A NEW DEDUCTION OF SINGULARITY CRITERIA FOR THE FIRST ORDER DIFFERENTIAL EQUATIONS

Katsumasa MATSUMOTO

(Received Jan. 30, 1957)

§ 1. The author, having been engaged in investigation of higher order singularities of differential equations, was, once on his way of study, struck with an idea to prove the singularity criteria hitherto known for the first order differential equations in a far easier way than those usually done. Until present, it seems to have been a tradition to deduce the criteria by the aid of matrices, transformations, characteristic equations, etc[1]. The author believes the methods adopted are too circuitous and seem awful for beginners, who will think it something mysterious. If it concerns only with the classification of singularities and with the criteria for distinguishing different types of them, there is no need of such circuitousness nor even necessity, for the most part, to solve the differential equations. As the differentiatiation is easier and always possible while the integration is not, we prefer the former to the latter in examining the properties of the solution curves. We shall develop here our idea mainly for the first order differential equations.

§ 2. Suppose that there is given a first order differential equation of the type

$$\frac{dy}{dx}=\frac{\alpha x+\beta y}{\gamma x+\delta y}, \quad \alpha\delta-\beta\gamma\neq 0. \qquad (1)$$

The origin is a singular point. As any other first order differential equations, the singular point of which lies at any other point than the origin, may be reduced to the form (1) by simple translation, we always consider the differential equation in the form (1). Let $y=f(x)$ be its solution, if any, satisfying the initial condition $x=0$, $y=0$. Imagine that we follow along the solution curve until we reach the origin. We examine the slope of the curve how it changes along it. We shall inspect preliminarily a few special cases and compare with the solutions themselves already known in the purpose of explaining our ider and making clear how simple it is.

(a)

$$\frac{dy}{dx}=\frac{y}{x}. \qquad (2)$$

Solving it gives $y=Cx$. There are an infinite number of straight lines through the origin, which is a node. But if we try to infer the property of the origin without solving the equation, we proceed as follows. $\frac{y}{x}$ is an indeterminate form if $x\to 0$, $y\to 0$. In both members of (2), passing to the limit as $x\to 0$, especially applying L'Hospital's rule to the right member, we obtain

---

[1] Cf., e. g., Bieberbach, L.: Differentialgleichungen; Niemytzki, V. V. and Stepanov, V, V.: Qualitative theory of differential equations.

$$\left(\frac{dy}{dx}\right)_0 = \left(\frac{dy}{dx}\right)_0.$$

This means that any value whatever of $\left(\frac{dy}{dx}\right)_0$ may be had by the solution curves of (2), i. e., solution curves may enter into the origin along any direction. The origin is thus a node. Differentiating (2) again, we have $\frac{d^2y}{dx^2} = 0$, which means that the solution curves are straight lines.

(b)

$$\frac{dy}{dx} = a\frac{y}{x}. \quad \cdots\cdots\cdots (3)$$

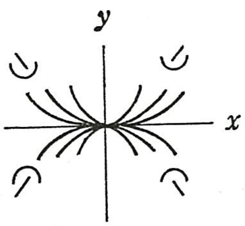
Solving (3) yields $y = Cx^a$. All integral curves either (i) pass through the origin and are tangent to the y-axis there except $y = 0$ $(0 < a < 1)$ or (ii) pass through the origin and are tangent to the x-axis there except $x = 0$ $(a > 1)$ or (iii) are asymptotic to the axes, only $x = 0$ and $y = 0$ passing through the origin, which all others pass by $(a < 0)$. But we prefer to proceed as follows, not integrating (3) but differentiating it. Passing to the limit as $x \to 0$ in (3), we obtain

$$\left(\frac{dy}{dx}\right)_0 = a\left(\frac{dy}{dx}\right)_0,$$

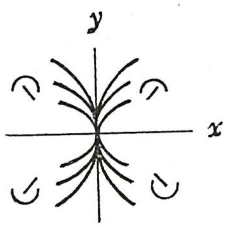
which is satisfied by $\left(\frac{dy}{dx}\right)_0 = 0$ or ∞, indicating that infinitely many solution curves touch the coordinate axes at the origin or no curves pass through it except $x = 0$ and $y = 0$. Differentiating (3) again, we have

$$\frac{d^2y}{dx^2} = \frac{a(a-1)y}{x}. \quad \cdots\cdots\cdots (4)$$

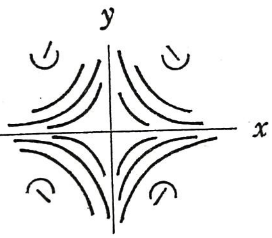
Taking (3) – the slope – and (4) – the curvature – into account, and considering the uniqueness theorem, we obtain the following possible distributions of curves (Figs. 1, 2 and 3), indicating that the origin is either a node or a saddle point. In the figures, the marks / and \ signify that the slopes are positive and negative there respectively and the marks ∪ and ∩ that the curvatures are positive and negative there, respectively.



$a > 1$
Fig. 1



$0 < a < 1$
Fig. 2



$a < 0$
Fig. 3

(c)

$$\frac{dy}{dx} = \frac{x + y}{x}. \quad \cdots\cdots\cdots (5)$$

Solving (5) yields $y=x\log|x|+Cx$, which is a little hard to be graphed at a glance. Let us graph it not integrating (5) but differentiating it. Firstly, passing to the limit as $x\to 0$ in (5), we obtain

$$\left(\frac{dy}{dx}\right)_0=1+\left(\frac{dy}{dx}\right)_0,$$

which is satisfied by $\left(\frac{dy}{dx}\right)_0=\infty$. Next, differentiating (5) gives $\frac{d^2y}{dx^2}=\frac{1}{x}$. Noting the signs of slopes and curvatures, we easily obtain the possible distribution of curves near the origin as shown in Fig. 4, indicating that the origin is a node. See how simple it is as compared with the integration method.

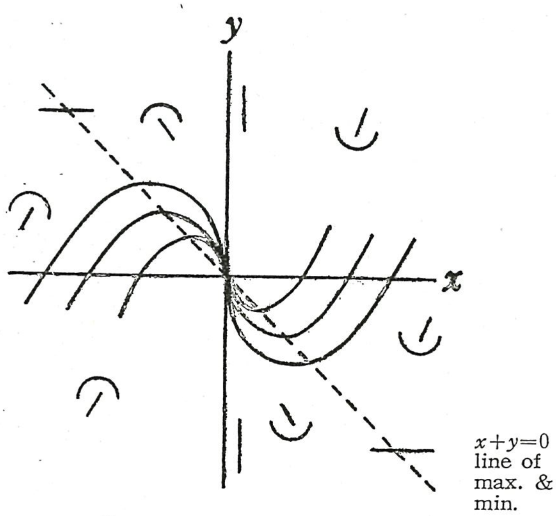


Fig. 4

(d)

$$\frac{dy}{dx}=-a^2\frac{x}{y}. \quad \cdots\cdots (6)$$

Solving (6) gives $y^2+a^2x^2=C$, which are ellipses. If we proceed as in the previous cases, making $x\to 0$ as well as $y\to 0$, we obtain

$$\left(\frac{dy}{dx}\right)_0^2=-a^2,$$

which no real directions satisfy. This means that we cannot make $y\to 0$ as $x\to 0$, i. e., no real curves pass through the origin. Combining the two imaginary straight lines entering the origin, which have the imaginary slopes $\pm ia$, we obtain $y^2+a^2x^2=0$, i. e., a point ellipse, which hints at the fact that the ellipse-like solution curves pack the neighborhood of the origin, which must therefore be either a center or a focus. We shall discuss this case in more general form later.

(e)

$$\frac{dy}{dx}=\frac{x+ay}{ax-y} \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (7)$$

Solving (7) in polar coordinates gives $r=Ce^{a\vartheta}$, which are logarithmic spirals. Proceeding as in (d), we obtain

$$\left(\frac{dy}{dx}\right)_0^2=-1.$$

Remark is the same as in (d).

§ 3. Now we shall turn to the general case (1). Assuming that $y\to 0$ as $x\to 0$, making use of L'Hospital's rule and arranging in $\left(\frac{dy}{dx}\right)_0$, we obtain the following quadratic equation in $\left(\frac{dy}{dx}\right)_0$:

$$\delta\left(\frac{dy}{dx}\right)_0^2-(\beta-\gamma)\left(\frac{dy}{dx}\right)_0-\alpha=0,$$

the discriminant of which is

$$\triangle=(\beta-\gamma)^2+4\alpha\delta.$$

We distinguish the following three cases.

(i) $\triangle>0$. We have two real distinct directions in which the solution curves enter the origin. Differentiating (1), we obtain

$$\frac{d^2y}{dx^2}=(\alpha\delta-\beta\gamma)\frac{\delta y^2-(\beta-\gamma)xy-\alpha x^2}{(\gamma x+\delta y)^3}. \qquad (8)$$

$$m_{1,2}=\frac{\beta-\gamma\pm\sqrt{(\beta-\gamma)^2+4\alpha\delta}}{2\delta}$$

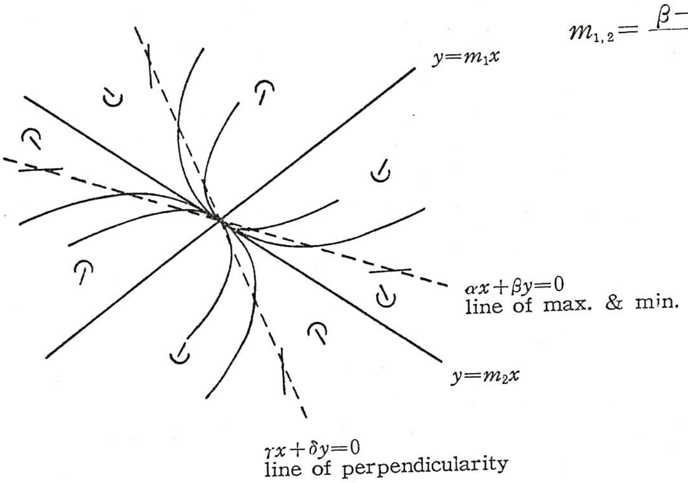


Fig. 5

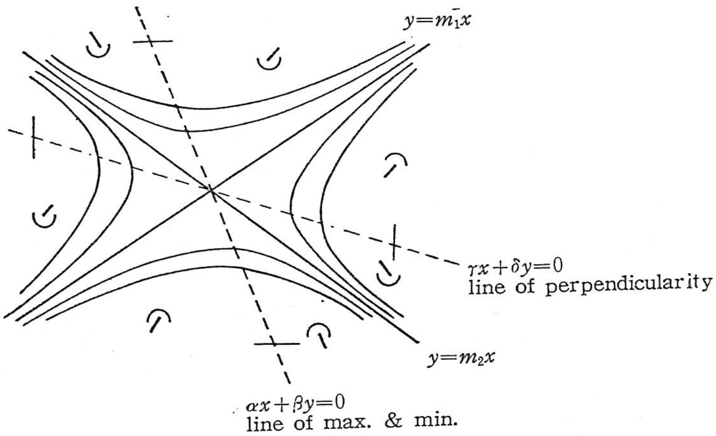


Fig. 6

Considering the slopes and curvatures, we draw all possible distributions of integral curves near the origin as shown in Figs. 5 and 6, which indicate that the origin is either a node ($\alpha\delta-\beta\gamma<0$) or a saddle point ($\alpha\delta-\beta\gamma>0$).

(ii) $\triangle=0$. We have one real direction in which the solution curves enter the origin. Regardless of the sign of $\alpha\delta-\beta\gamma$, we obtain a figure indicating that the origin is a node (Fig. 7).

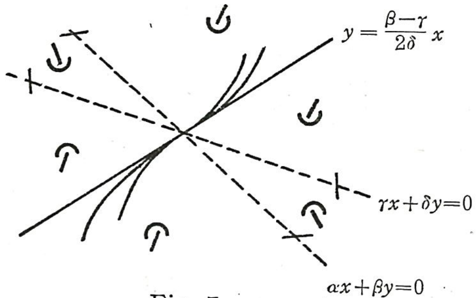


Fig. 7

(iii) $\triangle<0$. Combining into one the two imaginary straight lines

$$y=\frac{\beta-\gamma\pm i\sqrt{-\triangle}}{2\delta}x$$

which enter the origin, we obtain

$$\delta y^2-(\beta-\gamma)xy-\alpha x^2=0,$$

which is a point ellipse, indicating that the neighborhood of the origin is packed with ellipse-like integral curves. Integral curves appear to surround the origin in this case. To see how they surround the origin, we imagine a point moving along them, introduce time concept, rewriting (1) in the form of simultaneous equations

$$\begin{cases} \dfrac{dx}{dt}=\gamma x+\delta y \\ \dfrac{dy}{dt}=\alpha x+\beta y \end{cases} \qquad \cdots\cdots (9)$$

the characteristic equation of which is

$$\begin{vmatrix} D-\gamma & \delta \\ \alpha & D-\beta \end{vmatrix}=0$$

having the roots

$$D=\frac{\beta+\gamma\pm i\sqrt{-\triangle}}{2}$$

The solutions of (9) are

$$\begin{cases} x=e^{\frac{\beta+\gamma}{2}t}\left(C_1\cos\frac{\sqrt{-\triangle}}{2}t+C_2\sin\frac{\sqrt{-\triangle}}{2}t\right) \\ y=e^{\frac{\beta+\gamma}{2}t}\left\{C_1\left(\frac{\beta-\gamma}{2\delta}\cos\frac{\sqrt{-\triangle}}{2}t-\frac{\sqrt{-\triangle}}{2\delta}\sin\frac{\sqrt{-\triangle}}{2}t\right)\right. \\ \qquad\left.+\,C_2\left(\frac{\sqrt{-\triangle}}{2\delta}\cos\frac{\sqrt{-\triangle}}{2}t+\frac{\beta-\gamma}{2\delta}\sin\frac{\sqrt{-\triangle}}{2}t\right)\right\} \end{cases} \qquad \cdots\cdots (10)$$

where C_1 and C_2 are arbitrary constants.

We see from (10) that the solution curves encircle the origin either periodically

$(\beta+\gamma=0)$ or spirally $(\beta+\gamma\neq 0)$. The origin is thus a center $(\beta+\gamma=0)$ or a focus $(\beta+\gamma\neq 0)$.

§ 4. In conclusion, a few words about Poincaré's theorem. Let the equation be of the form:

$$\frac{dy}{dx}=\frac{\alpha x+\beta y+P(x,y)}{\gamma x+\delta y+Q(x,y)}$$

where $P(x,y)$ and $Q(x,y)$ are $O(x^2+y^2)$.

Proceeding in the previous way, we have, when $x\to 0$, $y\to 0$,

$$\left(\frac{dy}{dx}\right)_0=\frac{\alpha+\beta\left(\frac{dy}{dx}\right)_0}{\gamma+\delta\left(\frac{dy}{dx}\right)_0}$$

showing that the terms of higher order than the first do not affect the character of singularity of the origin, which hints at Poincaré's theorem. Case when $\beta+\gamma=0$ must be considered separately as one sees clearly from the fact that the nonlinear form like

$$\frac{dy}{dx}=\frac{\alpha x^2+\beta xy+\gamma y^2}{\alpha' x^2+\beta' xy+\gamma' y^2}$$

belongs to this case, which will not be touched here.