REMARKS ON DUALITY IN LINEAR SPACES

Keishirō ŌHIRA

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1. Introduction. Given any locally convex topological real or complex linear (Hausdorff) space E, a topology t for its conjugate space E^* is called reflexive if E is algebraically isomorphic to its second conjugate space E^{*t*} under the *"natural" mapping* (cf. [1][1]). It is well known that if the closed convexification of a compact subset in E is again compact, the k-topology for E^* (i.e. the topology for E^* according to which convergence of functionals means uniform convergence on the compact subsets of E) is reflexive. In this paper we will note that this condition is also *necessary* for reflexivity of the k-topology for E^*. Therefore, it becomes clear that the following conditions are equivalent:

(1) The k-topology for E^* is reflexive.

(2) The closed convexification of a compact subset in E is again compact.

(3) The k-topology for E^* is equivalent to the c- topology i.e. the topology of uniform convergence on the convex, compact subsets of E.

(4) The k-topology for E^* is weaker than the κ- topology i.e. the topology of uniform convergence on the convex, weakly compact subsets of E.

M.F. Smith considered the following question in his foregoing paper (cf. [2]). *"For what class of topological groups does the Pontrjagin duality theorem hold?"* He answered this question in part, by showing that in reflexive real linear spaces and real Banach spaces the Pontrjagin Duality Theorem is valid. By applying our result, we shall answer the above question completely in the case that the considered topological group is a locally convex topological real linear (Hausdorff) space. Our answer is stated as follows. Let E be a locally convex topological real linear (Hausdorff) space. In order that the Pontrjagin Duality Theorem should be valid in E as a topological group, the following conditions are necessary and sufficient:

(1) The closed convexification of a compact subset in E is again compact.

(2) Any k-compact subset of the conjugate space E^* is equicontinuous.

2. Preliminaries.

Let E be a locally convex topological real or complex (Hausdorff) space, E^* its conjugate space, i.e. the space of continuous linear functionals on E. The zero vector in E is denoted by θ, the zero functional on E by θ^*. Let E^{*t} be the space E with the topology t. When we wish to talk of a notion in the t-topology, we shall refer to t-notion—— e.g.

1) Numbers in brackets refer to the bibliography at the end of the paper.

t-compactness, t-neighborhood. A topology t for E^* is called *reflexive* if E is algebraically isomorphic to E^{*t*} under the "*natural*" *mapping* N defined by

$N(x) = X$ if and only if $f(x) = X(f)$ for all $f \varepsilon E^*$. $(x \varepsilon E, \ X \varepsilon E^{**}.)$

The polar in E^* of a set K in E, denoted by K°, is the set of all f in E^* such that, if x is in K, then $|f(x)| \leqq 1$. *The polar in* E of a set K in E^* is defined as the set of all x in E such that, if f is in K, then $|f(x)| \leqq 1$. Furthrmore, a set K such that, if $\lambda + \mu = 1$, and if $\lambda, \mu \geqq 0$, then $\lambda K + \mu K \subset K$, is convex, a set K such that, if $|\lambda| = 1$, then $\lambda K \subset K$, is *circled*. The smallest convex enveloped of a set K is called the *convexification* of the set K, and denoted by K_{conv}. The closure of a set K is denoted by K^b.

We use the notions of the k-, p-, c-, *weak*, κ-*topologies* and the topology which we call k^*-*topology*. The topology for E^* of uniform convergence on compact subsets of E is called k-*topology*, and k-neighorhoods of θ^* are the sets K°, where K is a compact subset of E. The topology (for E^*) of simple convergence, is called p-*topology*, and p-neighborhoods of θ^* are the sets K°, where K is a finite set of E. Furthermore, we call c-*topology* the topology for E^* of uniform convergence on convex, compact subsets of E, and c- neighborhoods of θ^* are the set K°, where K is a convex compact subset of E. The *weak topology* for E has neighborhoods of θ of the form K°, where K° is the polar in E of a set K in E^*. The topology for E^* of uniform convergence on convex, weakly compact subsets of E, is called κ-*topology*, and has neighborhoods of θ^* of the form K°, where K is a convex, weakly compact subset of E. The k^*-*topology* for E has neighborhoods of θ of the form K° where K is the polar in E of a k-compact set in E^*. This k^*-topology is stronger than original topology for E.

Next, assume that E is a locally convex topological real linear (Hausdorff) space, and that $\bar{E}$ is the character group of E as a topological group. The Pontrjagin Duality Theorem, known to be true for locally compact groups, asserts that the given group E and the character group $\bar{\bar{E}}$ of its charácter group $\bar{E}$ are isomorphic under the "*natural*" *mapping* M defined by

$M(x) = \Phi$ if and only if $\varphi(x) = \Phi(\varphi)$ for all $\varphi \varepsilon \bar{E}$. $(x \varepsilon E, \ \Phi \varepsilon E.)$

We shall answer the following question completely in section 4: "*For what class of linear spaces as topological groups does the Pontrjagin theorem hold?*" We shall denote the set of all f satisfying the condition X by $\{f; X\}$.

3. Reflexivity of k-topology.

Let E be a locally convex topological real or complex linear (Hausdorff) space, E^* its conjugate space. We prove at first the following

LEMMA. *If A is a k-neigoborhood in E^*, then k^*-topology is equivalent with weak topology* *on A°, where A° is the polar in E.*

Proof. To prove this lemma, it is sufficient to show that, corresponding to each k^*-neighborhood $N^{k^*}(x_0)=x_0+C^\circ$ of a point $x_0 \varepsilon A$, where C is a k-compact set in E^*, a weak neighborhood $N^w(x_0)$ of a point x_0 can be found, such that

$$N^w(x_0)\cap A^\circ \subset N^{k^*}(x_0)\cap A^\circ.$$

Now, let $N^k(f)=f+(1/4)A\ (f\,\varepsilon\, E^*)$, and let $N_f^w(\theta)=\{4f\}^\circ$. Then, if $g\,\varepsilon\, K^k(f)$, $x\,\varepsilon\, N_f^w(\theta)\cap A^\circ$, then $4(g-f)\,\varepsilon\, A$, $x\,\varepsilon\, A^\circ$, and $x\,\varepsilon\,\{4f\}^\circ$. Therefore, we have two inequalities:

$$|g(x)-f(x)|\leqq 1/4, \qquad |f(x)|\leqq 1/4.$$

Hence, $|g(x)|\leqq|g(x)-f(x)|+|f(x)|\leqq 1/2$.

That is, corresponding to each $f\,\varepsilon\, C$, there exists a k-neighborchood $N^k(f)$ and a weak neighborhood $N_f^w(\theta)$, such that $|g(x)|\leqq 1/2$ for all $g\,\varepsilon\, N^k(f)$ and $x\,\varepsilon\, N_f^w(\theta)\cap A^\circ$. Let $f_1,\ldots,f_n$ be a finite number of points in C such that $\cup_{\alpha=1}^n N^k(f_\alpha)$ covers C, and let $N^w(x_0)=x_0+\cup_{\alpha=1}^n 2N_{f_\alpha}^w(\theta)$. Now, assume that $x\,\varepsilon\, N^w(x_0)\cap A^\circ$. If $f\,\varepsilon\, C$, then $f\,\varepsilon\, N^k(f_\alpha)$ for some α, and $1/2\ x-1/2x_0\,\varepsilon\, N_{f_\alpha}^w(\theta)$. Since A° is a convex circled set, we have the relation $1/2\ x-1/2\ x_0\,\varepsilon\, A^\circ$. Therefore, $1/2\ x-1/2\ x_0\,\varepsilon\, N_{f_\alpha}^w(\theta)\cap A^\circ$. Hence $|f(1/2\ x-1/2x_0)|\leqq 1/2$. That is, $|f(x-x_0)|\leqq 1$. Therefore, $x-x_0\,\varepsilon\, C^\circ$. That is, $x\,\varepsilon\, x_0+C^\circ=N^{k^*}(x_0)$. Hence, $N^w(x)\cap A^\circ\subset N^{k^*}(x_0)\cap A^\circ$.

Next, using the above lemma, we prove our main

Theorem 1. *Let E be a locally convex topological real or complex liuear (Hausdorff) space, E^* its conjugate space. If the k-topology for E^* is reflexive, then the closed convexification of a compact subset in E is again compact.*

Proof. Assume that the k-topology for E^* is reflexive, and let K be a compact set in E. Then the polar $A=K^\circ$ in E^* is a k-neighborhood. Since the polar $A^\circ=K^{\circ\circ}$ in E^{*k*} is p-compact, and the k-topology for E^* is reflexive, the polar $A^\circ=K^{\circ\circ}$ in E is weakly compact. Therefore, by the above lemma, $A^\circ=K^{\circ\circ}$ is k^*-compact. Hence, $K^{\circ\circ}$ is a convex, circled, compact set in E. Since $K_{conv}^b\subset K^{\circ\circ}$, we see that K_{conv}^b is compact.

From the above theorem 1, we have the following

Corollary. *Let E be a locally convex topological real or complex linear (Hausdorff) space, E^* its conjugate space. The following four conditions are equivalent:*

(1) *The k-topology for E^* is reflexive.*

(2) *The closed convexificaiton of a compact subset in E is again compact.*

(3) *The k-topology for E^* is equivalent to the c-topology.*

(4) *The k-topology for* E^* *is weaker than the* κ-*topology.*

4. The Pontriagin Duality Theorem in linear space.

From theorem 1, we have the following

THEOREM 2. *Let* E *be a locally convex topological real linear (Hausdorff) space. In order that the Pontrjagin Dulity Theorem should be valid in* E *as a topological group, the following conditions are necessary and sufficient:*

(1) *The closed convexification of a compact subset in* E *is again compact.*

(2) *Any k-compact subset of the coujugate space* E^{*k} *is equicontinuous.*

Proof. The character group $\overline{E}$ of E as a topological group and E^{*k} are algebraically and topologically equivalent as topological groups under the mapping T of E^{*k} into $\overline{E}$ defined by

$$Tf = \chi \text{ where } \chi(x) = \exp [if(x)] \text{ for all } x \in E.$$

(cf. [2]). Indeed, by lemma 1 in [2], E^{*k} and E are algebraically isomorphic groups. It is clear that T is continuous. For a open k-neighborhood $(K, \varepsilon) = \{f \in E^*;\ |f(x)| < \varepsilon, x \in K, K \text{ compact}, K \subset E.\}$ of θ^* in E^*, let $K_1 = \bigcup_{n=-\infty}^{\infty} \lambda_n K$, where $\lambda_n = (2n\pi - \varepsilon)/(2n\pi + \varepsilon)$ $(n \neq 0)$, $\lambda_0 = 1$, and let $\varepsilon_1 < \varepsilon$. Then, we see that

$$\{f \in E^*;\ |e^{if(x)} - 1| < \varepsilon_1,\ x \in K_1\} \subset (K, \varepsilon),$$

and K_1 is a compact set in E. Therefore the inverse mapping T^{-1} is continuous.

Therefore, to prove this theorem 2, it is sufficient to show that, in order that E^{*k*k} and E should be algebraically and topologically equivalent, two conditions (1), (2) of this theorem 2 are necessary and sufficient. By theorem 1, we see that the condition (1) is necessary and sufficient for algebraical equivalence of E^{*k*k} and E. We can show easily that the condition (2) is necessary and sufficient for topological equivalence of E^{*k*k} and E.

BIBLIOGRAPHY

1. R. Arens, *Duality in linear spaces,* Duke Math. J. vol. 14 (1947), 787-793.
2. M. F. Smith, *The Pontrjagin duality theorem in linear spaces,* Ann. of Math. vol. 56 (1952), 248-253.