infty$. Then every element $(u, t)(\in \overline{\mathfrak{G}})$ has a neighborhood W which is in one-to-one correspondence with a sufficiently small neighborhood U of $x(u, t)$. Hence we can define locally and isometrically a metric on $\overline{\mathfrak{G}}$ by making use of the above correspondence between W and U. Therefore we can define a homeomorphism Φ: $W \to U$. We can extend the definition of this homeomorphism on all $\overline{\mathfrak{G}}$. Thus we have a homomrphism Φ: $\overline{\mathfrak{G}} \to \mathfrak{G}$. Hence $\overline{\mathfrak{G}}$ is a covering space for $\mathfrak{G}$. We can define a curve length of any continuous curve connecting two elements (points) in $\overline{\mathfrak{G}}$. From this the distance of any two points is defined by the least upper bound of lengths of all continuous curves connecting these two points. Then any two points (u_0, t), (v_0, s_0) can be connected by the arcs (u_0, t), $0 \leqq t \leqq t_0$; (v_0, s), $0 \leqq s \leqq s_0$,. Hence the distance $(u_0, t_0)(v_0, s_0)$ is finite and symmetric. Hence the axiom **A** is satsfied in $\overline{\mathfrak{G}}$. Since $\overline{\mathfrak{G}}$ is a covering space for $\mathfrak{G}$ and $(u, t)(v, s) \geqq x(u, t)\, x(v, s)$ for any two points of $\overline{\mathfrak{G}}$, if a set of points $\{(u_\nu, t_\nu)\}$ is bounded, then $\{q_\nu\}$ $(q_\nu = x(u_\nu, t_\nu))$ is bounded in $\overline{\mathfrak{G}}$. Therefore $\{q_\nu\}$ has an accumulation point q. From this, we can easily prove that the axiom **B** is satisfied in $\overline{\mathfrak{G}}$. It is also clear that the axioms **C**, **D** are satisfied in $\overline{\mathfrak{G}}$. Hence $\overline{\mathfrak{G}}$ is a **G**-space.

We shall next prove that $\overline{\mathfrak{G}}$ is simply connected. If we consider a simple continuous curve $\mathfrak{C}$ in $\overline{\mathfrak{G}}$, then we can establish a continuous correspondence between the pairs (u, t) and the points of $\mathfrak{G}$. At this time, $\mathfrak{G}$ is contractible to p. Hence $\overline{\mathfrak{G}}$ is simply connected· and is a universal covering space for $\mathfrak{G}$.

We shall next prove that $\overline{\mathfrak{G}}$ is straight. To do this, we shall show that for any fixed u the curve (u,t), $0\leqq t<\infty$, is a ray or that the arc (u,t), $0\leqq t\leqq t_0$, is a shortest connection of $\bar{p}=(u, 0)$ and (u,t_0) for every t_0, because t_0 is the arc length of (u,t), $0\leqq t\leqq t_0$, from the defintion. Let α be the least upper bound of those t_0. Then $\alpha\geqq\alpha_p>0$. Therefore we have to show $\alpha=+\infty$. If α is finte, then there exists a half extremal $x(u,t)$, $0\leqq t<+\infty$, such that (u,t), $0\leqq t\leqq\alpha+\rho$, $(\rho>0)$ is not segment. But in this case a segment connecting $(u,\alpha+\rho)$ to $\bar{p}(u,0)$ can be represented by (v,t), $0\leqq t\leqq$ the distance $(u, 0)(u,\alpha+\rho)$, which contradicts the definition of $\overline{\mathfrak{G}}$.

From the fact mentioned above, every half extremal issuing from the point $\bar{p}$ is a ray in $\overline{\mathfrak{G}}$. Let $\bar{q}$ $(=(u_0, t_0))$ be a point of $\overline{\mathfrak{G}}$ different from $\bar{p}$. Because $\mathfrak{G}$ has non-positive curvature and $\overline{\mathfrak{G}}$ is a universal covering space for $\mathfrak{G}$, every point of a half extremal with the initial point $\bar{q}$ has a neighborhood simply covered by a system of half extremals issuing from the point $\bar{q}$. Let two half extremals $\mathfrak{x}, \mathfrak{x}'$ having the same initial point $\bar{q}$ interect at $\bar{r}$. At this time, we may assume $\bar{r}\neq\bar{p}$. Then $\bar{r}$ has a neighborhood U simply covered by the two systems of half extremals passing through neighborhoods of $\mathfrak{x}, \mathfrak{x}'$. Consider a continus curve $\mathfrak{C}'$ connecting $\bar{p}$ to $\bar{r}$, we can from these systems pick up two half extremals, which intersect at a point $\bar{r}'$ in $U\cap\mathfrak{C}'$. We can apply the same consideration to $\bar{r}'$. Thus when $\bar{r}'$ tends to $\bar{p}$, these two half extremals don't coincide because $\mathfrak{G}$ has non-positive curvature and $\overline{\mathfrak{G}}$ has the same local property as $\mathfrak{G}$. There exist through $\bar{p}$ two distinct extremals with a point $\bar{q}$ in common. Hence every half extremal with the initial point $\bar{q}$ is a ray. We see that $\overline{\mathfrak{G}}$ is straight.

Next, we shall prove the last part of (2.1).

Let $\bar{x}(\bar{t})$, $\alpha_1\leqq\bar{t}\leqq\alpha_2$; $\bar{y}(\bar{s})$, $\beta_1\leqq\bar{s}\leqq\beta_2$, be two extremal subarcs in $\overline{\mathfrak{G}}$. As we can see from the above proof, there exists a real number $\bar{\chi}$ for the segment $\bar{x}(\bar{t})$, $\alpha_1\leqq\bar{t}\leqq\alpha_2$ such that if any extremal subarc $\bar{z}(\bar{t})$, $\gamma_1\leqq\bar{t}\leqq\gamma_2$, which is a segment in this case, is contained in $\bigcup_{\alpha_2\leqq\bar{t}\leqq\alpha_1} S(\bar{x}(\bar{t}), \bar{\chi})$ and $\bar{z}(\gamma_i)\in S(\bar{x}(\alpha_i), \bar{\chi})$ $(i=1, 2)$, then the function $f(\bar{t}) =\bar{x}(\bar{t})\,\bar{z}(a\bar{t}+b)$ is semi-convex, where $a\gamma_i+b=\alpha_i$ $(i=1. 2)$. If we put $\bar{\varepsilon}$ the lower bound of those $\bar{\chi}$ for all segments connecting two points which divide the segments $T(\bar{x}(\alpha_i), \bar{y}(\beta_i))$ $(i=1, 2)$ into same ratio, then $\bar{\varepsilon}>0$. We divide two segments $T(\bar{x}(\alpha_i), \bar{y}(\beta_i))$ $(i=1, 2)$ into n parts of equal lengths d_i/n $(i=1, 2)$ by points $\bar{x}(\alpha_1)=\bar{x}_0^1$, $\bar{x}_1^1$, ..., $\bar{x}_1^{n-1}$, $x_1^n=\bar{y}((\beta_1)$ and by points $\bar{x}(\alpha_2)=\bar{x}_2^0$, $\bar{x}_2^1$, ..., $\bar{x}_2^{n-1}$, $\bar{x}_2^n=\bar{y}(\beta_2)$, where d_1, d_2 are respectively the arc-lengths of $T(\bar{x}(\alpha_1), \bar{y}(\beta_1))$, $T(\bar{x}(\alpha_2), \bar{y}(\beta_2))$. If n is sufficiently large, then the $\bar{\varepsilon}$-neighborhood of the segment $T(\bar{x}_1^i, \bar{x}_2^i)$ contains the segments $T(\bar{x}_1^{i-1}, \bar{x}_2^{i-1})$, $T(\bar{x}_1^{i+1}, \bar{x}_2^{i+1})$ for every $i=1, 2,$ ...., $n-1$. Let $\bar{x}^0, \bar{x}^1,$ ...., $\bar{x}^n$ be points dividing the segments $T(\bar{x}_1^i, \bar{x}_2)$ $(i=0, 1, 2,$ ...., $n)$ into a ratio $\bar{t}-\alpha_1 : \alpha_2-\bar{t}$ $(\alpha<\bar{t}$

$<\alpha_2)$ where $\bar{x}^0=\bar{x}(\bar{t})$, $\bar{x}^n=\bar{y}(\bar{a}\bar{t}+\bar{b})$ and $\alpha_i=\bar{a}\beta_i+\bar{b}\,(i=1,2)$. Then we have

$$\bar{x}(\bar{t})\,\bar{y}(\bar{a}\bar{t}+\bar{b})\leqq\sum_{i=0}^{n-1}\bar{x}^i\,\bar{x}^{i+1}\leqq max\,(d_1,\,d_2).$$

The above proof is applicable to any subinterval of $[\alpha_1,\,\alpha_2]$. Hence we can see that if $\mathfrak{G}$ has (negative) non-positive curvature $\overline{\mathfrak{G}}$ also has (negative) non-positive curvature. Thus we complete the proof of (2.1).

(2.3) *If a simply connected* **G**-*space* $\mathfrak{G}$ *has non-positive curvature and satisfies the property of domain invariance, then* $\overline{\mathfrak{G}}$ *is straight.*

This theorem easily follows from (2.1). The following theorem is clear from the proof of (1.2).

(2.4) *In a* **G**-*space* $\mathfrak{G}$, *if, for any point* $p\,(\in\mathfrak{G})$ *and for any bounded extremal subarc issuing from* p, *a neighborhood of every point except* p *of this subarc is simply covered by a system of half extremals with the same initial point* p, *then the universal covering spaces for* $\mathfrak{G}$ *are straight.*

(2.5) *If a compact* **G**-*space* $\mathfrak{G}$ *has non-positive curvature and satisfies the property of domain invariance, then* $\mathfrak{G}$ *is not simply connected but has finite connec tivity.*

This theorem is clear from (2.1). The last part also can be easily proved.

3. ***Asymptotes and some remarks.*** E. Cartan [6] had dealt with the relation between a ray and its coray in a hyperbolic space. The theory of asymptotes is also due to his work. Later H. Busemann systematically studied these objects. In an **E**-space $\mathfrak{E}$, let $\mathfrak{l}$ be a ray whose parametric representation is $x(t)$, $0\leqq t<+\infty$, let $\{t_n\}$ be a sequence of real numbers such that $t_n\to+\infty$ as $n\to\infty$ and let $\{p_n\}$ be a Cauchy sequence which converges to a point p. Then the sequence of segments $\{T(p_n,\,x(t_n))\}$ contains a subsequence coverging to a ray $\mathfrak{x}$ and having the initial point p, and $\mathfrak{x}$ is said a coray from p to $\mathfrak{l}$. H. Busemann [2] proved that the above limit is equivalent to the closed limit introduced by Hausdorff.

It is clear that there is not necessarily a unique coray $\mathfrak{x}$ from p to $\mathfrak{l}$. But if $q\in\mathfrak{x}-p$, then there is one and only one coray from q to $\mathfrak{l}$, and the coray is a subray of $\mathfrak{x}$ [3]. The union $\mathfrak{A}$ of all the cariers of the corays to $\mathfrak{l}$ which contain $\mathfrak{x}$ as subray is said an asymptote to $\mathfrak{l}$. If the asymptote $\mathfrak{A}$ carries a straight line, then there does not exist the initial point of $\mathfrak{A}$ but generally exists. The initial point of $\mathfrak{A}$ is said an asymptotic conjugate point to $\mathfrak{l}$. There arise many interesting problems for the set of asymptotic conjugate points, and some of the results [8] obtained by the author will be published elsewhere.

Specialy, if $\mathfrak{G}$ is straight, then the set of asymptotic conjugate points to a ray $\mathfrak{l}$ is empty. In other words, every asymptote carries a straight line. In this case, the theory of asymptotes is simple and the sequence $\{p_n\}$ in the defintion of corays can be substituted by only one point p. If a G-space $\mathfrak{G}$ has non-positive curvature in H. Busemann's sense and is straight, then the concept of asymptotes is symmetric and transitive, in other words, if $\mathfrak{A}$ is an asymptote to $\mathfrak{l}$, then the straight line carrying $\mathfrak{l}$ is an asymptote to $\mathfrak{A}$. H. Busemann [3] obtained the many results by making use of this property. But even if $\mathfrak{G}$ has non-positive curvature in the present sense and is straight, the concept of asymptotes is not necessarily symmetric and transitive. Therefore many results corresponding to the Busemann's cannot be obtained. We also come to the same conclusion in a straight G-space with non-positive curvature in the sense of F. P. Pederson. F. P. Pederson assumed in [4] the divergence property: "*Two rays* $\mathfrak{X}$ and $\mathfrak{X}'$ *having the same initial point satisfy the conditions:*

$$\lim_{t\to\infty} x(t)\,\mathfrak{X}' = \lim_{t\to\infty} x'(t')\,\mathfrak{X} = +\infty,$$

where $x(t)$, $0 \leqq t < +\infty$,; $x'(t')$, $0 \leqq t' < +\infty$, *denote parametric representations of* $\mathfrak{X}$, $\mathfrak{X}'$ *respectively*".

Under this assumption, he obtained many results analogous to the Busemann's. We can also in the present sense obtain these results. But it is important to investigate more precisely the properties of divergence and domaine invariance. We hope to study these two properties and moreover the property of set of the asymptotic conjugate points to a ray in differentiable Riemannian or Finsler manifolds in a later paper.

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