ON DIFFERENTIAL EQUATIONS IN LOCALLY CONVEX SPACES OF SOME TYPES.

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(Received January 10, 1955)

1. The widest generalization of the theorems on existence of solutions of the differential equation:

$$\frac{dx}{dt} = f(x, t) \cdots\cdots \qquad (1)$$

where x denotes a point of the n-dimensional vector space R^n and t a real number, to the case of an infinite-dimensional vector space E, seems not to have ever been given. It has already been classical that Piacrd's existence theorem, i.e. the existence theorem under Lipschitz condition holds true in any Banach space. But Peano's existence theorem, i.e. the existence theorem under the condition of continuity alone of the functon $f(x, t)$ cannot always be valid. In fact, an example to the contrary was given by J. Diudonné [1][(1)].

On the other hand it was proved by A. Tychonoff [2] that Peano's existence theorem holds true in case E is a product space $\prod_{\lambda \in \Lambda} R^1_\lambda$ of one-dimensional spaces R^1_λ with product topology. Therefore, in our case of generalization of n-dimensional space to infinite-dimensional ones, it might be more natural to consider E as a product space than as a Banach space.

Now, as is easily seen, Peano's existence theorem bases its validity on that of Arzelà's theorem, whose proof is essentially based on the two properties, i.e. the completeness of the space E and the property, which shall be called here temporarily *property* (M) and is stated as follows: "every bounded subset of the space is relatively compact." These two properties are preserved on the procedure of product of spaces [3, Chap. I, 10, Th. 2, Cor. and Chap. II, 5, Prop. 4]. A Banach space provided with these two properties is no other than finite-dimensional, which is trivial. Among infinite-dimensional vector spaces with these two properties, most particular is a space (M), i.e. a locally convex, metrisable, and complete space, with these two properties [4, p. 79]. This space has almost the same character as a finite-dimensional space R^n. Hence it will be natural to expect the validity of Peano's existence theorem in a space (M), and moreover, after Tyconoff's procedure [2], in a product space of spaces (M) with product topology, which will be verified in the following,

2. Let E be a space (M). Since E is metrisable, its topology is defined by a denombrable sequence of semi-norms $\{p_n\}$ [5, Chap. II, 5, Prop. 6]. Let $x(t)$ denote a function

(1) Numbers in brackets refer to the bibilography at the end of the paper.

on the closed bounded interval $I=[a, b]$ to E.

Lemma 1. *If the sequence of fuuctions* $\{x^m(t)\}$ *is equicontinuous in the interval* I *and is convergent on a subset* M *dense in the interval* I, *then it converges uniformly in the whole interval* I.

The proof is analogous to that of the one-dimensional space [6, p. 61, Hilfsatz 2], with slight modifications on uniform continuity as follows: for an arbitrary positive number ε and an arbitrary set of semi-norms $\{p_{n_1}, p_{n_2}, \cdots\cdots\cdots, p_{n_k}\}$, there exists a positive integer N, such that the inequalities

$$m, m' \geqq N$$

imply those

$$p_{n_i}(x^m(t) - x^{m'}(t)) < \varepsilon, \qquad (i = 1, 2, \cdots\cdots\cdots, k)$$

for every $t \in I$.

Lemma 2. *If functions of the sequeuce* $\{x^m(t)\}$ *are equicontinuous in the interval* I *and bounded at every point of the interval* I, *then there exisits such a subsequence* $\{x^{m'}(t)\}$ *as converges uniformly in the whole interual* I.

The proof follows, by virtue of the property (M), from Lemma 1, analogously to the case of the one-dimensional space [6, p. 60, Hilfsatz 1].

We denote the set of all functions $x(t)$ on I to E, continuous in the interval I by $\boldsymbol{C}(I; E)$ or simply $\boldsymbol{C}$, and each function $x(t)$ considered as element of $\boldsymbol{C}$ by $\boldsymbol{x}$. Let $\boldsymbol{x}+\boldsymbol{y}$ denote the sum function $x(t)+y(t)$ and $\alpha\boldsymbol{x}$, where α denotes a real (or complex) number, the product function $\alpha x(t)$ multiplied by α, and then the set $\boldsymbol{C}$ will be a vector space. Moreover, if we define $q_n(\boldsymbol{x})$ as follows:

$$q_n(\boldsymbol{x}) = \sup_{t \in I} p_n(x(t)), \qquad \cdots\cdots (2)$$

then each q_n is, as is easily seen, semi-norm of the vector space $\boldsymbol{C}$. Thus the vector space $\boldsymbol{C}$ will be topologized by a denombrable sequence of semi-norms $\{q_n; n = 1, 2, \cdots\cdots\cdots\}$ and then shall be denote by $\boldsymbol{C}(I, E; q_n, n = 1, 2, \cdots\cdots\cdots)$ or simply by $\boldsymbol{C}(I, E; q_n)$. The completeness of the space E implies that of the space $\boldsymbol{C}(I, E; q_n)$.

Since the space $\boldsymbol{C}(I, E; q_n)$ is metrisable, compactness and sequential compactness are coincident, and threfore we shall have the following fundamental

Lemma 3. [*Arzelà*] *A subst* F *of the space* $\boldsymbol{C}(I, E; q_n)$, *consisting of functions* $x(t)$ *equicontinuous iu ihe interval* I *and uniformly bounded there, is relatively compact.*

3. Let E be a space (M), and $f(x, t)$ a function on $E \times I$ to E, where $I=[t_0, t_0 + \tau]$.

Theorem. Let the function $f(x, t)$ be continuous and bounded in $E\times I$, i, e. for every n let there exists a positive constant M_n such that

$$p_n(f(x, t))\leqq M_n.$$

Then there exists at least one solution of the differential equation (1):

$$\frac{dx}{dt}=f(x, t)$$

which satisfies the initial condition: $x=x^0$ for $t=t_0$ and is defined in the interval I.

Proof. The problem can be reduced to the existence of solutions of the integral equation:

$$x(t)=x^0+\int_{t_0}^{t} f(x(t), t)dt, \qquad x(t)\in \mathbf{C}(I, E; q_n) \quad \cdots\cdots\cdots\cdots\cdots\cdots (3)$$

Now, let $\boldsymbol{X}$ the set of all functions $\mathbf{C}(I, E; q_n)$ which satisfy the following conditions:

$$(a)_n \qquad p_n(x(t)-x^0)\leqq \tau M,$$

$$(b)_n \qquad \limsup_{h\to 0}\ p_n\left(\frac{x(t+h)-x(t)}{h}\right)\leqq M_n.$$

Then the set $\boldsymbol{X}$ will be easily found to satisfy the conditions of Lemma 3, and therefore it is relatively compact. Since it is closed, it is compact, and obviously convex.

The relation

$$y(t)=x^0+\int_{t_0}^{t} f(x(t), t)\, dt \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (4)$$

will define a transformation Φ:

$$y=\Phi(x)$$

which transforms the set $\boldsymbol{X}$ into $\boldsymbol{X}$:

$$\Phi(X)\subset X.$$

Indeed,

$$p_n(y(t)-x^0)=p_n\left(\int_{t_0}^{t} f(x(t), t)\, dt\right)\leqq \int_{t_0}^{t} p_n(f(x(t), t)\, dt$$
$$\leqq \tau M_n,$$

and

$$\limsup_{h\to 0}\ p_n\left(\frac{y(t+h)-y(t)}{h}\right)=\limsup_{h\to 0}\ p_n\left(\frac{1}{h}\int_{t}^{t+h} f(x(t), t)\, dt\right)$$

$$= \limsup_{h\to 0} \frac{1}{|h|} \left(\int_t^{t+h} p_n (f(t), t)\, dt \right)$$

$$\leqq \limsup_{h\to 0} \frac{1}{|h|} \cdot |h| . \ M_n = M_n$$

The transformation is continuous. Indeed, let

$$\bar{y} = \Phi(\bar{x})$$

and then

$$p_n (y(t) - \bar{y}(t)) = p_n \left(\int_{t_0}^{t} \{ f(x(t), t) - f(\bar{x}(t), t) \}\, dt \right)$$

$$\leqq \int_{t_0}^{t} p_n \{ f(x(t), t) - f(\bar{x}(t), t) \}\, dt$$

Now let ε, ε' be arbitrarily given positive numberss, such that $\varepsilon' < \varepsilon$. Because of the continuity of the function $f(x, t)$, there exist a finite set of positive integers $\{m_1^n, m_2^n, \cdots, m_{l_n}^n\}$ and that of positive numbers $\{\delta_1^n, \delta_2^n, \cdots\cdots\cdots, \delta_{l_n}^n\}$ such that the inequalities:

$$p m_j^n (x(t) - \bar{x}(t)) < \delta_j^n \qquad (j = 1, 2, \cdots\cdots\cdots, l_n)$$

imply those

$$p_n \{ f(x(t), t) - f(\bar{x}(t), t) \} < \frac{\varepsilon'}{\tau}. \qquad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \quad (5)$$

Hence much more the inequalities

$$q m_j^n (x - \bar{x}) < \delta_j^n \qquad (j = 1, 2, \cdots\cdots\cdots, l_n)$$

imply (5), accordingly

$$p_n (y(t) - \bar{y}(t)) < \varepsilon',$$

therefore

$$q_n (y - y) \leqq \varepsilon' < \varepsilon.$$

According to the above argument, for any pair of a finite set of positive integers $\{n_1, n_2, \cdots\cdots\cdots, n_k\}$ and that of positive numbers $\{\varepsilon_1, \varepsilon_2, \cdots\cdots\cdots, \varepsilon_k\}$, there exist a finite set of positive integers $m_j^{n_i} \begin{pmatrix} i = 1, 2, \cdots\cdots\cdots, k \\ j = 1, 2, \cdots\cdots\cdots, l_i \end{pmatrix}$ and that of positive numbers $\delta_j^{n_i}$ $\begin{pmatrix} i = 1, 2, \cdots\cdots\cdots, k \\ j = 1, 2, \cdots\cdots\cdots, l_i \end{pmatrix}$ such that the inequalities

$$q m_j^{n_i} (x - \bar{x}) < \delta_j^{n_i}$$

imply those

$$q_{n_i}(y-\bar{y})<\varepsilon_i,$$

which shows the continuity of the transformation $y=\Phi(x)$. Hence from Tychonoff's fixed points theorem follows that the transformation $y=\Phi(x)$ of the compact convex subset X into itself has at least one fixed point $x\varepsilon X$: $x=\Phi(x)$, which proves the theorem.

4. Let E_λ be a space (M) for each index $\lambda\varepsilon\Lambda$, and its defining sequence of semi-norms $\{p_{\lambda,1}, p_{\lambda,2}, \cdots\cdots\cdots, p_{\lambda,n}, \cdots\cdots\cdots\}$. The product space $E=\prod_{\lambda\varepsilon\Lambda}E_\lambda$ with product topology is complete, since so is each E_λ $(\lambda\varepsilon\Lambda)$, and is provided with the property (M) (but not necessarily a space (M) except in case the set of index Λ is at most denombrable). Each point x of the space E can be represented by

$$x=(\cdots\cdots\cdots, x_\lambda, \cdots\cdots\cdots), \qquad x_\lambda\varepsilon E_\lambda, \qquad \lambda\varepsilon\Lambda.$$

Let $f(x,t)$ denote a function on the product space $E\times I$ to E and $f_\lambda(x,t)$ its projection on E_λ, and then the function $f(x,t)$ is represented as follows:

$$f(x,t)=(\cdots\cdots\cdots, f_\lambda(x,t), \cdots\cdots\cdots), \qquad \lambda\varepsilon\Lambda.$$

The boundedness of the function $f(x,t)$ is represented as follows: for every λ, n there exists a positive number $M_{\lambda,n}$ such that

$$p_{\lambda,n}(f(x,t))\leqq M_{\lambda,n}.$$

The theorem will be found to hold true in the product space E by some modifications of Tychonoff's procedure [2] as follows. Let C, X, p_n, M_n corresponding to each E_λ be denoted by C_λ, X_λ, $p_{\lambda,n}$, $M_{\lambda,n}$, and then X_λ is compact and convex. Each point x of C corresponding to product space $E=\prod_{\lambda\varepsilon\Lambda}E_\lambda$ is represented by

$$x=(\cdots\cdots\cdots, x_\lambda, \cdots\cdots\cdots), \qquad x_\lambda\,\varepsilon\, C_\lambda,$$

and C itself is a product space of C_λ $(\lambda\,\varepsilon\,\Lambda)$ with product topology [3, Chap, I, 8, Prop. 2, Cor. 1]:

$$C=\prod_{\lambda\varepsilon\Lambda}C_\lambda.$$

The space C is complete as product space of complete spaces. Let X denote the product space $\prod X_\lambda$ of subsets X_λ $(\lambda\,\varepsilon\,\Lambda)$, and then X is also compact and convex as product of such ones. On the other hand X is considered as a set of point x of C satisfying the following condition:

$$(a)_{\lambda,n} \qquad p_{\lambda,n}(x(t)-x^0)\leqq M_{\lambda,n};$$

$$(b)_{\lambda,n} \qquad \limsup_{h\to 0}\, p_{\lambda,n}\left(\frac{x(t+h)-x(t)}{h}\right)\leqq M_{\lambda,n}.$$

Since the topology of the product space $E=\prod_{\lambda\in\Lambda} E_\lambda$ is defined by the family of semi-norms $\{p_{\lambda,n};\ \lambda\in\Lambda,\ n=1,2,\cdots\cdots\cdots\}$, the remaining process is quite analogous to that of Tychonoff [2].

Bilbiography

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