ON THE NORMALITY IN MINKOWSKIAN SPACES

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In an n-dimensional Euclidean space $\mathbf{R}$, let a function $F(\xi)$ defined on vectors $X=(\xi^1, \xi^2, \cdots\cdots, \xi^n)$ be positive homogeneous of the first order in the variables ξ's. If C: $F(\xi)=1$ is convex, then we can define a Minkowskian space $\overline{\mathbf{R}}$ by making use of C as unit sphere. Specially, if C is convex and differentiable, a transversal hyperplane to a vector $X_0=(\xi_0^1, \xi_0^2, \cdots\cdots\cdots, \xi_0^n)$ is given by

$$\sum_{i=1}^{n}\left(\frac{\partial F}{\partial \xi^i}\right)(\xi^i-\xi_i^0)=0. \tag{0.1}$$

In this case, we say that the vector X_0 is normal to the hyperplane (0,1) or that the hyperplane (0,1) is transversal to X_0. H. Busemann extended the concepts of the normality and the transversaltiy by making use of the Sm-function defined in his work [1] and he briefly proved A. E. Taylor's theorem (see [2]). On the other hand, W. Barthel [3] studied Minkowskian spaces from the standpoint of volume and area defined by H. Busemann. If C is merely strictly convex, we can introduce the concepts of curvature and curve length by using Minkowskian metric, area, and volume in place of Euclidean metric, area, and volume respectively. We shall devote this note to extend (0,1) to the conditon of normality and transversality in H. Busemann's sense. W. Barthel [3] incompletely studied this problem. We shall deal with the part overlooked by him. We assume in this note that C has a point as center, i. e., $F(\xi)=F(-\xi)$.

1. Let V be a set on an a-flat A in $\overline{\mathbf{R}}$, then the Minkowskian area $|V|^M$ is defined by

$$|V|^M=\sigma(A)\,|V|^L, \quad \sigma(A)=\omega^{(a)}/|U(A)|^L, \tag{1,1}$$

where $\omega^{(a)}=\pi^{a/2}/\Gamma\left(\frac{a}{2}+1\right)$ is the volume of the Euclidean unit sphere in A, $|V|^L$ denotes the Lebesgue measure of V, and $U(A)$ denotes the point set in which the a-flat parallel to A through the center of C intersects the interior of C.

Let the a-flat A intersects the b-flat B in the d-flat D where d is given by $min(a,b)=d+1$. At this time, $sin(A,B)$ is defined and $b\,(=a+b-d)$ is the dimension of the flat Q spanned by A and B. We shall call m-box an m-dimensional parallelepiped, and denote it by P_m. Let P_d be a proper d-box in D and P_a, P_b proper a and b-boxes in A and B respectively, which contain P_d as face. Let P_q be the q-box spanned by the boxes P_a and P_b. Then we define the function $Sm(A, B)$ as follows:

$$Sm(A, B) = |P_d|^M |P_q|^M / |P_a|^M |P_b|^M = \sin(A, B)\,\sigma(D)\,\sigma(Q)/\sigma(A)\sigma(B). \tag{1,2}$$

We can easily show that the above definition does not depend on the choice of the boxes P_a, P_b, P_d. *We say that* A *is normal to* B *in* Q *at* D, *or* B *is transversal to* A *in* Q *at* D, *when* $Sm(A^*, B) \leqq Sm(A, B)$ *for any a-flat* A^* *through* D *in* Q.

If we take a system of orthogonal unit vectors $X_i = (\xi_i^1, \xi_i^2, \cdots\cdots, \xi_i^n)(i=1, 2, \cdots, a)$ parallel to A, then $\sigma(A)$ is represented by

$$\sigma(A) = \omega^{(a)} \Big/ \int_{F(\xi_i u^i) \leqq 1} du^1 \cdots du^a \tag{1,3}$$

Hereafter we shall put $\sigma(A) = F^{(a)}(X_1, \cdots\cdots, X_a)$. Let X'_i $(i=1, 2, \cdots\cdots, a)$ be other independent vectors parallel to A, then we have

$$X'_k = \sum_i C_k^i X_i, \quad det \| C_k^i \| \neq 0.$$

By putting $u^i = \sum_k C_k^i u^{k'}$, we have

$$F^{(a)}(X'_1 \cdots, X'_a) = \omega^{(a)} \Big/ \int_{F(\xi'_i u^{i'}) \leqq 1} du^{1'} \cdots du^{a'} = F^{(a)}(X_1, \cdots\cdots, X_a) \left| det \| C_k^i \| \right|. \tag{1,4}$$

Hence we have the function $F^{(a)}(X_1, X_2, \cdots\cdots\cdots, X_a)$ defined on linealy independent vectors $X_1, X_2, \cdots\cdots, X_a$. If $X_1, X_2, \cdots\cdots, X_a$ are in partciular linearly dependent, then we assume $F^{(a)}(X_1, X_2, \cdots\cdots, X_a) = 0$. Next, we shall deal with the relation between the differentiability of $F^{(a)}(X_1, X_2, \cdots\cdots, X_a)$ with respect to the cempomnents of vectors and that of the indicatrix function $F(\xi)$ with respect to the variables ξ' s.

(1,5). *Let* $X_1, X_2, \cdots\cdots\cdots, X_{a-1}$ *be a system of orthogonal unit vectors parallel to an* $(a-1)$*-flat* Φ *containing the center of* C; *let* C_Φ *be the intersection of the interior of* C *and* Φ; *let a vector* X_a *be perpendicular* to* Φ; *let* Ψ *be a half a-flat spanned by the vector* X_a *and* Φ; *and* Π *be a 2-flat perpendicular to* Ψ *at* O *which contains* X_a. *If* $|\Psi \cap C|^L$ *is laid off on the ray* $\Psi \cap \Pi$ *from* O *when* Ψ *varies under the above condition and* Φ, Π *are fixed, then the resulting curve* Γ *in* Π *is differentiable when* $F(\xi)$ *is dfferentiable.*

* Hereafer we use the term "perpendicular" in the Euclidean sense.

Furthermore the order of differentiability of Γ is not less than that of $F(\xi)$. *If* C *is (strictly) convex, then* Γ *is also (strictly) convex.*

We introduce rectangular coordinates $(x, y, u^1, \cdots\cdots, u^{a-1})$ with O as origin, such that Π is the (x, y)-plane and Φ is the $(u^1, \cdots\cdots, u^{a-1})$-plane. We also introduce polar coordinates (γ, φ) in Π. Then the equations of Γ are given by

$$x=\Big(\int_{C_\Phi} r(\varphi\,;\,u^1,\,\cdots\cdots,\,u^{a-1})\,du^1\cdots du^{a-1}\Big)\cos\varphi,$$

$$y=\Big(\int_{C_\Phi} r(\varphi\,;\,u^1,\,\cdots\cdots,\,u^{a-1})\,du^1\cdots du^{a-1}\Big)\sin\varphi,$$

where $r(\varphi\,; u^1, \cdots\cdots, u^{a-1})$ is the length of segment cut off by $C\cap\Psi$ on the ray perpendicular to C_Φ at $(o, o, u^1, \cdots\cdots, u^{a-1})$ and φ is the angle between the ray $\Pi\cap\Psi$ and x-axis. If C is (strictly) convex, then the functions $\gamma(\varphi\,; u^1, \cdots\cdots, u^{a-1})\cos\varphi$ and $\gamma(\varphi\,; u^1, \cdots\cdots, u^{a-1})\sin\varphi$ are both (strictly) convex as φ varies and $u^1, \cdots\cdots, u^{a-1}$ are fixed. This fact yields that, if C is (strictly) convex, then Γ is also (strictly) convex.

On the other hand, $r(\varphi\,; u_0^1, \cdots\cdots\cdots, u_0^{a-1})$ is the moving radius of the intersection of C and the 2-flat parallel to Π at $(o, o, u_0^1, \cdots\cdots, u_0^{a-1})$ whose parameter is φ. If we take a rectangular coordinate system $(x, y, u^1, \cdots\cdots, u^{a-1}, u^a, \cdots\cdots u^{n-2})$ with O as origin, then $\xi^1, \cdots\cdots, \xi^n$ are represented by the linear combination:

$$\xi^i = a_1^i x + a_2^i y + \sum_{k=3}^{n-2} a_k^i u^k \ (i=1, 2, \cdots\cdots, n).$$

In the equaton of C, if we substitute $(x, y, u^1, \cdots\cdots, u^{n-2})$ *for* $(\xi^1, \cdots\cdots, \xi^n)$ and put $x=\gamma\cos\varphi$, $y=\gamma\sin\varphi$, $u^1=u_0^1$, $u^2=u_0^2$. $\cdots\cdots$, $u^{a-1}=u_0^{a-1}$, $u^a=\cdots\cdots=u^{n-2}=0$, we then have the equation of the intersection of C and the 2-flat parallel to Π at $(o, o, u_0^1, \cdots\cdots, u_0^{a-1})$ as an implicit function.

From the theorem of implicit functions and the differentiability of $F(\xi)$, we see that the order of the differentiability of Γ with respect to φ is not less than that of $F(\xi)$ in the ξ's. Thus we complete the proof of (1.5).

We can easily see from the explanation in the next paragraph that $F^{(a)}(X_1, X_i, \cdots\cdots\cdots, X_a)$ is differentiable with respect to the components of vectors, when $F(\xi)$ is differentiable, and that the order of differentability is not less than that of $F(\xi)$.

2. In this paragraph, we shall directly extend (0.1) to the condition of normality and transversality in H. Busemann's sense. Generally that A is normal to B does not

imply that B is normal to A. Therefore we must distinguish the two cases $a=d+1$ and $b=d+1$ for our purpose.

I $a=d+1$. Let A^* be an a-flat perpendicular to B at D; *let* z be a vertex of P_d, and let zz^* be the ray perpendicular to B at z. Then zz^* is contained in A^*. Let B^* be the b-flat containing the b-face parallel to B. Then the a-box P_a is cut off by B and B^* on the a-flat A through D in Q, Let θ be the angle between the ray zz^* and the ray zx perpendicular to D in A. Then we have

$$\left|P_a\right|^L = sec\,\theta \left|P_a^*\right|^L,$$

where P_a^* is the a-box cut off by B and B^* on A^*. From the definition of $Sm(A, B)$, we can easily see that A is normal to B, when $sec\theta\ \sigma(A)$ is minimal (or $|U(A)|^L\ cos\theta$ is maximal). Let $(X_1, \cdots\cdots, X_{a-1}, X_a^*), (X_1, \cdots\cdots X_{a-1}, X_{a+1}, \cdots\cdots X_q)(X_1, \cdots\cdots, X_{a-1})$ be the system of orthogonal unit vectors with the directions of edges of P_a^*, P_b, P_d at z respectively. The unit vector X_a with the direction zx is represented by

$$X_a = u^a X_a^* + u^{a+1} X_{a+1} + \cdots\cdots\cdots + u^q X_q \quad \left(\sum_{c=a}^{q} (u^c)^2 = 1\right).$$

By virtue of the definition of $F^{(a)}(X_1, \cdots =, X_a)$ if we put $\lambda = |U(A)|^L$, we get

$$\omega^{(a)} = \lambda F^{(a)}(X_1, \cdots\cdots\cdots, X_a) = F^{(a)}(X_1, \cdots\cdots\cdots, X_{a-1}, \lambda X_a).$$

Hence the $(q-d+1)$-dimensional hypersurface

$$T:\quad F^{(a)}(X_1, \cdots\cdots, X_{a-1}, u^a X_a^* + u^{a+1} X_{a+1} + \cdots\cdots\cdots + u^q X_q) = \omega^{(a)}$$

is the locus of the points x such that $zx = |U(A)|^L$, when A varies. If C is strictly convex then so is T, and its order of differentiability with respect to the variables u's is not less than that of C. Therefore the supporting plane of T parallel to B in Q is unique and has only one common point with T, when C is strictly convex. Let this common point be x_0. Then the a-flat A spanned by zx_0 and D is normal to B. The direction perpendicular to this spupporting plane is given by

$$\left(\left(\frac{\partial F^{(a)}}{\partial u^a}\right)_0, \left(\frac{\partial F^{(a)}}{\partial u^{a+1}}\right)_0, \cdots\cdots\cdots\cdots\cdots, \left(\frac{\partial F^{(a)}}{\partial u^q}\right)_0\right)$$

where $(\ \ )_0$ denotes the value at x_0, and this vector is proportional to $(1, 0, \cdots\cdots\cdots\cdots, 0)$ which coincides with X_a^*. *Hence, in order that* A *is normal to* B, *when* C *is strictly* *convex and differentiable, the condition*

$$(2.2)\quad \begin{cases} \dfrac{\partial F^{(a)}}{\partial \xi^i_a}\, \xi^{*i}_a \neq 0, \\[2ex] \dfrac{\partial F^{(a)}}{\partial \xi^i_a}\, \xi^i_c = 0 \quad (c=a+1, \cdots\cdots, q) \end{cases}$$

is necessary and sufficient. If $a=1$ and $b=n-1$, then B is a hyperplane and $d=0$. In this case, T coincides with C: $F(\xi)=1$ and we have the condition (0.1) as a special case of (2.2).

II $b=d+1$. In this case, let A^*, B, D be spanned by the systems of orthogonal unit vectors $(X_1, \cdots\cdots, X_{b-1}\ X^*_{b+1}, \cdots\cdots, X^*_q)$, $(X_1, \cdots\cdots X_{b-1}, X_b)$, $(X_1, \cdots\cdots, X_{b-1})$ respectively, and let z be a vertex of P_d. The ray zz^* perpendicular to A^* at z is contained in B. Accordingly, we know that zz^* has the same direction as X_b. Let G be the set of all lines perpendicular to A^* at every point of P^*_a. We shall consider in Q the a-box P_a cut off by G on the a-flat A through D. We can easily find that $|P_a|^L = |P^*_a|^L \sec\theta$, where θ is the angle between the ray zx perpendicular to A at z and the ray zz^*, and $Sm(A, B)$ has the maximal value if and only if $|U(A)|^L \cos\theta$ is maximal, when A varies and P_b is fixed. This fact easily follows from the definition of $Sm(A, B)$. Therefore we know that an a-flat perpendicular to such ray zx at D is normal to B. Next we shall assume that C is differentiable and strictly convex. Let A be spanned by a system of orthogonal unit vectors $(X_1, \cdots X_{b-1}, X_{b+1} \cdots, X_q)$. Then the vector X perpendicular to A is expressible by

$$(2.3)\quad X = u^b X_b + u^{b+1} X^*_{b+1} + \cdots\cdots\cdots; + u_q X^*_q \quad \left(\sum_{c=b}^{q} (u^c)^2 = 1\right).$$

The components of the vector X is also expressible in terms of components of the unit vectors $X_1, X_2, \cdots\cdots\cdots, X_{b-1}, X_{b+1}, \cdots\cdots X_q, Y_1 \cdots\cdots\cdots, Y_{n-q}$ where $Y_1 . \cdots\cdots\cdots, Y_{n-q}$ is a system of orthogonal unit vecotors perpendicular to Q in R. Therefore $F^{(a)}(X_1, \cdots\cdots, X_{b-1}, X_{b+1}, \cdots\cdots\cdots, X_q)$ is a functon on the unit vectors expressed by (2.3). Hence we can put

$$(2.4)\quad F^{(a)}(X_1, \cdots\cdots, X_{b-1}, X_{b+1}, \cdots\cdots, X_q) = \overline{F}^{(a)}(u^b X_b + u^{b+1} X^*_{b+1} + \cdots\cdots\cdots + u^q X^*_q).$$

From this, we can easily prove that $\overline{F}^{(a)}(X)$ is represented by $\overline{F}^{(a)}(\xi^1, \cdots\cdots, \xi^n)$ where

$$\xi^i=(-1)^{i-1}\begin{vmatrix} \xi_1^1 & \cdots\cdots & \xi_1^n \\ \cdots & \cdots\cdots & \cdots \\ \xi_{b-1}^1 & \cdots\cdots & \xi_{b-1}^n \\ \cdots & \cdots\cdots & \cdots \\ \xi_{b+1}^1 & \cdots\cdots & \xi_{b+1}^n \\ \cdots & \cdots\cdots & \cdots \\ \eta_1^1 & \cdots\cdots & \eta_1^n \\ \cdots & \cdots\cdots & \cdots \\ \eta_{n-q}^1 & \cdots\cdots & \eta_{n-q}^n \end{vmatrix}$$

in other words, $F^{(a)}(\mathrm{X}_1, \cdots\cdots, \mathrm{X}_{b-1}, \mathrm{X}_{b+1}, \cdots\cdots, \mathrm{X}_q)$ is also a function on $\mathrm{X}=(\xi^1, \cdots, \xi^n)$. Furthermore, $\overline{F}^{(a)}(\mathrm{X})$ is positve homogeneous of the first order in the variables ξ's. Hence by virtue of the definition of the function $\overline{F}^{(a)}(\mathrm{X})$ we have

$$(2.4)' \qquad \overline{F}^{(a)}(u^b \mathrm{X}_b + u^{b+1}\mathrm{X}_{b+1}^* + \cdots\cdots + u^q \mathrm{X}_q^*) = \omega^{(a)},$$

if we use $(u^b, u^{b+1}, \cdots\cdots, u^q)$ in place of $(\lambda u^b, \lambda u^{b+1}, \cdots\cdots, \lambda u^q)$ where λ denotes $|U(A)|^L$. We can easily find that the order of differentiabilty of $F^{(a)}(\mathrm{X}_1 \cdots\cdots, \mathrm{X}_{b-1}, \mathrm{X}_{b+1}, \cdots\cdots, \mathrm{X}_q)$ with respect to the variables ξ's is not less than that of $F^{(a)}(\mathrm{X}_1, \cdots\cdots, \mathrm{X}_{b-1}, \mathrm{X}_{b+1}, \cdots\cdots, \mathrm{X}_q)$ with respect to the vector components.

If $|U(A)|^L$ is laid off on the ray zx perpendicular to A from z. the equation (2.4)′ is the locus of the end points in Q, We shall denote this locus by T. If C is strictly convex and differentiable then so is T. $|U(A)|^L \cos\theta$ is maximal when the supporting plane of T is perpendicular to zz^*. At this time, this supporting plane has only one common point x_0 with T. The a-flat perpendicular to zx_0 is normal to B at D. This fact yields that the vector

$$(2,5) \qquad \left(\left(\frac{\partial \overline{F}^{(a)}}{\partial u^b}\right)_0, \left(\frac{\partial \overline{F}^{(a)}}{\partial u^{b+1}}\right)_0, \cdots\cdots\cdots\cdots, \left(\frac{\partial \overline{F}^{(a)}}{\partial u^q}\right)_0\right)$$

is proportional to $(1, 0, \cdots\cdots\cdots 0)$ where $(\ )_0$ denotes the value at x_0. *Hence, in order that* A *is normal to* B *the following condition is necessary and sufficient.*

$$(2.6) \qquad \begin{cases} \dfrac{\partial F^{(a)}}{\partial \xi^i}\, \xi_b^i \neq 0, \\[2ex] \dfrac{\partial F^{(a)}}{\partial \xi^i}\, \xi_c^{*i} = 0 \quad (c=b+1, \cdots\cdots, q). \end{cases}$$

W. Barthel did not distinguish the two cases **I, II.** and his theory was also ambiguous because he did not make use of the property of the locus T in **I, II.**

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References.

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