DIFFERENTIAL EQUATIONS IN COORDINATED SPACES.

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1. *Preliminary.* We consider a sequence space E, which is an (abstract) linear space and consists of infinite sequences $x=\{x_2, x_2, \cdots\cdots, x_n, \cdots\cdots)$, called points of the space E, where each x_n $(n=1,2,\cdots\cdots)$ is an element of each linear convex space E_n and is called the n-th coordinate of the point x. A sequence space E is called a coordinated space if it is topologized locally convex with a fundamental system of neighbourhoods $\mathfrak{U}=\{U\}$ such that each mapping $x_n(x)$: $x \rightarrow x_n$ $(n=1,2,\cdots\cdots)$ is linear and continuous. For any point $x=\{x_1, x_2, \cdots, x_n, \cdots\cdots\}$, the point, which is constructed by equating some of the coordinates of x to zero, is called a projection of the point x, and especially that which is constructed by equating the coordinates with indices greater than n to zero, is denoted by $x^{[n]}$.

We call a coordinated space E to have the property (P), if, for every neighbourhood of the origin U (of the fundamental system $\mathfrak{U}$), the relation $x \varepsilon U$ implies that [projection of x] εU. For the future we shall assume this property; indeed, a great number of usual abstract spaces have it, as most Banach spaces (e.g. space (l^p) $(p>1)$, space (c_0), space (m), and space (c)), the product space of an enumerable number of spaces of real numbers, and its subspaces. For an point x of E there is considered a sequence of points $\{x^{[n]}\}$. The space E is called to have the property "Abschnittskonvergenz" or simply (AK) [1][(1)] if, for every point x, the corresponding sequence $\{x^{[n]}\}$ converges to x, that is, if, for an arbitrary neighbourhood of the origin $U \varepsilon \mathfrak{U}$, there exists a positive integer N such that $x-x^{[n]} \varepsilon U$ for $n \geqq N$. Among the spaces above indicated, the first two and the product space have this property, but the second two have not. In the space E provided with the property (P), the property (AK) is stated simplied as follows: for an arbitrary neighbourhood of the origin $U \varepsilon \mathfrak{U}$, there exists a positive integer N such that $x-x^{[N]} \varepsilon U$. For the future we shall consider only coordinated spaces with the two properties (P) and (AK).

Here we introduce a lemma on (AK) for continuous functions. Let I be a compact interval $[\tau, T]$, and $x(t)$ a continuous function on I to E. Then the n-th coordinate $x_n(t)=x_n(x(t))$ is also continuous as continuous function of a continuous function, and therefore so is $x^{[n]}(t)$ as finite sum of continuous functions.

Lemma 1. *Let $x(t)$ be a continuous function on I to E. Then, $x^{[n]}(t)$ converges to $x(t)$ uniformly for $t \varepsilon I$, that is, given an arbitrary neighbourhood $U \varepsilon \mathfrak{U}$, there exists a positive integer N_0 (dependent on U, but independent of t) such that $x(t)-x^{[n]}(t) \varepsilon U$ for $n \geqq N_0$.*

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(1) Numbers in brackets refer to the bibliograply at the end of the paper.

PROOF. For given $U \varepsilon \mathfrak{U}$, there exists a positive integer $N(U,t)$ (depending on U and t), such that for $n \geqq N$

$$x(t)-x^{[n]}(t) \varepsilon \frac{1}{2}U \quad \cdots\cdots \quad (1)$$

Because of the continuity of the function $x(t)-x^{[N]}(t)$, there exists a positive number δ (depending on U, t and N, hence on U and t) such that the inequality $|t-t'|<\delta$ implies the relation

$$\{x(t')-x^{[N]}(t')\}-\{x(t)-x^{[N]}(t) \varepsilon \frac{1}{2}U \quad \cdots\cdots \quad (2)$$

The relations (1) and (2) imply that

$$x(t')-x^{[N]}(t') \varepsilon U \quad \cdots\cdots \quad (3)$$

for $|t-t'|<\delta$. In other words, for every point t of I, there corresponds a pair of a positive integer N and an open interval $I_t = \{t'; |t-t'|<\delta\}$ (a half open interval, at the terminals τ and T), such that the relations (3) holds there. By virtue of Heine-Borel covering theorem, there exist a finite set of such intervals $I_{t_1}, I_{t_2}, \cdots\cdots, I_{t_m}$ as cover the whole interval I, and a finite set of corresponding postive intergers, namely $N_1, N_2, \cdots\cdots\cdots, N_m$. Let N_0 be the maximum of these positive integers. Then, for an an arbitrary t of I, there exixts an interval I_{t_i} of the set such that $t \varepsilon I_{t_i}$, and therefore

$$x(t)-x^{[N_i]}(t) \varepsilon U,$$

by the property (P), we have

$$x(t)-x^{[N_0]}(t) \varepsilon U$$

which proves the lemma.

2. *Continuous Dependency and Uniqueness.* In this paragraph, we shall assume only the completeness of the linear convex space E. Let I be a compact interval $[o, T]$ and $f(t,x)$ a function on $I \times E$ to E. For the future we shall assume the continuity of $f(t,x)$ with respect to (t,x). Lipschitz condition shall be stated in a more general form as follows: for the function $f(t, x)$ there exists a positive, (Riemann-) integrable function $\mu(t)$, such that $\int_0^T \mu(t)\,dt < \infty$ and that, for an arbitrary neighbourhood of the origin $U \varepsilon \mathfrak{U}$, the relation $x - x' \varepsilon U$ implies that $f(t,x)-f(t,x') \varepsilon \mu(t).U$ for every pair of points x, x' belonging to the space E (or, if necessary, to a given domain of E).

The problem of an ordinary differential equation:

$$\frac{dx}{dt} = f(t,x) \quad \cdots\cdots \quad (\mathrm{I})$$

under the initial conditions: $x=a$ for $t=o$, can be, as usually, reduced to the problem of the integral equation:

$$x(t) = a + \int_0^t f(t, x(t))\, dt \quad \cdots\cdots \quad \text{(II)}$$

In the present paper we shall assume the existence of a solution $x(t)$, instead of proving it, which may be difficult or, it seems, impossible under the conditions postulated in the present paper. The following theorem, although it does not immediately refer to the main subject of the paper, will afford the essential of the procedure of the proofs.

Theorem 1. [*Continuous dependency*] *Let* $f(t, x)$ *and* $g(t, x)$ *be functions on* $I \times E$ *to* E *and continuous, and we assume that the latter satisfies Lipschitz condition. Further, let* $x(t)$ *be a solution of the equation* (I) *and* $y(t)$ *that of the equaiion*

$$\frac{dx}{dt} = g(t, x) \quad \cdots\cdots \quad (4)$$

both under the same initial condition.

Then, for a given neighbourhood of the origin $U \in \mathfrak{U}$, *the relation*

$$f(t, x) - g(t, x) \in U_0 \quad for \quad t \in I, \quad x \in E$$

implies that $x(t) - y(t) \in \sigma U_0$, *where* σ *is an arbitrarily given positive number such that* $\sigma > Te^{\int_0^T \mu(t) dt}$.

PROOF From the definition of $x(t)$ and $y(t)$, it follows that

$$\begin{aligned} x(t) - y(t) &= \int_0^t [f(t, x(t)) - g(t, y(t))]\, dt \\ &= \int_0^t [f(t, x(t)) - g(t, x(t))]\, dt \\ &\quad + \int_0^t [g(t, x(t)) - g(t, y(t))]\, dt \quad \cdots\cdots \quad (5) \end{aligned}$$

From one of the assumptions, we have

$$f(t, x(t)) - g(t, x(t)) \in U_0, \quad \cdots\cdots \quad (6)$$

therefore

$$\int_0^t [f(t, x(t)) - g(t, x(t))]\, dt \in t\, U_0$$

The functions $x(t)$ and $y(t)$ are, as solutions of the differential equations, continuous functions of t, and so is the difference $x(t) - y(t)$. Since the set of points $\{x(t) - y(t);\ t \in I\}$ is compact as continuous image of the compact set I, there exists a positive number λ such that

$$x(t) - y(t) \in \lambda U_0 \quad for \quad t \in I \quad \cdots\cdots \quad (7)$$

Lipschitz condition upon the function $g(t, x)$ and the relation (7) give the relation

$$g(t, x(t)) - g(t, y(t)) \varepsilon \lambda\mu(t)\cdot U_0, \quad \cdots\cdots \quad (8)$$

therefore we have

$$\int_0^t [g(t, x(t)) - g(t, y(t))]\, dt \,\varepsilon\, \lambda\left(\int_0^t \mu(t)\, dt\right)\cdot U_0, \quad \cdots\cdots \quad (9)$$

hence from the equality (5),

$$x(t) - y(t) \,\varepsilon\, (t + \lambda\int_0^t \mu(t)\, dt)\cdot U_0 \quad \cdots\cdots \quad (10)$$

Here, if we substitute the relation (10), instead of that (7), into the left-hand side of the relation (9), we shall have, instead of the relation (9), the relation

$$\int_0^t [g(t, x(t)) - g(t, y(t))]\, dt \,\varepsilon\, \left(\int_0^t t\mu(t)dt + \lambda\int_0^t \mu(t)\int_0^t \mu(t)dt\, dt\right)\cdot U_0 \quad \cdots\cdots \quad (9)'$$

hence, from the equality (5), we shall have the relation

$$x(t) - y(t) \,\varepsilon\, \left[t + \int_0^t t\mu(t)dt + \lambda\int_0^t \mu(t)\int_0^t \mu(t)dt\, dt\right]\cdot U_0 \quad \cdots\cdots \quad (10)'$$

instead of the relation (10).
After p times iterrations of such procedure, we have

$$x(t) - y(t) \,\varepsilon\, \Big[t + \int_0^t t\mu(t)dt + \cdots + \int_0^t \mu(t)\int_0^t \cdots \int_0^t t\mu(t)dt \cdots\cdots dt$$
$$+ \lambda\int_0^t \mu(t)\int_0^t \cdots\cdots \int_0^t \cdots\cdots \int_0^t \mu(t)dt \cdots\cdots dt\Big]\cdot U_0 \quad \cdots\cdots \quad (11)$$

By virture of simple calculations and an evaluation, the relation (11) can be written as follows:

$$x(t) - y(t) \,\varepsilon\, \Big[t\Big\{1 + \frac{1}{1!}\int_0^t \mu(t)\, dt + \frac{1}{2!}\left(\int_0^t \mu(t)\, dt\right)^2 + \cdots\cdots.$$
$$+ \frac{1}{(p-1)!}\left(\int_0^t \mu(t)\, dt\right)^{p-1}\Big\} + \frac{\lambda}{p!}\left(\int_0^t \mu(t)\, dt\right)^p\Big]\cdot U_0 \quad \cdots\cdots \quad (12)$$

The expression enclosed with the bracket [ ] on the right-hand side of the relation (12) can be taken smaller than the beforehand given number σ, if we take the number p sufficiently large, accordingly we have

$$x(t) - y(t) \,\varepsilon\, \sigma U_0 \quad \cdots\cdots \quad (13)$$

which proves the theorem.

The case, in which $f(t, y) = g(t, x)$, gives the following

Corollary. [*Uniqueness.*] *Let the function $f(t, x)$ be continuous and satisfy Lipschitz condition. Then the differential equation* (I) *has at most one solution under the given* *initial condition*: $x=a$ *for* $t=o$.

Remark. The relation (11) and (12) are significant in the point of view of approximations of the solution $x(t)$ of (I) by that $y(t)$ of (4). If, especially, the function $\mu(t)$ of Lipschitz condition is constant, then the situation can be more simplied, namely, the relation (11) can be replaced by the relation

$$x(t)-y(t)\,\varepsilon\left[\frac{1}{\mu}\left[\frac{\mu t}{1!}+\frac{(\mu t)^2}{2!}+\cdots\cdots+\frac{(\mu t)^p}{p!}+\lambda\frac{(\mu t)^p}{p!}\right]\cdot U_0\right. \qquad (14)$$

Hence we have a more concrete estimation

$$x(t)-y(t)\,\varepsilon\,\sigma\cdot U_0$$

where σ is a positive function, arbitrarily taken but subject to the condition

$$\sigma>\frac{e^{\mu t}-1}{\mu} \qquad (15)$$

3. *Théorème des Réduites.* In this paragraph we assume that the space E is a coordinated space provided with the two properties (P) and (AK) and moreover that each space E_n, space of coordinate x_n, admits Picard or Peano existence theorem of the differential equation $\frac{dx_n}{dt}=f_n(t,\ x_n)$, as e. g. finite-dimensional spaces, Banach spaces, or spaces (M) and product spaces of spaces (M) [2].

We assume the initial condition:

$$x=a \quad \text{for} \quad t=o \qquad (16)$$

With a function $f(t,\ x)$ defined on $I\times E$, where $I=[o,\ T]$, we associate three functions

$$f^{[n]}(t,\ x),\ f(t,\ x^{[n]}),\ f^{[n]}(t,\ x^{[n]})$$

and denote each by

$$\hat{f}_{[n]}(t,\ x),\ \check{f}_{[n]}(t,\ x).\ \check{\hat{f}}_{[n]}(x)$$

respectively. As is easily seen, the continuity of the function $f(t,\ x)$ implies that of the three associated functions, and the fulfillment of Lipschitz condition on the former implies that on the three associated functions with common $\mu(t)$.

Now, with a given differential equation

$$\frac{dx}{dt}=f(t,\ x) \qquad (\mathrm{I})$$

we associate following three differential equations:

$$\frac{dx}{dt}=\hat{f}_{[n]}(t,\ x) \qquad (\mathrm{I.A})$$

$$\frac{dx}{dt} = \tilde{f}_{[n]}(t, x) \qquad \text{(I.B)}$$

$$\frac{dx}{dt} = \tilde{\tilde{f}}_{[n]}(t, x) \qquad \text{(I.C)}$$

Let each solution of (I), (I.A), (I.B) and (I.C) under the same initial condition (16) be denoted by $x(t)$, $\hat{x}_{[n]}(t)$, $\tilde{x}_{[n]}(t)$ and $\tilde{\tilde{x}}_{[n]}(t)$ respectively.

Theorem 2. *Let $f(t, x)$ be a function on $I \times E$ to E, where $I = [o, T]$, and continuous with respect to (t, x), and satisfy Lipschitz condition. Moreover we assume the existence of a solution $x(t)$ of* (I) *under the initial condition* (16).

Then the solution $x(t)$ can be approximated uniformly on the interval I by each solution $\hat{x}_{[n]}(t)$, $\tilde{x}_{[n]}(t)$, $\tilde{\tilde{x}}_{[n]}(t)$ as $n \to \infty$, i. e. for an arbitrary neighbourhood of the origin $U \in \mathfrak{U}$, there exists a positive integer N such that

$$n \geqq N \text{ implies } x(t) - \hat{x}_{[n]}(t) \in U_2 \text{ for } t \in I$$

and similarly for $\tilde{x}_{[n]}(t)$, $\tilde{\tilde{x}}_{[n]}(t)$.

PROOF (i). Let U be an arbitrary neighbourhood of the origin, and $\varkappa$ an arbitrarily chosen but fixed positive number such that

$$\varkappa < \frac{1}{T} e^{-\int_0^T \mu(t)dt}$$

where $\mu(t)$ is the function characterizing Lipschitz condition on the function $f(t, x)$. Since $x(t)$ is continuous, so is $f(t, x(t))$ and therefore by virtue of Lemma 1, there exists a positive integer N such that we have for $n \geqq N$

$$f(t, x(t)) - f^{[n]}(t, x(t)) \in U_0$$

or

$$f(t, x(t)) - \hat{f}_{[n]}(t, x(t)) \in U_0, \qquad (17)$$

where $U_0 = \varkappa U$. Because of the continuity of the solutions $x(t)$ and $\hat{x}_{[n]}(t)$, there exists a positive number λ such that

$$x(t) - \hat{x}_{[n]}(t) \in \lambda U_0 \qquad (18)$$

By virtue of Lipschitz condition upon $f(t, x)$ and the property (P), the relation (18) implies that

$$f^{[n]}(t, x(t)) - f^{[n]}(t, \hat{x}_{[n]}(t)) \in \lambda\mu(t) \cdot U_0$$

or

$$\hat{f}_{[n]}(t, x(t)) - \hat{f}_{[n]}(t, \hat{x}_{[n]}(t)) \in \lambda\mu(t) \cdot U_0 \qquad (19)$$

Now, replacing the functions $g(t, x)$ and $y(t)$ in the proof of theorem 1 by those $\hat{f}_{[n]}(t, x)$ and $\hat{x}_{[n]}(t)$ respectively, we have the relations (17), (18) and (19) corresponding to those (6), (7) and (8) respectively. Hence, according to an analogous argument there, we have

$$x(t) - \hat{x}_{[n]}(t) \varepsilon \left[t \left\{ 1 + \frac{1}{1!} \int_0^t \mu(t)\, dt + \frac{1}{2!} \left(\int_0^t \mu(t)\, dt \right)^2 \cdots\cdots \right.\right.$$

$$\left.\left. + \frac{1}{(p-1)!} \left(\int_0^t \mu(t)\, dt \right)^{p-1} \right\} + \frac{\lambda}{p!} \left(\int_0^t \mu(t)\, dt \right)^p \right] \cdot U_0 \quad \cdots\cdots\cdots\cdots \quad (20)$$

For sufficiently large p, we have

$$x(t) - \hat{x}_{[n]}(t) \varepsilon \varkappa U_0 = U \quad \cdots\cdots\cdots\cdots \quad (21)$$

which provides the proof for $\hat{x}_{[n]}(t)$.

(ii) Again analogously to the proof of Theorem 1, we can proceed as follows. From the definition

$$x(t) - \hat{x}_{[n]}(t) = \int_0^t \Big[f(t, x(t)) - \tilde{f}_{[n]}(t, x(t)) \Big] dt$$

$$+ \int_0^t \Big[\tilde{f}_{[n]}(t, x(t)) - \tilde{f}_{[n]}(t, \tilde{x}_{[n]}(t)) \Big] dt \quad \cdots\cdots\cdots \quad (22)$$

Let U be an arbitrary neighbourhood of the origin, and $\varkappa'$ an arbitrarily chosen but fixed positive number such that

$$\varkappa' < e^{-\int_0^T \mu(t) dt}$$

By virute of the continuity of the solution $x(t)$ and Lemma 1, there exists a positive integer N such that we have, for n $\geqq N$,

$$x(t) - x_{[n]}(t) \varepsilon U_0 \quad \cdots\cdots\cdots\cdots \quad (23)$$

where $\varkappa' U_0 = U$. Since $\tilde{f}_{[n]}(t, x(t)) = f(t, x_{[n]}(t))$, by virtue of Lipschitz condition on the function $f(t, x)$, the function (23) implies that

$$f(t, x(t)) - \tilde{f}_{[n]}(t, x(t)) \varepsilon \mu(t) \cdot U_0 \cdot \quad \cdots\cdots\cdots\cdots \quad (24)$$

Therefore

$$\int_0^t \Big[f(t, x(t)) - \tilde{f}_{[n]}(t, x(t)) \Big] dt \varepsilon \left(\int_0^t \mu(t)\, dt \right) \cdot U_0 \cdot \quad \cdots\cdots\cdots\cdots \quad (25)$$

Because of the continuity of $x(t) - \tilde{x}_{[n]}(t)$, as frequently observed, there exists a positive integer λ such that

$$x(t) - \tilde{x}_{[n]}(t) \varepsilon \lambda U_0 , \quad \cdots\cdots\cdots\cdots \quad (26)$$

accordingly

$$x^{[n]}(t) - \tilde{x}^{[n]}_{[n]}(t) \varepsilon \lambda U_0 .$$

This relation implies, since $\widehat{f}_{[n]}(t, x) = f(t, x^{[n]})$ again, that

$$\widehat{f}_{[n]}(t, x(t)) - \widehat{f}_{[n]}(t, \tilde{x}_{[n]}) \varepsilon \lambda \mu(t) \cdot U_0 \quad \cdots\cdots\cdots \quad (27)$$

Therefore

$$\int_0^t \left[\widehat{f}_{[n]}(t, x(t)) - \widehat{f}_{[n]}(t, \tilde{x}_{[n]}(t))\right] dt \varepsilon \lambda \left(\int_0^t \mu(t)\, dt\right) \cdot U_0 \quad \cdots\cdots \quad (28)$$

The relations (22), (25) and (28) give the relation

$$x(t) - \tilde{x}_{[n]}(t) \varepsilon (1+\lambda) \left(\int_0^t \mu(t)\, dt\right) \cdot U_0$$

Analogously to the proof of Theorem 1, after p times iterrations, we have

$$x(t) - \tilde{x}_{[n]}(t) \varepsilon \left[\frac{1}{1!}\int_0^t \mu(t)\, dt + \frac{1}{2!}\left(\int_0^t \mu(t)\, dt\right)^2 + \cdots\cdots\right.$$
$$\left. + \frac{1}{p!}\left(\int_0^t \mu(t)\, dt\right)^p + \frac{\lambda}{p!}\left(\int_0^t \mu(t)\, dt\right)^p\right] \cdot U_0 ,$$

accordingly, since $\varkappa' U_0 = U$, we have

$$x(t) - \tilde{x}_{[n]}(t) \varepsilon U.$$

which proves the second part of the theorem.

(iii) Let U be an arbitrary neighbourhood of the origin. Then, by virtue of the Lemma 1, the just proved, and the property (AK), there exists a positive integer N such that we have for $n \geqq N$

$$x(t) - x_{[n]}(t) \varepsilon \frac{1}{3} U, \quad \cdots\cdots\cdots \quad (30)$$

$$x(t) - \tilde{x}_{[n]}(t) \varepsilon \frac{1}{3} U, \quad \cdots\cdots\cdots \quad (31)$$

$$a - a^{[n]} \quad \varepsilon \frac{1}{3} U \quad \cdots\cdots\cdots \quad (32)$$

By virtue of the property (P) the relation (31) implies that

$$x^{[n]}(t) - \tilde{x}^{[n]}_{[n]}(t) \varepsilon \frac{1}{3} U \quad \cdots\cdots\cdots \quad (33)$$

Since $\tilde{x}^{[n]}_{[n]}(t) = \tilde{\tilde{x}}^{[n]}_{[n]}(t)$ and $\tilde{\tilde{x}}_{[n]}(t) - \tilde{\tilde{x}}^{[n]}_{[n]}(t) = a - a^{[n]}$
it follows that

$$\tilde{\tilde{x}}_{[n]}(t) - \tilde{x}^{[n]}_{[n]} x(t) \varepsilon \frac{1}{3} U \quad \cdots\cdots\cdots \quad (34)$$

The relations (30), (33) and (34) give the relation

$$x(t) - \hat{\hat{x}}_{[n]}(t) \in U \quad \cdots\cdots (35)$$

which proves the last part of the theorem. Thus the proof has been completed.

Remark. In case $a = 0$, clearly $\hat{\hat{x}}_{[n]}(t) = \hat{x}_{[n]}\, x(t)$, hence the last proof (iii) is superfluous.

Example 1. We consider a differential equation

$$\frac{dx}{dt} = x \quad \cdots\cdots (36)$$

under the initial condition (16). This satisfies the continuity and Lipschitz condition, considered as equation both in the space (m) and in the product space, and its solution is given as follows:

$$x = e^t.\ a \quad \cdots\cdots (37)$$

In case the sequence $\{a_n\}$ of elements of $a = \{a_1, a_2, \cdots\cdots, a_n, \cdots\cdots\}$ is not convergent to zero, Theorem 2 holds in the latter space, but does not in the former, which, of course, has not the property (AK). On the contrary, in case the sequence $\{a_1, a,_2 \cdots\cdots, a_n, \cdots\cdots\}$ is convergent to zero, considered as equation in the space (c_0), Theorem 2 holds, and clearly this space has the property (AK).

Example 2. We consider the differential equation (I), where $t > 0$, $|x| \leqq l$. We assume that each component $f_n(t, x)$ of the right-hand side of (I) is continuous with respect to t, uniformly for both the coordinate number n and x, and moreover satisfies the condition:

$$|f_n(t, x) - f(t, x')| \leqq \mu(t) \sum_{i=1}^{\infty} c_{ni} |x_i - x'_i| \quad \cdots\cdots (38)$$

where each series $\sum_{i=1}^{\infty} c_{ni}$ $(c_{ni} \geqq 0)$ is convergent, uniformly for the number n, and each sum is bounded by a positive constant L, and $\int_0^T \mu(t)\, dt < \infty$ for every value of T ([3], whose summary only we we know through the review of Mathematical Review.).

Considered as an equation in the space (m), the equation (I) satisfies Lipschitz condition in the form:

$$\|f(t, x) - f(t, x')\| \leqq L \mu(t) \|x - x'\|, \quad \cdots\cdots (39)$$

but the property (AK) is not fulfilled, and therefore the result of Theorem 2 cannot be guarranteed. Now we define a norm as follows:

$$\|x\| = \sup_n \sum_{i=i}^{\infty} c_{ni} |x|$$

then we have $\|x\| \leqq Ll$, and Lipschitz condition in the same form as (39). The sequence space, topologized by this norm, has the property (AK). Indeed, for an arbitrary positive number ε, there exists a positive integer N such that, by virtue of the uniformity of the convergence of the series $\sum_{i=1}^{\infty} c_{mi}$, $\sum_{i=n+1}^{\infty} c_{mi} < \frac{\varepsilon}{Ll}$ for $n \geqq N$, hence it follows

$$\| x - x^{[n]} \| = \sup_m \sum_{i=n+1}^{\infty} c_{mi} \cdot | x_\iota | \leqq \frac{\varepsilon}{l} l = \varepsilon.$$

Thus the result of Theorem 2 is ascertained.

4. *Case of General Coordinated spaces.* Let Γ be a family of indices, ι an element or index of Γ, and for each $\iota \varepsilon \Gamma$, E_ι a linear convex space. We consjder a set E whose elements x are $\{x_\iota\}_{\iota \varepsilon \Gamma}$, where $x_\iota \varepsilon E_\iota$ and is called the ι-coordinate of x. E is called a (general) coordinated space, if it is toplogized locally convex by a fundamental system of neighbourhoods $\mathfrak{U} = \{U\}$ such that each mapping $x_\iota(x)$: $x \rightarrow x_\iota$ is (i) linear, i. e. for every α, β, $x_\iota(\alpha x + \beta x') = \alpha x_\iota(x) + \beta x_\iota(x')$ and (ii) continuous, i. e. for an arbitrary neighbourhood of the origin $U \varepsilon \mathfrak{U}$ in E, there exists a neighbourhood of the origin U_ι in E_ι, such that the relation $x \varepsilon U$ implies that $x_\iota \varepsilon U_\iota$. A projection of x and the property (P) are defined analogously to the case of special coordinated spaces (Cf. 1).

Let a finite set of indices $\{\iota_1, \iota_2, \cdots\cdots, \iota_n\}$ be denoted by J, and ordering $(\leqq)$ by set-inclusion, the totality of J forms a directed system $\{J\}$, denoted by $\mathfrak{J}$. A set of points directed by $\mathfrak{J}$ is denoted by $\{x_J\}_{J \varepsilon \mathfrak{J}}$. $\mathfrak{J}$-convergence aud $\mathfrak{J}$-limit are defined as fllows: $\{x_J\}_{J \varepsilon \mathfrak{J}}$ is $\mathfrak{J}$-convegent to x (called $\mathfrak{J}$-limit of $\{x_J\}_{J \varepsilon \mathfrak{J}}$), if, for an arbitrary neighbourhood of the origin $U \varepsilon \mathfrak{U}$, there exists an element $J_0 \varepsilon \mathfrak{J}$ such that the relation $J_0 \leqq J$ implies that $x_J - x \varepsilon U$. For an element J of $\mathfrak{J}$ and a point x of E, there corresponds a projection of x, donoted by $x^{[J]}$, which is constructed dy equating all ι-coordinates $(\iota \bar{\varepsilon} J)$ of x to zero. The space E is called to have the property "$\mathfrak{J}$-Abschnittkonvergenz" or simply "Abschnittkonvergenz", if for every point to x, the corresponding directed set $\{x^{[J]}\}_{J \varepsilon \mathfrak{J}}$ is $\mathfrak{J}$-convergent to x. Specially in the coordinated space E provided with the propety (P), (AK) is stated simplied as follows: for an arbitrary neighbourhood of the origin $U \varepsilon \mathfrak{U}$, there exists an element J_0 of $\mathfrak{J}$, such that $x - x^{[J_0]} \varepsilon U$.

A lemma analogous to Lemma 1 is given, whose proof is quite the same.

Lemma 2. *Let* $x(t)$ *be a continuoús function on a compact interval* I *to* E, *which has the property* (P) *and that of* (AK).

Then $\{x^{[J]}(t)\}_{J \varepsilon \mathfrak{J}}$ *is* $\mathfrak{J}$-*convergent to* $x(t)$ *uniformly for* $t \varepsilon I$, *i. e. given an arbitrary neighbourhdod* U, *there exists an element* J_0 *of* $\mathfrak{J}$ *(dependent on* U, *but indepent of* t*) such that* $x(t) - x^{[J]}(t) \varepsilon U$ *for* $J_0 \leqq J$.

We shall assume also in this paragraph that each space E_ι, space of coordinate x_ι, admits Picard or Peano existence theorem of the differential equation $\frac{dx_\iota}{dt} = f_\iota(t, x_\iota)$. As done in the preceding paragraph, with a function $f(t, x)$ defined on $I \times E$, associate we, for eah $J \varepsilon \mathfrak{J}$, following three functions

$$f^{[J]}(t, x),\ f(t, x^{[J]}),\ f^{[J]}(t, x^{[J]})$$

and denote each by

$$\hat{f}_{[J]}(t, x),\ \tilde{f}_{[J]}f(t, x_{[J]}),\ \tilde{\tilde{f}}_{[J]}(t, x).$$

Correspondingly, with a given differential equations

$$\frac{dx}{dt} = f(t, x) \cdots\cdots \quad (\mathrm{I})$$

we can associate following three differential equations:

$$\frac{dx}{dt} = \hat{f}_{[J]}(t, x) \cdots\cdots \quad (\mathrm{I}.\alpha)$$

$$\frac{dx}{dt} = \tilde{f}_{[J]}(t, x) \cdots\cdots \quad (\mathrm{I}.\beta)$$

$$\frac{dx}{dt} = \tilde{\tilde{f}}_{[J]}(t, x) \cdots\cdots \quad (\mathrm{I}.\gamma)$$

and we denote each solution of (I), (I. α), (I. β) and (I. γ) under the same initial condition (16) by $x(t)$, $\hat{x}_{[J]}(t)$, $\tilde{x}_{[J]}(t)$ and $\tilde{\tilde{x}}_{[J]}(t)$ respectively.

The argument in the preceding paragraph, is, as is easily seen, quite applicable to the general coordinated space and $\mathfrak{J}$-convergence without almost any modifications. Thus we can state the following more general

Theorem 3. *Let* E *be a complete, general coordinated space provided with two properties* (P) *and* (AK), *and* I *a compact interval* $[o, T]$. *Let* $f(t, x)$ *be a function on* $I \times E$ *to* E, *continuous with respect to* (t, x), *and satisfy Lipschitz condition. Moreover we assume the existence of a solution* $x(t)$ *of* (I) *under the initial conditions* (16).

Then the solution $x(t)$ *can be approximated uniformly on the interval* I *by each solutions* $\hat{x}_{[J]}(t)$, $\tilde{x}_{[J]}(t)$, $\tilde{\tilde{x}}_{[J]}(t)$, *or more precisely, for an arbitrary neighbourhood of the origin* $U \in \mathfrak{U}$, *there exists an element* $J_0 \in \mathfrak{J}$ *such that* $J_0 \leqq J$ *implies* $x(t) - \hat{x}_{[J]}(t) \in U$ *for* $t \in I$ *and similarly for* $\tilde{x}_{[J]}(t)$ *and* $\tilde{\tilde{x}}_{[J]}(t)$.

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