NOTE ON BLOCKS OF GROUP CHARACTERS

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By Prof. R. BRAUER a number of theorems concerning the blocks of group characters were given without detailed proofs ([2], [3], [4]). Recently, Prof. M. OSIMA determined the primitive idempotents of the center of a modular group ring and gave many properties of the idempotents. Making use of them, he proved some of Brauer's theorems ([7]).

The purpose of this paper is to verify directly some of Brauer's theorems and to derive some properties of idempotents of the center of the modular group ring. § 1 concerns with the elementary divisors of Cartan matrices corresponding to blocks and § 2 with the defect groups of blocks. In § 3 we deal with some properties of the idempotents.

Notations.

$\mathfrak{G}$: A group of finite order g.

p: A fixed rational prime number.

a: The exponent of the highest power of p dividing g.

$K_1, K_2, \cdots\cdots, K_n$: The classes of conjugate elements of $\mathfrak{G}$. K_{ν^*} denotes the class which consists of elements reciprocal to those of K_ν.

$\mathfrak{C}(\ )$: The centralizer of a subgroup in $\mathfrak{G}$.

$\mathfrak{N}(\ )$: The normalizer of a subgroup or an element in $\mathfrak{G}$.

n_ν : The order of the normalizer $\mathfrak{N}(G_\nu)$ of an element G_ν in K_ν.

$\mathfrak{H}_\nu$: The defect group of K_ν for p, i.e., a p-Sylow subgroup of the normalizer $\mathfrak{N}(G_\nu)$ of an element G_ν in K_ν, which is uniquely determined by the class K_ν up to conjugacy.

ρ_ν : The defect of K_ν for p, i. e., the order of $\mathfrak{H}_\nu$.

g_ν : The number of elements in K_ν, $g_\nu = g/n_\nu$.

K : The field of the g-th roots of unity.

$\mathfrak{p}$: A prime ideal of K dividing p. We also denote by $\mathfrak{p}$ the ideal $\mathfrak{p}\mathfrak{o}_\mathfrak{p}$ in the ring $\mathfrak{o}_\mathfrak{p}$ of $\mathfrak{p}$-integers of K.

K^*: The residue class field $\mathfrak{o}_\mathfrak{p}/\mathfrak{p}$.

$\Gamma(\ )$: The group ring of a subgroup of $\mathfrak{G}$ over K.

$\Lambda(\ )$: The center of $\Gamma(\ )$.

$\Gamma^*(\ )$: The group ring of a subgroup of $\mathfrak{G}$ over K^*.

$\Lambda^*(\ )$: The center of $\Gamma^*(\ )$.

$\zeta^{(1)}, \zeta^{(2)}, \cdots\cdots, \zeta^{(n)}$: The absolutely irreducible ordinary characters of $\mathfrak{G}$. $\zeta_\nu^{(i)}$ denotes the value of $\zeta^{(i)}$ for the class K_ν.

z_i : The degree of $\zeta^{(i)}$.

$\omega^{(i)}$: The linear character of $\Lambda(\mathfrak{G})$ belonging to $\zeta^{(i)}$. $\omega_\nu^{(i)}$ denotes the value of $\omega^{(i)}$ for the class K_ν, $\omega_\nu^{(i)}=g_\nu \zeta_\nu^{(i)}/z_i$.

$\varphi^{(1)}, \varphi^{(2)}, \cdots\cdots, \varphi^{(k)}$: The absolutely irreducible modular characters of $\mathfrak{G}$ for p defined as complex numbers. $\varphi_\nu^{(\kappa)}$ denotes the value of $\varphi^{(\kappa)}$ for the p-regular class K_ν.

Z : The matrix $(\zeta_\nu^{(i)})_{i,\nu}$ of ordinary characters of $\mathfrak{G}$.

Φ : The matrix $(\varphi_\nu^{(\kappa)})_{\kappa,\nu}$ of modular characters of $\mathfrak{G}$.

$d_{i\kappa}$: The decomposition numbers of $\mathfrak{G}$ for p.

$c_{\kappa\lambda}$: The Cartan invariants of $\mathfrak{G}$ for p.

D : The matrix $(d_{i\kappa})_{i,\kappa}$ $(i=1, 2, \cdots\cdots, n;\ \kappa=1, 2, \cdots\cdots, k)$.

C : The matrix $(c_{\kappa\lambda})_{\kappa,\lambda}$ $(\kappa, \lambda=1, 2, \cdots\cdots, k)$.

$B_1, B_2, \cdots\cdots, B_s$: The blocks of characters of $\mathfrak{G}$ for p. Corresponding to $B_1, B_2, \cdots\cdots, B_s$ if the rows and columns are arranged suitably, C and D have the forms

$$C=\begin{pmatrix} C_1 & & 0 \\ & C_2 \ddots & \\ 0 & & C_s \end{pmatrix}, \quad D=\begin{pmatrix} D_1 & & 0 \\ & D_2 \ddots & \\ 0 & & D_s \end{pmatrix}.$$

From now on we assume that C and D are of these forms.

1. Let $K_1, K_2, \cdots\cdots\cdots, K_k$ be the p-regular classes. Let x_τ and y_τ denote the numbers of $\zeta^{(i)}$ and $\varphi^{(\kappa)}$ in B_τ, respectively. Consider the k columns of Z which belong to $K_1, K_2, \cdots\cdots, K_k$. Then we can select a minor Δ of degree k which is not divisible by $\mathfrak{p}$. As is well known, y_τ of the rows in Δ must belong to $\zeta^{(i)}$ in B_τ. It is then possible to associate y_τ of the p-regular classes with B_τ in such a way, that each p-regular class is associated with one and only one block and that the y_τ rows of Δ belonging to $\zeta^{(i)}$ in B_τ and the y_τ columns belonging to the p-regular classes associated with B_τ form a minor which is not divisible by $\mathfrak{p}$.

[1.1] *If the p-regular classes $K_\lambda, K_\mu, \cdots\cdots, K_\sigma$ are associated with B_τ, then the elementary divisors of C_τ are $p^{\rho_\lambda}, p^{\rho_\mu}, \cdots\cdots, p^{\rho_\sigma}$.*

PROOP. Corresponding to the form of C we write

$$\Phi=\begin{pmatrix} \Phi_1 & & * \\ & \Phi_2 \ddots & \\ * & & \Phi_s \end{pmatrix}$$

where the columns containing Φ_τ correspond to y_τ p-regular classes associated with B_τ.

Then we see that

$$|\Phi_\tau| \not\equiv 0 \pmod{\mathfrak{p}} \qquad (\tau = 1, 2, \cdots\cdots, s).$$

Similarly we write

$$\Psi = (\varphi_{\nu^*}^{(\kappa)})_{\kappa,\nu}^{-1} = \begin{pmatrix} \Psi_1 & & & * \\ & \Psi_2 & & \\ & & \ddots & \\ * & & & \Psi_s \end{pmatrix}.$$

Since $|\Phi| \not\equiv 0 \pmod{\mathfrak{p}}$, all coefficients of Ψ are $\mathfrak{p}$-integers. From the orthogonal relations for modular characters of $\mathfrak{G}$ we have

$$C_\tau \Phi_\tau = \Psi'_\tau \begin{pmatrix} n_\lambda & & & 0 \\ & n_\mu & & \\ & & \ddots & \\ 0 & & & n_\sigma \end{pmatrix}$$

where Ψ'_τ is the transpose of Ψ_τ. Therefore

$$|C_\tau| \geqq p^{\rho_\lambda + \rho_\mu + \cdots\cdots + \rho_\sigma} \qquad (\tau = 1, 2, \cdots\cdots, s).$$

On the other hand

$$|C| = \prod_\tau |C_\tau| = p^{\rho_1 + \rho_2 + \cdots\cdots + \rho_k}.$$

Hence

$$|C_\tau| = p^{\rho_\lambda + \rho_\mu + \cdots\cdots + \rho_\sigma}.$$

Thus we have

$$|\Psi_\tau| \not\equiv 0 \pmod{\mathfrak{p}},$$

and hence the elementary divisors of C_τ are $p^{\rho_\lambda}, p^{\rho_\mu}, \cdots\cdots, p^{\rho_\sigma}$.

[1.2] *If B_τ is a block of defect d, then the corresponding Cartan matrix C_τ has one elementary divisor p^d while all other elementary divisors are powers of p with exponents smaller than d.*

PROOF.[1] Let $p^{\varepsilon_1}, p^{\varepsilon_2}, \cdots\cdots, p^{\varepsilon_y}$ $(\varepsilon_1 \leqq \varepsilon_2 \leqq \cdots\cdots \leqq \varepsilon_y,\ y = y_\tau)$ be the elementary divisors of C_τ and consider

$$S_{ij} = \sum_{\nu=1}^{k} g_\nu \zeta_\nu^{(i)} \zeta_{\nu^*}^{(j)} \qquad (\zeta^{(i)}, \zeta^{(j)} \text{ in } B_\tau).$$

Then we see that

$$S_\tau = (S_{ij})_{i,j} = D_\tau g C_\tau^{-1} D'_\tau.$$

All coefficients of S_τ are rational integers and the elementary divisors of S_τ are $p^{a-\varepsilon_y}, \cdots,$

1) This proof is due to [1]. Cf. [1], footnote 18 and § 8.

$p^{a-\varepsilon_2}$, $p^{a-\varepsilon_1}$, 0, ..., 0. Since every $\omega_\nu^{(i)}$ is an algebraic integer and the degree z_i of each $\zeta^{(i)}$ in B_τ is divisible at least by p^{a-d}, we have

(1.1) $$S_{ij} = S_{ji} \equiv 0 \pmod{p^{a-d}}.$$

Hence we have $\varepsilon_y \leqq d$. On the other hand, ε_y is not less than d.[2] Consequently we have

$$\varepsilon_y = d.$$

Let $\zeta^{(i_1)}$ be a character in B_τ such that z_{i_1} is exactly divisible by p^{a-d}. Since the congruences

(1.2) $$\omega_\nu^{(i)} \equiv \omega_\nu^{(j)} \pmod{\mathfrak{p}} \qquad (\zeta^{(i)},\ \zeta^{(j)} \text{ in } B_\tau)$$

hold for all classes K_ν of $\mathfrak{G}$, we have

(1.3) $$S_{ij} \equiv \frac{z_i}{z_{i_1}} S_{i_1 j} \pmod{p^{a-d+1}}.$$

In particular

$$S_{i_1 j} = S_{j i_1} \equiv \frac{z_j}{z_{i_1}} S_{i_1 i_1} \pmod{p^{a-d+1}},$$

and therefore

$$S_{ij} \equiv \frac{z_i\, z_j}{z_{i_1}^2} S_{i_1 i_1} \pmod{p^{a-d+1}}.$$

Thus we have

$$S_\tau = S_{i_1 i_1}\left(\frac{z_i\, z_j}{z_{i_1}^2}\right) + H_\tau .$$

H_τ is a matrix in which all coefficients are divisible by p^{a-d+1}. Since every minor of degree 2 of S_τ is divisible by $p^{(a-d)+(a-d+1)}$, we have

$$\varepsilon_{y-1} < d.$$

[1.3] *A character* $\zeta^{(i)}$ *belongs to a block of defect larger than or equal to a given number d, if and only if*

(1.4) $$\omega_\nu^{(i)} \equiv 0 \pmod{\mathfrak{p}}$$

for all p-regular classes K_ν with $\rho_\nu < d$.

PROOF. Since the sufficiency is obvious, we prove the necessity. Suppose that $\zeta^{(i)}$ belongs to a block B_τ of defect d_τ smaller than d. By [1.1] and [1.2] we can find a p-regular class K_μ with $\rho_\mu = d_\tau$ among the p-regular classes associated with B_τ in the sense of [1.1]. Then we have by (1.4)

2) See [5], §7.

$$\frac{g_\mu \zeta_\mu^{(j)}}{z_j} = \omega_\mu^{(j)} \equiv \omega_\mu^{(i)} \equiv 0 \pmod{\mathfrak{p}}$$

for all $\zeta^{(j)}$ in B_τ. Since g_μ is exactly divisible by p^{a-d_τ} and z_j is divisible at least by p^{a-d_τ},

$$\zeta_\mu^{(j)} \equiv 0 \pmod{\mathfrak{p}}$$

holds for all $\zeta^{(j)}$ in B_τ, which yields a contradiction. Therefore we have

$$d_\tau \geqq d.$$

[1.4] *Two characters $\zeta^{(i)}$ and $\zeta^{(j)}$ belonging to blocks of defect d appear in the same block, if and only if*

(1.5) $$\omega_\nu^{(i)} \equiv \omega_\nu^{(j)} \pmod{\mathfrak{p}}$$

for all p-regular classes K_ν with $\rho_\nu = d$.

PROOF. Since the sufficiency is obvious, we prove the necessity. Suppose that $\zeta^{(i)}$ and $\zeta^{(j)}$ belong to different blocks B_σ and B_τ, respectively. We now consider

$$S_{lm} = \sum_{\nu=1}^{k} g_\nu \zeta_\nu^{(l)} \zeta_{\nu^*}^{(m)}$$

for $\zeta^{(l)}$ and $\zeta^{(m)}$ belonging to either B_σ or B_τ. Arranging the rows and columns suitably we have

$$(S_{lm})_{l,m} = \begin{pmatrix} D_\sigma g C_\sigma^{-1} D'_\sigma & 0 \\ 0 & D_\tau g C_\tau^{-1} D'_\tau \end{pmatrix}.$$

Therefore by [1.2] two of the elementary divisors of the matrix (S_{lm}) are equal to p^{a-d}. On the other hand, from (1.5) we can deduce the following congruences

$$S_{lm} \equiv \frac{z_l}{z_{i_1}} S_{i_1 l} \pmod{p^{a-d+1}}$$

where z_{i_1} is exactly divisible by p^{a-d}. Thus we can conclude in the same way as in the proof of [1.2] that the elementary divisors of (S_{lm}) are larger than p^{a-d} except one, which yields a contradiction.

2. We also denote by K_ν the sum of all elements in the class K_ν. Then $K_1, K_2, \cdots\cdots, K_n$ form a K-basis of $\Lambda = \Lambda(\mathfrak{G})$ and we have

(2.1) $$K_\alpha K_\beta = \sum_\gamma a_{\alpha\beta\gamma} K_\gamma$$

where $a_{\alpha\beta\gamma}$ are non-negative rational integers.

Let $\mathfrak{H}$ be any subgroup of order p^h of $\mathfrak{G}$ and consider a subgroup $\mathfrak{N}$ satisfying

(2.2) $$\mathfrak{H}\mathfrak{C}(\mathfrak{H}) \subseteqq \mathfrak{N} \subseteqq \mathfrak{N}(\mathfrak{H}).$$

Denote by K_ν^0 the part of K_ν which lies in $\mathfrak{C}(\mathfrak{H})$. K_ν^0 is a sum of complete classes of $\mathfrak{N}$ or zero according as K_ν contains any element of $\mathfrak{C}(\mathfrak{H})$ or not. From (2.1) we can see that

$$K_\alpha^0 K_\beta^0 = \sum_\gamma a_{\alpha\beta\gamma} K_\gamma^0 \pmod{p}. \tag{2.3}$$

We show this. If we set $K_\nu = K_\nu^0 + K_\nu^1$, (2.1) yields

$$K_\alpha^0 K_\beta^0 + K_\alpha^0 K_\beta^1 + K_\alpha^1 K_\beta^0 + K_\alpha^1 K_\beta^1 = \sum_\gamma a_{\alpha\beta\gamma} K_\gamma^0 + \sum_\gamma a_{\alpha\beta\gamma} K_\gamma^1.$$

Let m_G denote the coefficient of an element G in $K_\alpha^1 K_\beta^1$ and for each element R of $\mathfrak{C}(\mathfrak{H})$ consider the set $\mathfrak{P}_R$ of all elements P in K_α^1 such that $PQ = R$ for a suitable element Q in K_β^1. It is then easy to see that m_R is equal to the number of elements of $\mathfrak{P}_R$. In order to see that m_R is divisible by p, we consider the elements P', P'', ... which are the transforms of an element P' in $\mathfrak{P}_R$ by the elements of $\tilde{\mathfrak{N}}(R) = \mathfrak{N}(R) \cap \mathfrak{N}$. Obviously all elements $P^{(i)}$ belong to $\mathfrak{P}_R$ and the number of elements $P^{(i)}$ is equal to the index $[\tilde{\mathfrak{N}}(R): \mathfrak{N}(P') \cap \tilde{\mathfrak{N}}(R)]$. This index is divisible by p, since $\mathfrak{H}$ is a normal subgroup of $\tilde{\mathfrak{N}}(R)$ while $\mathfrak{H}$ is not contained in $\mathfrak{N}(P') \cap \tilde{\mathfrak{N}}(R)$. Continuing the same precess for the remaining P of $\mathfrak{P}_R$, we can finally see that m_R is equal to a sum of such indices. Hence m_R is divisible by p, from which (2.3) follows.

If $\mathfrak{H}_\alpha \not\supseteqq \mathfrak{H}$[3], then by (2.3) we have

$$a_{\alpha\beta\gamma} \equiv 0 \pmod{p} \tag{2.4}$$

for any class K_γ with $\mathfrak{H}_\gamma \supseteqq \mathfrak{H}$. Therefore K_ν with $K_\nu^0 = 0$ form a K^*-basis of an ideal T^* in $\Lambda^* = \Lambda^*(\mathfrak{G})$ and K_ν^0 ($\neq 0$) form a K^*-basis of a subring R^* of $\tilde{\Lambda}^* = \Lambda^*(\mathfrak{N})$. Thus we have

$$R^* \cong \Lambda^*/T^*. \tag{2.5}$$

Let $\tilde{K}_1$, $\tilde{K}_2$, ... be the classes of conjugate elements of $\mathfrak{N}$. Then $\tilde{K}_\nu$ either does not contain any elements of $\mathfrak{C}(\mathfrak{H})$ or consists of elements of $\mathfrak{C}(\mathfrak{H})$. The classes which consist of elements of $\mathfrak{C}(\mathfrak{H})$ form a K^*-basis of a subring $\tilde{R}^*$ of $\tilde{\Lambda}^*$ and the others form a K^*-basis of an ideal $\tilde{T}^*$ in $\tilde{\Lambda}^*$. We see that $\tilde{\Lambda}^*$ is the direct sum

$$\tilde{\Lambda}^* = \tilde{R}^* + \tilde{T}^*. \tag{2.6}$$

[2.1] *If $\mathfrak{G}$ contains a normal subgroup $\mathfrak{H}$ of order p^d, then there exists at least one character $\zeta^{(i)}$ in each B_τ which represents the elements of $\mathfrak{H}$ by the unit matrix and all blocks of $\mathfrak{G}$ have at least the defect d.*

PROOF. We have

$$\sum_{H \text{ in } \mathfrak{H}} \sum_{i \text{ in } B_\tau} d_{i\kappa} \zeta^{(i)}(H) = \sum_{i \text{ in } B_\tau} d_{i\kappa} z_i \neq 0,$$ [4]

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[3]) "$\mathfrak{H}_\mu \supseteqq \mathfrak{H}$" means "$\mathfrak{H}$ is conjugate to a subgroup of $\mathfrak{H}_\mu$".

because

$$\sum_{i \text{ in } B_\tau} d_{i\kappa}\zeta^{(i)}(S) = 0$$

holds for all p-singular elements S of $\mathfrak{G}$. Hence there exists at least one $\zeta^{(i)}$ in B_τ such that the 1-character of $\mathfrak{H}$ appears in the character $\zeta^{(i)}(H)$ of $\mathfrak{H}$. Then $\zeta^{(i)}$ represents all elements of $\mathfrak{H}$ by the unit matrix.

Let denote by $\omega_\tau{}^*$ the linear character of Λ^* corresponding to B_τ.

[2.2] *If $\mathfrak{G}$ contains a normal subgroup $\mathfrak{H}$ of order p^h, then any linear character of Λ^* vanishes for all elements of T^*.*

PROOF. Let K_μ be a class belonging to T^* and $\overline{K}_\mu$ the class of conjugate elements of the group $\mathfrak{G}/\mathfrak{H}$ which contains the coset $\overline{G}_\mu$ of an element G_μ in K_μ. The defect group $\overline{\mathfrak{H}}_\mu$ of $\overline{K}_\mu$ contains $\mathfrak{H}_\mu\mathfrak{H}/\mathfrak{H}$ while $\mathfrak{H}_\mu$ does not contain $\mathfrak{H}$. Hence we have $\overline{\rho}_\mu > \rho_\mu - h$, where $\overline{\rho}_\mu$ is the defect of $\overline{K}_\mu$. Therefore, if we denote by $\overline{g}_\mu$ the number of elements in $\overline{K}_\mu$

$$\frac{g_\mu}{\overline{g}_\mu} \equiv 0 \pmod{p}. \tag{2.7}$$

On the other hand, if $\zeta^{(i)}$ is a character in B_τ which represents all elements of $\mathfrak{H}$ by the unit matrix, we have

$$\frac{\overline{g}_\mu\zeta^{(i)}(\overline{G}_\mu)}{z_i} = \frac{\overline{g}_\mu\zeta_\mu^{(i)}}{z_i}. \tag{2.8}$$

Since $\overline{g}_\mu\zeta^{(i)}(\overline{G}_\mu)/z_i$ is an algebraic integer, it follows from (2.7) and (2.8) that

$$\omega_\tau^*(K_\mu) = \omega_\mu^{(i)} \pmod{\mathfrak{p}} = 0.$$

We again consider a subgroup $\mathfrak{N}$ satisfying (2.2). Denote by $\tilde{B}_1, \tilde{B}_2, \cdots\cdots\cdots$ the blocks of characters of $\mathfrak{N}$ for p and by $\tilde{\omega}_\sigma^*$ the linear character of $\tilde{\Lambda}^*$ corresponding to $\tilde{B}_\sigma$. $\tilde{\omega}_\sigma^*$ induces a linear character of R^* and therefore induces a linear character ω_τ^* of Λ^* which vanishes for all elements of T^*. Thus we have

$$\omega_\tau^*(K_\nu) = \sum\tilde{\omega}_\sigma^*(\tilde{K}_\rho) \tag{2.9}$$

where the sum extends over the classes $\tilde{K}_\rho$ of $\mathfrak{N}$ which are contained in K_ν. If the defect group of K_ν is $\mathfrak{H}$, K_ν contains one and only one class of $\mathfrak{N}$ and hence there appears only one $\tilde{K}_\rho$ in the sum in (2.9). The block B_τ associated with ω_τ^* is called *the block of $\mathfrak{G}$ determined by the block $\tilde{B}_\sigma$ of $\mathfrak{N}$* ([3]). (2.9) yields immediately

$$h \leqq \tilde{d}_\sigma \leqq d_\tau \tag{2.10}$$

where d_τ and $\tilde{d}_\sigma$ are the defects of B_τ and $\tilde{B}_\sigma$, respectively.

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4) "i in B_T" means "$\zeta^{(i)}$ in B_T".

[2.3] *Let* $\{\mathfrak{H}^{(\alpha)}\}$ *be a complete system of representatives for the classes of conjugate subgroups of order* p^d *in* $\mathfrak{G}$ *and* $\mathfrak{N}^{(\alpha)}$ *the normalizer* $\mathfrak{N}(\mathfrak{H}^{(\alpha)})$ *of* $\mathfrak{H}^{(\alpha)}$. *Then*

1) *All blocks of* $\mathfrak{N}^{(\alpha)}$ *have at least the defect d. Each block of defect d of* $\mathfrak{N}^{(\alpha)}$ *determines a block of defect d of* $\mathfrak{G}$ *and the different blocks of defect d of* $\mathfrak{N}^{(\alpha)}$ *determine different blocks of* $\mathfrak{G}$.

2) *Each block of defect d of* $\mathfrak{G}$ *is determined by a block of defect d of one and only one* $\mathfrak{N}^{(\alpha)}$. (The p-subgroup $\mathfrak{H}^{(\alpha)}$ corresponding to this $\mathfrak{N}^{(\alpha)}$ is called *the defect group* of the block of $\mathfrak{G}$, which is uniquely determined by the block up to conjugacy ([3]).)

PROOF. 1) Let $\mathfrak{H}=\mathfrak{H}^{(\alpha)}$ be any subgroup in $\{\mathfrak{H}^{(\alpha)}\}$ and $\tilde{B}_\sigma$ a block of defect d of $\mathfrak{N}=\mathfrak{N}^{(\alpha)}$. There exists a p-regular class $\tilde{K}_\rho$ of defect d such that $\tilde{\omega}_\sigma^*(\tilde{K}_\rho)\neq 0$. Since $\mathfrak{H}$ is the defect group of $\tilde{K}_\rho$, the class K_ν containing $\tilde{K}_\rho$ also has the defect group $\mathfrak{H}$. If B_τ is the block determined by $\tilde{B}_\sigma$, then $\omega_\tau^*(K_\nu)=\tilde{\omega}_\sigma^*(\tilde{K}_\rho)\neq 0$. It follows from [1.3] and (2.10) that the defect of B_τ is d.

The last part of 1) is readily seen from [1.4].

2) Let B_τ be a block of defect d. There exists a p-regular class K_ν of defect d such that $\omega_\tau^*(K_\nu)\neq 0$. Let $\mathfrak{H}=\mathfrak{H}^{(\alpha)}$ be the defect group of K_ν. If K_μ is any class belonging to T^*, then by (2.4) $a_{\mu\nu\lambda}\equiv 0 \pmod p$ for the classes K_λ such that $\mathfrak{H}_\lambda\not\subseteq\mathfrak{H}_\mu\cap\mathfrak{H}$. Since $\mathfrak{H}_\mu\cap\mathfrak{H}$ is a proper subgroup of $\mathfrak{H}$, [1.3] yields $\omega_\tau^*(K_\mu)\,\omega_\tau^*(K_\nu)=0$ and hence

$$\omega_\tau^*(K_\mu)=0. \tag{2.11}$$

Thus we can consider ω_τ^* as a linear character of R^* for $\mathfrak{N}=\mathfrak{N}^{(\alpha)}$. Then there exists a linear character $\tilde{\omega}_\sigma^*$ of $\tilde{\Lambda}^*$ which coincides with ω_τ^* on R^*.[5] The block $\tilde{B}_\sigma$ of $\mathfrak{N}$ associated with $\tilde{\omega}_\sigma^*$ determines B_τ and the defect of $\tilde{B}_\sigma$ is d because of [2.1] and (2.10).

The remaining part of the proof follows immediately from [1.4] and (2.11).

From the proof of [2.3] we obtain the following [2.4] and [2.5].

[2.4] *A subgroup of order* p^d *of* $\mathfrak{G}$ *is the defect group of a block* B_τ *of defect d, if and only if it is the defect group of a* p*-regular class* K_ν *of defect d such that* $\omega_\tau^*(K_\nu)\neq 0$.

[2.5] *If the defect group of* K_ν *does not contain the defect group of* B_τ, *then*

$$\omega_\tau^*(K_\nu)=0.$$

[2.6] *If* $\mathfrak{H}$ *is a normal* p*-subgroup of* $\mathfrak{G}$, *then the defect group of each block* B_τ *contains* $\mathfrak{H}$.

PROOF. Let $\mathfrak{D}_\tau$ be the defect group of B_τ. $\mathfrak{D}_\tau$ is the defect group of a p-regular class K_ν such that $\omega_\tau^*(K_\nu)\neq 0$. Hence [2.2] yields $\mathfrak{H}\subseteq\mathfrak{D}_\tau$.

5) See [6], §1.

[2.7] *Let $\mathfrak{H}$ be a subgroup of order p^h of $\mathfrak{G}$ and $\mathfrak{N}$ a subgroup of $\mathfrak{G}$ satisfying (2.2). A block B_τ with the defect group $\mathfrak{D}_\tau$ is determined by a block of $\mathfrak{N}$, if and only if $\mathfrak{H}\subseteqq\mathfrak{D}_\tau$. If B_τ is determined by a block $\tilde{B}_\sigma$ of $\mathfrak{N}$ with the defect group $\tilde{\mathfrak{D}}_\sigma$, then $\mathfrak{H}\subseteqq\tilde{\mathfrak{D}}_\sigma\subseteqq\mathfrak{D}_\tau$.*

PROOF. If $\mathfrak{D}_\tau\supseteqq\mathfrak{H}$, then by virtue of [2.5] B_τ can be determined by a block of $\mathfrak{N}$. If B_τ is determined by $\tilde{B}_\sigma$, then a p-regular class K_ν with the defect group $\mathfrak{D}_\tau$ such that $\omega_\tau^*(K_\nu)\neq 0$ must contain at least one p-regular class $\tilde{K}_\rho$ of $\mathfrak{N}$ such that $\tilde{\omega}_\sigma^*(\tilde{K}_\rho)\neq 0$ because of (2.9). Hence from [2.5] we have $\mathfrak{D}_\tau\supseteqq\tilde{\mathfrak{D}}_\sigma$.

Combining [2.5] with [1.4], we obtain

[2.8] *Two characters $\zeta^{(i)}$ and $\zeta^{(j)}$ belonging to blocks with the defect group $\mathfrak{D}$ appear in the same block, if and only if*

$$\omega_\nu^{(i)}\equiv\omega_\nu^{(j)} \pmod{\mathfrak{p}}$$

for all p-regular classes K_ν with the defect group $\mathfrak{D}$.

From [2.1] and [2.3] or from [2.4] we have

[2.9] *If $\mathfrak{H}$ is not a maximal normal p-subgroup of $\mathfrak{N}(\mathfrak{H})$, then $\mathfrak{H}$ cannot be the defect group of any block B_τ.*

[2.10] *If $\mathfrak{G}$ contains a normal p-subgroup $\mathfrak{H}$ and if $\mathfrak{C}(\mathfrak{H})$ is also a p-group, then $\mathfrak{G}$ possesses only one block and the defect of this block is a.*

PROOF. Let $\mathfrak{D}_\tau$ be the defect group of a block B_τ. Since $\mathfrak{D}_\tau$ contains $\mathfrak{H}$, $\mathfrak{C}(\mathfrak{D}_\tau)$ is contained in the p-group $\mathfrak{C}(\mathfrak{H})$. Hence $\mathfrak{C}(\mathfrak{D}_\tau)$ does not contain any p-regular class except K_1 which consists of the unit element. Therefore $\mathfrak{D}_\tau$ is a p-Sylow subgroup of $\mathfrak{G}$. Since $\omega^*(K_1)=1$ holds for any linear character ω^* of Λ^*, by [1.4] $\mathfrak{G}$ possesses only one block of defect a.

3. [6] Let $e_1, e_2, \cdots\cdots, e_n$ be the primitive idempotents of $\Lambda=\Lambda(\mathfrak{G})$ corresponding to $\zeta^{(1)}, \zeta^{(2)}, \cdots\cdots, \zeta^{(n)}$, respectively. Each e_i is given by

$$e_i=\frac{1}{g}\sum_{\nu=1}^{n} z_i\,\zeta_{\nu^*}^{(i)}\,K_\nu \tag{3.1}$$

and we have

$$K_\nu e_i=\omega_\nu^{(i)}e_i \text{ and } \omega^{(i)}(e_i)=1. \tag{3.2}$$

For each block B_τ we consider the idempotent

$$E_\tau=\sum_{i \text{ in } B_\tau} e_i\,.$$

From (3.1) we have

6) Cf. [7].

$$E_\tau = \sum_\nu b_\nu^{(\tau)} K_\nu \tag{3.3}$$

where $b_\nu^{(\tau)}$ are $\mathfrak{p}$-integers. For all p-singular classes K_σ

$$b_\sigma^{(\tau)} = 0.$$

Put $E_\tau \pmod{\mathfrak{p}} = E_\tau^*$ and $b_\nu^{(\tau)} \pmod{\mathfrak{p}} = b_\nu^{(\tau)*}$. We then have

$$E_\tau^* = \sum_\nu b_\nu^{(\tau)*} K_\nu , \tag{3.4}$$

$$\omega_\tau^*(E_\tau^*) = \sum_\nu b_\nu^{(\tau)*}(K_\nu) = 1 \tag{3.5}$$

and

$$1 = \sum_\tau E_\tau^*. \tag{3.6}$$

Hence we see that the elements E_τ^* are the mutually orthogonal primitive idempotents of $\Lambda^* = \Lambda^*(\mathfrak{G})$.

Let denote by $\mathfrak{D}_\tau$ the defect group of B_τ.

[3.1] *Each E_τ^* is a linear combination of p-regular classes K_ν with $\mathfrak{H}_\nu \subseteqq \mathfrak{D}_\tau$.*

PROOF. If K_ν is a class such that $b_\nu^{(\tau)*} \neq 0$, then by (3.4) B_τ is determined by a block of $\mathfrak{N}(\mathfrak{H}_\nu)$ since for $\mathfrak{H} = \mathfrak{H}_\nu$ the elements $E_\sigma^* \pmod{T^*} \neq 0$ form the complete system of mutually orthogonal primitive idempotents of Λ^*/T^*. Hence [2.7] yields $\mathfrak{H}_\nu \subseteqq \mathfrak{D}_\tau$.

[3.2] *There exists at least one p-regular class K_ν with the defect group $\mathfrak{D}_\tau$ such that both $b_\nu^{(\tau)*}$ and $\omega_\tau^*(K_\nu)$ do not vanish.*

PROOF. By [1.3] we have

$$\omega_\tau^*(K_\mu) = 0$$

for all classes K_μ with $\mathfrak{H}_\mu \subset \mathfrak{D}_\tau$. Therefore, by [3.1] and (3.5) there exists at least one p-regular class K_ν with $\mathfrak{H}_\nu = \mathfrak{D}_\tau$ such that

$$\omega_\tau^*(K_\nu) \neq 0 \quad \text{and} \quad b_\nu^{(\tau)*} \neq 0.$$

We now consider the elements $K_\alpha E_\tau$ and put

$$K_\alpha E_\tau = \sum a_{\alpha\gamma}^{(\tau)} K_\gamma . \tag{3.7}$$

For any p-regular class K_α the above sum extends over the p-regular classes only. If the p-regular classes $K_{\tau,1}, K_{\tau,2}, \cdots\cdots, K_{\tau,y_\tau}$ are associated with B_τ in the sense of [1.1], then the elements $K_{\tau,1}E_\tau, K_{\tau,2}E_\tau, \cdots\cdots, K_{\tau,y_\tau}E_\tau$ are linearly independent (mod $\mathfrak{p}$). We show this. From (3.1) and (3.2) we have

$$(K_{\tau,1}E_\tau \;\; K_{\tau,2}E_\tau \cdots\cdots K_{\tau,y_\tau}E_\tau) = (K_1 \;\; K_2 \cdots\cdots K_k)A_\tau^{(0)}$$

where

$$A_\tau^{(0)} = (\varphi_{\nu^*}^{(\kappa)})' \, C_\tau \, (\varphi_{\tau,l}^{(\kappa)}) \begin{pmatrix} n_{\tau,1}^{-1} & & & 0 \\ & n_{\tau,2}^{-1} & & \\ & & \ddots & \\ 0 & & & n_{\tau,y_\tau}^{-1} \end{pmatrix}^{7)}$$

$$(\varphi^{(\kappa)} \text{ in } B_\tau ; \ \nu = 1, 2, \cdots\cdots, k ; \ l = 1, 2, \cdots\cdots, y_\tau).$$

Hence by [1.1] the rank of $A_\tau^{(0)} (\text{mod } \mathfrak{p})$ is y_τ.

Conversely if the p-regular classes $K'_{\tau,1}, K'_{\tau,2}, \cdots\cdots, K'_{\tau,y'_\tau}$ are associated with B_τ in such a way that each p-regular class is associated with one and only one block and y'_τ elements $K'_{\tau,l} E_\tau$ are linearly independent (mod $\mathfrak{p}$), then we have $y'_\tau = y_\tau$ and y'_τ p-regular classes $K'_{\tau,l}$ can be associated with B_τ in the sense of [1.1].

Let P be an element of order p^h belonging to the center of $\mathfrak{G}$. Any class K_α which contains elements with the p-factor[8] P is of the form $K_\alpha = PK_\beta$ where K_β is a p-regular class. Therefore, for any class K_α which consists of elements with the p-factor P the sum in (3.7) extends over the classes K_γ which consist of elements with the p-factor P only.[9] Further, if the p-regular classes $K_{\tau,1}, K_{\tau,2}, \cdots\cdots, K_{\tau,y_\tau}$ are associated with B_τ in the sense of [1.1], then y_τ elements $PK_{\tau,l} E_\tau$ are linearly independent (mod $\mathfrak{p}$).

Let $\mathfrak{H}$ be any p-subgroup of $\mathfrak{G}$ and consider (2.5) and (2.6) for $\mathfrak{N} = \mathfrak{N}(\mathfrak{H})$. Further let $\widetilde{P}_0 = 1, \widetilde{P}_1, \widetilde{P}_2, \cdots\cdots$ be a complete system of representatives for the classes of conjugate elements of $\mathfrak{N}$ whose orders are powers of p and $\widetilde{E}_\sigma^*$ the primitive idempotent of $\Lambda^*(\mathfrak{N})$ corresponding to a block $\widetilde{B}_\sigma$ of $\mathfrak{N}$. We then show the following: *If $\widetilde{K}_1^{(j)} \widetilde{K}_2^{(j)}, \cdots\cdots$ are the classes of $\mathfrak{N}$ which have the defect group $\mathfrak{H}$ and contain elements with the p-factor $\widetilde{P}_j$, then we have*

(3.8) $$\widetilde{K}_\lambda^{(j)} \widetilde{E}_\sigma^* = \sum_\mu a_{\lambda\mu}^{(\sigma)*} \widetilde{K}_\mu^{(j)}.$$

Since $\mathfrak{H}$ is the defect group of $\widetilde{K}_\lambda^{(j)}$, the p-factor of any element in $\widetilde{K}_\lambda^{(j)}$ belongs to $\mathfrak{H} \cap \mathfrak{C}(\mathfrak{H})$ contained in the center of $\mathfrak{C}(\mathfrak{H})$. Evidently $\widetilde{K}_\lambda^{(j)}$ is a sum of complete classes of $\mathfrak{C}(\mathfrak{H})$ which consist of elements with the p-factors conjugate to $\widetilde{P}_j$ in $\mathfrak{N}$. Further $\widetilde{E}_\sigma^*$ belongs to $\Lambda^*(\mathfrak{C}(\mathfrak{H}))$[10] and is a sum of primitive idempotents of $\Lambda^*(\mathfrak{C}(\mathfrak{H}))$. Therefore applying the above result, the element $\widetilde{K}_\lambda^{(j)} \widetilde{E}_\sigma^*$ of $\Lambda^*(\mathfrak{C}(\mathfrak{H}))$ is a linear combination of the classes of $\mathfrak{C}(\mathfrak{H})$ which consist of elements with the p-factors conjugate to $\widetilde{P}_j$ in $\mathfrak{N}$.

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7) $\varphi_{\tau,l}^{(\kappa)}$ denotes the value of $\varphi^{(\kappa)}$ for the class $K_{\tau,l}$ and $n_{\tau,l}$ denotes the order of the normalizer $\mathfrak{N}(G_{\tau,l})$ of an element $G_{\tau,l}$ in $K_{\tau,l}$.

8) Let Q be an element of $\mathfrak{G}$ such that the order of Q is a power of p. If G is an element of the form QR where R is a p-regular element of $\mathfrak{N}(Q)$, then Q is called the *p-factor* of G.

9) From Theorem 2 in [4], we see that this fact remains valid even though P does not belong to the center of $\mathfrak{G}$. But the writer does not know the proof of this theorem.

10) This can be seen immediately from (2.6) and [2.2]. Cf. [7], §3.

Consequently the element $\tilde{K}_\lambda^{(j)}\tilde{E}_\sigma^*$ of $\Lambda^*(\mathfrak{N})$ is a linear combination of the classes $\tilde{K}_\rho$ of $\mathfrak{N}$ which contain elements with the p-factor $\tilde{P}_j$. Since every element in each $\tilde{K}_\rho$ belongs to $\mathfrak{C}(\mathfrak{H})$, the defect group $\tilde{\mathfrak{H}}_\rho$ of $\tilde{K}_\rho$ contains $\mathfrak{H}$. On the other hand, by (2.4) $\tilde{\mathfrak{H}}_\rho$ is contained in $\mathfrak{H}$. From the above (3.8) follows.

We now consider the classes $\tilde{K}_\lambda$ of $\mathfrak{N}$ with the defect group $\mathfrak{H}$. We then show the following: *It is possible to associate these classes $\tilde{K}_\lambda$ with the blocks $\tilde{B}_\sigma$ in such a way that each class $\tilde{K}_\lambda$ is associated with one and only one block $\tilde{B}_\sigma$ and for the classes $\tilde{K}_{\sigma,l}$ ($l = 1, 2, \cdots\cdots, \tilde{r}_\sigma$) associated with $\tilde{B}_\sigma$ $\tilde{K}_{\sigma,l}\tilde{E}_\sigma^*$ are linearly independent.*

By (3.8) it is sufficient to consider only the classes $\tilde{K}_\lambda^{(j)}$ for a fixed $\tilde{P}_j$. Since $\tilde{P}_j$ belongs to the center of $\mathfrak{C}(\mathfrak{H})$, $\tilde{\mathfrak{N}}(\tilde{P}_j) = \mathfrak{N}(\tilde{P}_j)\cap\mathfrak{N}$ contains $\mathfrak{C}(\mathfrak{H})$ and hence every element in any $\tilde{K}_\lambda^{(j)}$ belongs to $\tilde{\mathfrak{N}}(\tilde{P}_j)$. Evidently the elements in each $\tilde{K}_\lambda^{(j)}$ with the p-factor $\tilde{P}_j$ form a complete class $\hat{K}_\lambda^{(j)}$ of $\tilde{\mathfrak{N}}(\tilde{P}_j)$ and the defect group of $\hat{K}_\lambda^{(j)}$ is also $\mathfrak{H}$. Conversely if a class of $\tilde{\mathfrak{N}}(\tilde{P}_j)$ has the defect group $\mathfrak{H}$ and consists of elements with the p-factor $\tilde{P}_j$, then this class is contained in one class $\tilde{K}_\lambda^{(j)}$. On the other hand, since $\tilde{E}_\sigma^*$ belongs to $\Lambda^*(\mathfrak{C}(\mathfrak{H}))$, it belongs to $\Gamma^*(\tilde{\mathfrak{N}}(\tilde{P}_j))$ and hence to $\Lambda^*(\tilde{\mathfrak{N}}(\tilde{P}_j))$. Denoting by $\hat{B}^{(\sigma)}$ the collection of blocks of $\tilde{\mathfrak{N}}(\tilde{P}_j)$ corresponding to the idempotent $\tilde{E}_\sigma^*$ of $\Lambda^*(\tilde{\mathfrak{N}}(\tilde{P}_j))$, we can associate the classes $\hat{K}_\lambda^{(j)}$ with the collections $\hat{B}^{(\sigma)}$ in such a way that each $\hat{K}_\lambda^{(j)}$ is associated with one and only one collection $\hat{B}^{(\sigma)}$ and for the classes $\hat{K}_{\sigma,l}$ associated with $\hat{B}_{(\sigma)}$ $\hat{K}_{\sigma,l}\tilde{E}_\sigma^*$ are linearly independent. As is easily seen, the corresponding elements $\tilde{K}_{\sigma,l}\tilde{E}_\sigma^*$ are linearly independent.

If $\tilde{s}_\sigma^{(j)}$ is the number of the classes $\tilde{K}_{\sigma,l}$ which contain elements with the p-factor $\tilde{P}_j$, then $\tilde{s}_\sigma^{(j)}$ depends only on $\mathfrak{H}$, $\tilde{B}_\sigma$ and $\tilde{P}_j$.

These facts remain valid, if $\tilde{B}_\sigma$ are collections of blocks of $\mathfrak{N}$ such that each block of $\mathfrak{N}$ belongs to one and only one collection. In this case the classes associated with a collection can be associated with the blocks belonging to this collection in the above sense.

Let $\{\mathfrak{H}\}$ be a complete system of representatives for the classes of conjugate subgroups of orders p^h ($h = 0, 1, 2, \cdots\cdots$) in $\mathfrak{G}$. If the defect group of a block B_τ contains a subgroup $\mathfrak{H}$ in $\{\mathfrak{H}\}$, then for $\mathfrak{N}=\mathfrak{N}(\mathfrak{H})$ the element $\tilde{E}^{(\tau)*}$ of R^* corresponding to E_τ^* (mod T^*) in (2.5) is a sum of primitive idempotents $\tilde{E}_\sigma^*$ of $\Lambda^*(\mathfrak{N}(\mathfrak{H}))$. The blocks $\tilde{B}_\sigma$ belonging to these idempotents $\tilde{E}_\sigma^*$ form the collection $\tilde{B}^{(\tau)}(\mathfrak{H})$ of blocks of $\mathfrak{N}(\mathfrak{H})$ which determine the block B_τ.

For each subgroup $\mathfrak{H}$ in $\{\mathfrak{H}\}$ consider the classes $\tilde{K}_{\tau,l}(\mathfrak{H})$ of $\mathfrak{N}(\mathfrak{H})$ which have the defect group $\mathfrak{H}$ and are associated with $B^{(\tau)}(\mathfrak{H})$, $l = 1, 2, \cdots\cdots, r_\tau(\mathfrak{H})$. Then we have

[3.3] *If $K_{\tau,l}(\mathfrak{H})$ is the class of $\mathfrak{G}$ containing $\tilde{K}_{\tau,l}(\mathfrak{H})$, all the classes of $\mathfrak{G}$ are given by*

(3.9) $K_{\tau,l}(\mathfrak{H})$ $(l = 1, 2, \cdots\cdots, r_\tau(\mathfrak{H});\ \tau = 1, 2, \cdots\cdots, s;\ \mathfrak{H}$ *in* $\{\mathfrak{H}\})$

and the elements

(3.10) $K_{\tau,l}(\mathfrak{H})\, E_\tau$ $(l = 1, 2, \cdots\cdots, r_\tau(\mathfrak{H});\ \tau = 1, 2, \cdots\cdots, s;\ \mathfrak{H}$ *in* $\{\mathfrak{H}\})$

are linearly independent (mod $\mathfrak{p}$). *Conversely let* $K'_{\tau,l}$ *be the classes of* $\mathfrak{G}$ *associated with* B_τ *in such a way that each class* K_ν *is associated with one and only one block of* $\mathfrak{G}$ *and* $K'_{\tau,l} E_\tau$ *are linearly independent (mod* $\mathfrak{p}$). *Then this selection of classes of* $\mathfrak{G}$ *can be obtained as above.*[11]

PROOF. Arrange all subgroups $\mathfrak{H}$ in $\{\mathfrak{H}\}$ as follows: $\mathfrak{H}^{(1)}, \mathfrak{H}^{(2)}, \cdots\cdots$, where the order of $\mathfrak{H}^{(i)}$ is not less than that of $\mathfrak{H}^{(j)}$ if $i<j$. Considering (2.5) for $\mathfrak{H} = \mathfrak{H}^{(i)}$ and $\mathfrak{N}=\mathfrak{N}(\mathfrak{H})$, $i = 1, 2, \cdots\cdots$, we see that the elements in (3.10) are linearly independent (mod $\mathfrak{p}$).

We now prove the converse. If for $\mathfrak{N} = \mathfrak{N}(\mathfrak{H})$ we consider (2.5), then $K'^{0}_{\tau,l}\,(\neq 0)$ form a K^*-basis of R^* and hence for the classes $K'^{0}_{\tau,l} \neq 0$ $K'^{0}_{\tau,l}\widetilde{E}^{(\tau)*}$ also form a K^*-basis of R^*. In particular for the classes $K'_{\tau,l}$ with the defect group $\mathfrak{H}$ $K'_{\tau,l}E^{(\tau)*}$ are linearly independent. This completes the proof.

Let $P_0 = 1,\ P_1,\ P_2, \cdots\cdots$ be a complete system of representatives for the classes of conjugate elements of order p^h $(h = 0, 1, 2, \cdots\cdots)$ in $\mathfrak{G}$.

[3.4] *If, in the notation of* [3.3], *we denote by* $s_\tau^{(j)}(\mathfrak{H})$ *the number of classes* $K_{\tau,l}(\mathfrak{H})$ *which contain elements with the* p*-factor* P_j, *then* $s_\tau^{(j)}(\mathfrak{H})$ *depends only on* B_τ, $\mathfrak{H}$ *and* P_j. *In particular both* $s_\tau(\mathfrak{H})(= s_\tau^{(0)}(\mathfrak{H}))$ *and* $r_\tau(\mathfrak{H})(= \sum_j s_\tau^{(j)}(\mathfrak{H}))$ *depend only on* B_τ *and* $\mathfrak{H}$. *Further we have* $\sum_{\mathfrak{H}} r_\tau(\mathfrak{H}) = x_\tau$ *and* $\sum_{\mathfrak{H}} s_\tau(\mathfrak{H}) = y_\tau$.

[3.5] *The defect groups of the classes associated with* B_τ *in the sense of* [3.3] *are contained in the defect group* $\mathfrak{D}_\tau$ *of* B_τ. *For each* B_τ *exactly one of the* p*-regular classes associated with* B_τ *has the defect group* $\mathfrak{D}_\tau$.

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11) Such a selection is identical with that in [3], §5.