THE APPLICATION OF THE DIFFUSION EQUATIONS

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1) Introduction

It seems that the Fick's diffusion equations are applicable for the various problems which we meet in the chemical engineering laboratories, even for the case in which we hesitate to use the word "diffusion" in its proper molecular theoretical meaning. For instance, in problems such as the washing or leaching of any substances mechanically contained in any other solid, what we have to consider is merely the rate of extraction of substances, and very frequently this is far from the true molecular diffusion. But, I believe, it is not totally useless to see these problems as the application of the diffusion equations, and compare the calculated results with the experimental data, so long as the suitable interpretations are made of the terminology used in the equations which may not be strictly exact in the physical or chemical sense. For example, we say "solute" and "solvent". It is only a matter of convenience to say so, and it is not necessarily understood in the usual sense. Some times "solute" means only substances mechanically contained in any other liquids or solids which are "solvent". The diffusion equations depend upon the well known Fick's first law, which says that the flux of the solute substances in the solvent is proportional to the concentration gradient of itself. This proportional coefficient is denoted by D and as usual is called diffusion coefficient. As stated above, D sometimes loses its molecular theoretical meaning, and might in some cases be understood simply as a mobility of substances within other liquids or solids, while in other cases as an average "diffusional constant" of two or more vaguely defined boundary layers of the solid.

2) The zero surface concentration.

The one-dimensional equation of diffusion is

$$\frac{\partial C}{\partial t} = D \frac{\partial^2 C}{\partial x^2} \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (2,1)$$

where C is the concentration of the "solute" substance in the "solvent" and has the dimension $\frac{gr}{cm^3}$. The flux of the "solute" through unit area perpendicular to the moving direction per unit time is denoted by J and is given as follows

$$J = -D \frac{\partial C}{\partial x} \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (2,2)$$

We take a line perpendicular to the surface of the "solvent" as x-coordinate axis, whose origin is at this surface, with the "solvent" being in the positive side of the x-coordinate. First we consider the case with concentration function $C(x. t)$ satisfying the following conditions,

$$C(0, t) = 0 \text{ for all positive } t. \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (2,3)$$

$$C(x, 0) = \phi_0(x) \text{ for all positive } x. \quad\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \quad (2,4)$$

where $\phi_0(x)$ is a known function.

· These conditions correspond to a situation that, while initial distribution of the solute in the solvent is arbitrary, the surface concentration, that is the concentration of the solute immediately outside of the solvent, is kept always to zero.

We assume that $\phi_0(x)$, together with $C(x, t)$ is the function which has the Laplace transform with respect to t, and is subject to the condition;

$$\lim_{x\to\infty} \phi_0(x) = \lim_{x\to\infty} C(x, t) = C_0 = const. \quad\cdots\cdots\cdots\cdots\cdots\cdots\cdots \quad (2,5)$$

The Laplace transforms of any function $f(t)$ with respect to t will be denoted by $L[f(t)] = F(s)$ as usual.

Now, operating the L on both sides of the equations $(2,1)$ $(2,3)$ and $(2,5)$, we obtain

$$D \frac{\partial^2 Y}{\partial x^2} = sY - \phi_0(x) \quad\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \quad (2,6)$$

where $Y(x, s) = L[C(x,t)] \equiv \int_0^\infty e^{-st} C(x,t)\, dt \quad\cdots\cdots\cdots\cdots\cdots\cdots\cdots \quad (2,7)$

$Y(x, s)$ satisfies the folowing conditions according to $(2,3)$ and $(2,5)$

$$Y(0, s) = 0 \quad\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \quad (2,8)$$

$$\lim_{x\to\infty} Y(x, s) = \frac{C_0}{s} \quad\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \quad (2,9)$$

Here we have assumed that the order of two operations L and $\lim_{s\to\infty}$ can be changed freely.

From $(2,6)$ and $(2,8)$, $Y(x, s)$ is readily found as

$$Y(x, s) = A(s)\left(e^{\sqrt{\frac{s}{D}}\, x} - e^{-\sqrt{\frac{s}{D}}\, x}\right)$$
$$- \frac{1}{2\sqrt{sD}} \int_0^x \left\{ e^{\sqrt{\frac{s}{D}}\,(x-\xi)} - e^{-\sqrt{\frac{s}{D}}\,(x-\xi)} \right\} \phi_0(\xi)\, d\xi \quad\cdots\cdots\cdots \quad (2,10)$$

where $A(s)$ is the constant of intergration, and should be determined so as to satisfy the equation $(2,9)$. If one chooses so that $A(s)$ takes the form

$$A(s) = \frac{1}{2\sqrt{sD}} \int_0^\infty e^{-\sqrt{\frac{s}{D}}\,\xi}\, \phi_0(\xi)\, d\xi$$

then one can see after slight elementary calculation that $Y(x, s)$ has all the required properties, including $(2,9)$.

Inserting this value of $A(s)$ in the $Y(x, s)$, and operating the inverse operator L^{-1} on both sides, we get the required formula for $C(x, t)$ as follows

$$C(x, t) = \frac{1}{2\sqrt{\pi Dt}} \left\{ \int_x^\infty \phi_0(\xi)\, e^{-\frac{(\xi-x)^2}{4Dt}}\, d\xi - \int_0^\infty \phi_0(\xi)\, e^{-\frac{(x+\xi)^2}{4Dt}}\, d\xi \right.$$
$$\left. + \int_0^x \phi_0(\xi)\, e^{-\frac{(x-\xi)^2}{4Dt}}\, d\xi \right\} \quad\cdots\cdots\cdots\cdots\cdots\cdots\cdots \quad (2,11)$$

or

$$C(x,t) = \frac{1}{\sqrt{\pi}} \left\{ \int_0^\infty \phi_0(x+2\sqrt{Dt}\,\xi)\, e^{-\xi^2}\, d\xi - \int_{\frac{x}{2\sqrt{Dt}}}^\infty \phi_0(2\sqrt{Dt}\,\xi - x)\, e^{-\xi^2}\, d\xi \right.$$

$$\left. + \int_0^{\frac{x}{2\sqrt{Dt}}} \phi_0(x-2\sqrt{Dt}\,\xi)\, e^{-\xi^2}\, d\xi \right\} \quad \cdots\cdots\cdots\cdots\cdots \quad (2,12)$$

Here we used the well known inverse transformation

$$L^{-1}\left[\frac{1}{\sqrt{s}}\, e^{-k\sqrt{s}}\right] = \frac{1}{\sqrt{\pi t}}\, e^{-\frac{k^2}{4t}}$$

and assumed the changeability of the order of two operations L^{-1} and $\int^\infty d\xi$.

That $C(x,t)$ given above satisfies the original equation together with the conditions requ-ired, is readily seen by direct insertion of this in the corresponding equations.

The flux J is given by

$$J = -\frac{1}{4\sqrt{\pi Dt^3}} \left\{ \int_x^\infty (\xi - x)\, \phi_0(\xi)\, e^{-\frac{(\xi-x)^2}{4Dt}}\, d\xi \right.$$

$$\left. + \int_0^\infty (x+\xi)\, \phi_0(\xi)\, e^{-\frac{(x+\xi)^2}{4Dt}}\, d\xi - \int_0^x (x-\xi)\, \phi_0(\xi)\, e^{-\frac{(x-\xi)^2}{4Dt}}\, d\xi \right\} \cdots\cdots\cdots (2,13)$$

or

$$J = -\sqrt{\frac{D}{\pi t}} \left\{ \int_0^\infty \xi \phi_0(2\sqrt{Dt}\,\xi + x)\, e^{-\xi^2}\, d\xi \right.$$

$$\left. + \int_{\frac{x}{2\sqrt{Dt}}}^\infty \xi \phi_0(2\sqrt{Dt}\,\xi - x)\, e^{-\xi^2}\, d\xi - \int_0^{\frac{x}{2\sqrt{Dt}}} \xi \phi_0(x - 2\sqrt{Dt}\,\xi)\, e^{-\xi^2}\, d\xi \right\} \quad \cdots\cdots \quad (2,14)$$

The flux at the surface $(x=0)$ is denoted by J_{surf} and is given by

$$J_{surf} = -2\sqrt{\frac{D}{\pi t}} \int_0^\infty \xi e^{-\xi^2} \phi_0(2\sqrt{Dt}\,\xi)\, d\xi \quad \cdots\cdots\cdots\cdots\cdots\cdots \quad (2,14')$$

In the special case when the initial distribution of "solute" can be considered to be uniform and equal to C_0, then we can naturally replace $\phi_0(x)$ by constant C_0, and the results will be

$$C(x,t) = C_0 E(\lambda) \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \quad (2,15)$$

$$\lambda = \frac{x}{2\sqrt{Dt}}$$

where $E(\lambda) = \frac{2}{\sqrt{\pi}} \int_0^\lambda e^{-\xi^2}\, d\xi$,

$$J = -C_0 \sqrt{\frac{D}{\pi t}}\, e^{-\lambda^2} \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \quad (2,16)$$

and $J_{surf} = -C_0 \sqrt{\frac{D}{\pi t}} \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \quad (2,17)$

The total amount of the solute substance Q, which flows out through the unit area of the

surface $x = 0$ into the negative side of x in the interval $0 \leq t \leq T$, will be immediately obtained by integration, and is given by

$$Q = - \int_0^T J_{surf}\, dt \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \quad (2,18)$$

For the case when $\phi_0(x) = C_0$, Q becomes simply

$$Q = 2C_0 \sqrt{\frac{Dt}{\pi}} \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \quad (2,19)$$

3) Time dependent surface concentration.

Next we will treat the more general case, in which the surface concentration $C(0, t)$ varies arbitrarily with time. In this case we have to use the new initial condition

$$C(0, t) = \psi(t) \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \quad (3, 1)$$

in place of (2, 3), where $\psi(t)$ is some known function.
Mathematical treatment is almost the same as before, and therefore, it is convenient to distinguish the quantities appearing in this article from the corresponding one in the preceding article by adding the suffix 1. Then the Laplace transform $Y_1(x, s)$ of new $C_1(x, t)$ becomes

$$Y_1(x, s) = Y(x, s) + \Psi(s)\, e^{-\sqrt{\frac{s}{D}}\, x} \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \quad (3, 2)$$

where $\Psi(s)$ is the Laplace transform of $\psi(t)$,
and from this it follows that

$$C_1(x, t) = C(x, t) + L^{-1}\left[\Psi(s)\, e^{-\sqrt{\frac{s}{D}}\, x}\right]$$

Now from the convolution theorem of Laplace transformation, and from the formula

$$L\left[\frac{x}{2\sqrt{\pi D t^3}}\, e^{-\frac{x^2}{4Dt}}\right] = e^{-\sqrt{\frac{s}{D}}\, x}$$

it follows that

$$C_1(x, t) = C(x, t) + \frac{x}{2\sqrt{\pi D}} \int_0^t \frac{\psi(t-\tau)}{\sqrt{\tau^3}}\, e^{-\frac{x^2}{4D\tau}}\, d\tau \quad \cdots\cdots\cdots\cdots \quad (3, 3)$$

$$= C(x, t) + \frac{2}{\sqrt{\pi}} \int_{\frac{x}{2\sqrt{Dt}}}^{\infty} \psi\left(t - \frac{x^2}{4D\xi^2}\right) e^{-\xi^2}\, d\xi \quad \cdots\cdots\cdots\cdots \quad (3, 4)$$

$C(x, t)$ in the right hand side of both equations is already given in the equations (2, 11) or (2, 12). The flux at the surface is given by

$$J_{1surf} = - \frac{2\sqrt{D}}{\sqrt{\pi t}} \int_0^{\infty} \xi e^{-\xi^2} \phi_0(2\sqrt{Dt}\,\xi)\, d\xi + \sqrt{\frac{D}{\pi t}}\, \psi(0)$$

$$+ \lim_{x \to 0} \frac{x}{\sqrt{\pi}} \int_{\frac{x}{2\sqrt{Dt}}}^{\infty} \psi'\left(t - \frac{x^2}{4D\xi^2}\right) \frac{1}{\xi^2}\, e^{-\xi^2} d\xi \quad \cdots\cdots\cdots\cdots \quad (3, 5)$$

The third term of $(3,5)$ must be treated very carefully, because as the lower limit of integral tends to zero, the factor $\frac{1}{\xi^2}$ tends to infinity.

But if one put $t - \dfrac{x^2}{4D\xi^2} = \eta$

in the integral, and write η again ξ, then the intergral becomes

$$\sqrt{\frac{D}{\pi}} \int_0^t \frac{\psi'(\xi)}{\sqrt{t-\xi}}\, e^{-\frac{x^2}{4D(t-\xi)}}\, d\xi$$

which tends to

$$\sqrt{\frac{D}{\pi}} \int_0^t \frac{\psi'(\xi)}{\sqrt{t-\xi}}\, d\xi \qquad \text{as } x \to 0$$

and therefore the surface flux becomes

$$J_{1surf} = 2\sqrt{\frac{D}{\pi t}} \int_0^\infty \xi e^{-\xi^2} \phi_0(2\sqrt{Dt}\,\xi)\, d\xi + \sqrt{\frac{D}{\pi t}}\, \psi(0)$$
$$+ \sqrt{\frac{D}{\pi}} \int_0^t \frac{\psi'(\xi)}{\sqrt{t-\xi}}\, d\xi \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \quad (3, 6)$$

In the special case where $\phi_0(x) = C_0$, $(3,6)$ becomes

$$J_{1.surf} = (\psi(0) - C_0)\sqrt{\frac{D}{\pi t}} + \sqrt{\frac{D}{\pi}} \int_0^t \frac{\psi'(\xi)}{\sqrt{t-\xi}}\, d\xi \quad \cdots\cdots\cdots\cdots \quad (3, 7)$$

4) Problems of washing.

The equation $(3,7)$ of the preceding article will provide an interesting example of application in a problem of washing the substance with water. There are several different ways or systems to make the washing, and of these two are considered to be distinctive, and we call them provisionally the "continuous" and "batch" systems, respectively.

In the former system, continuously running fresh water washes the boundary surface and removes the emerging solute substance as soon as it comes out, whence, in this system $C(0, t)$ is considered to be always zero, and the results of former sections will be immediately applicable. On the contrary, in the latter system the water does not change continuously, but stays for definite time intervals in one tank, then flows to the next and so on. The water in the tank is effectively stirred so that the solute concentration becomes constant everywhere in the tank. But, of course, this concentration gradually increases with time, and consequently, the dissolving efficiency of the water becomes rather worse than in the former system.

To make the comparison of the two different systems clearer, we assume first that in the latter case there is only one tank, and the water remains sufficiently long time in it; and secondly, we assume that the initial C is constant C_0. If the volume of this tank is denoted by V, and the surface area by S, then evidently the next relation holds good.

$$-S \int_0^T J_{1surf}(t)\, dt = Q = V\psi(T) \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \quad (4, 1)$$

Inserting the J_{1surf} given in (3, 7), (4, 1) we obtain the equations.

$$2\left(\psi(0)-C_0\right)\sqrt{\frac{DT}{\pi}}+\sqrt{\frac{D}{\pi}}\int_0^T dt\int_0^t\frac{\psi'(\xi)}{\sqrt{t-\xi}}\,d\xi=-\frac{V}{S}\psi(T)$$

Differentiating both sides by T, we obtain the equation

$$\left(\psi(0)-C_0\right)\sqrt{\frac{D}{\pi T}}+\sqrt{\frac{D}{\pi}}\int_0^T\frac{\psi'(\xi)}{\sqrt{T-\xi}}\,d\xi=-\frac{V}{S}\psi'(T)\quad\cdots\cdots\cdots\cdots\cdots\cdots (4,\ 2)$$

which is an integral equation of Volterra type for unknown function ψ'. Replace T by t again and denote the Laplace transforms of $\psi'(t)$ and $\psi(t)$ by $\chi(s)$ and $\Psi(s)$, then,

$$\chi(s)=s\Psi(s)-\psi(0)$$

Operate L on both sides of (4, 2) and we obtain

$$\left(\psi(0)-C_0\right)\sqrt{\frac{D}{s}}+\sqrt{\frac{D}{s}}\chi(s)=-\frac{V}{S}\chi(s)$$

which is merely an algebraic equation for the $\chi(s)$ and also for the $\Psi(s)$. $\Psi(s)$ is given by

$$\frac{A}{s(a+\sqrt{s})}+\frac{\Psi(0)}{s}$$

where

$$a=\frac{S}{V}\sqrt{D}\quad\text{and}\quad A=\frac{S}{V}\sqrt{D}\,(C_0-\psi(0))$$

Using the well known formula

$$L^{-1}\left[\frac{1}{s(a+\sqrt{s})}\right]=\frac{1}{a}\left\{1-\frac{2}{\sqrt{\pi}}e^{a^2t}\int_{a\sqrt{t}}^\infty e^{-\xi^2}\,d\xi\right\}$$

we obtain $\psi(t)$ and $\psi'(t)$ together with J_{1surf} as foollows

$$\left.\begin{aligned}\psi(t)&=\frac{A}{a}\left\{1-\frac{2}{\sqrt{\pi}}e^{a^2t}\int_{a\sqrt{t}}^\infty e^{-\xi^2}\,d\xi\right\}+\psi(0)\\[2mm]\psi'(t)&=\frac{a^2C_0}{\sqrt{\pi}}\left\{\frac{1}{a\sqrt{t}}-2e^{a^2t}\int_{a\sqrt{t}}^\infty e^{-\xi^2}\,d\xi\right\}.\end{aligned}\right\}\quad\cdots\cdots\cdots\cdots (4,\ 3)$$

$$J_{1.surf}=-C_0\sqrt{\frac{D}{\pi t}}+2C_0a\sqrt{\frac{D}{\pi}}e^{a^2t}\int_{a\sqrt{t}}^\infty e^{-\xi^2}\,d\xi\quad\cdots\cdots\cdots\cdots (4,\ 4)$$

All these formulae are the required results. If we put $\psi(0)=0$, the total amount of solute dissolved in the water in the interval $0\leq t\leq T$ will be given as

$$Q=-S\int_0^T J_{1.surf}(t)\,dt=V\psi(T)\quad\cdots\cdots\cdots\cdots\cdots\cdots (4,\ 5)$$

To make the comparison easier it is convenient to adopt the following scheme of calculation together with new definitions and notations.

　i) Zero suffix refers to "continuous system", and no suffix refers to "batch system".

ii) $\kappa = Q/Q_0$ $1 - \eta = J/J_0$ $\mu = \dfrac{S}{V}\sqrt{Dt}$ .... (4, 6)

$$\Phi(\mu) = \frac{2}{\sqrt{\pi}}\int_0^\mu e^{-\xi^2}\,d\xi \qquad \Phi'(\mu) = \frac{2}{\sqrt{\pi}}e^{-\mu^2} \quad .... (4, 7)$$

iii) $\eta(\mu) = 2\mu\,\dfrac{1-\Phi(\mu)}{\Phi'(\mu)}$

$\left. \begin{aligned} Q(\mu) &= C_0 V\left\{1 - \frac{2}{\sqrt{\pi}}\frac{1-\Phi(\mu)}{\Phi'(\mu)}\right\} \\ \kappa(\mu) &= \frac{1}{\mu}\left\{\frac{\sqrt{\pi}}{2} - \frac{1-\Phi(\mu)}{\Phi'(\mu)}\right\} \\ J_0(\mu) &= -\frac{C_0 DS}{\sqrt{\pi}}\frac{1}{\mu} \end{aligned} \right\}$ .... (4, 8)
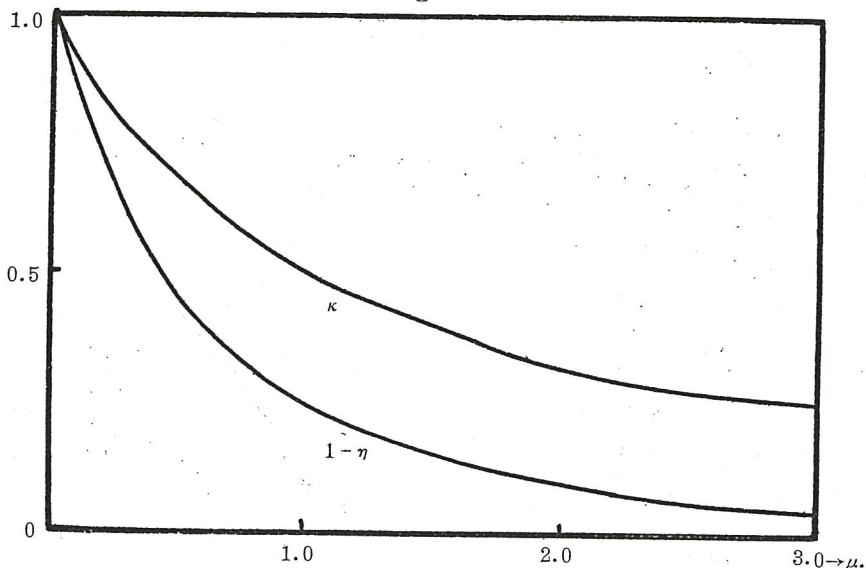
The numerical values of each function are shown in the Table 1 for the argument μ. Note that μ increases with $\sqrt{t}$. The Figures 1 and 2 show the graph of these functions.

Table 1

μ	$t\sqrt{V^2}/SD$	$\Phi(\mu)$	$\dfrac{1-\Phi(\mu)}{\Phi'(\mu)}$	$\dfrac{J_0(t)}{\frac{1}{\sqrt{\pi}}\frac{C_0SD}{V}}$	$\dfrac{J(t)}{\frac{1}{\sqrt{\pi}}\frac{C_0SD}{V}}$	$\eta(\mu)$	$1-\eta(\mu)$	$\dfrac{Q_0(t)}{C_0V}$	$\dfrac{Q(t)}{C_0V}$	$\kappa(\mu)$
0.0	0.00	0.0000	0.8862	∞	∞	0.0000	1.0000	0.0000	0.0000	1.0000
1	0.01	0.1125	7944	10.0000	8.4110	1589	8411	1128	1036	9180
2	0.04	0.2227	7170	5.0000	3.5660	2868	7132	2257	1909	8460
3	0.09	0.3286	6510	3.3333	2.0310	3906	6094	3385	2654	7840
4	0.16	0.4284	5945	2.5000	1.3110	4756	5244	4514	3292	7293
5	0.25	0.5205	5456	2.0000	0.9088	5456	4544	5642	3843	6812
6	0.36	0.6039	5032	1.6667	6603	6038	3962	6770	4322	6383
7	0.49	0.6778	4661	1.4286	4964	6525	3475	7899	4741	6001
8	0.64	0.7421	4334	1.2500	3833	6934	3066	9047	5110	5660
9	0.81	0.7969	4046	1.1111	3019	7283	2717	1.0156	5434	5351
1.0	1.00	0.8427	3789	1.0000	2422	7578	2422	1.1284	5724	5073
1.1	1.21	0.8802	3560	0.9091	1971	7832	2168	1.2412	5983	4820
1.2	1.44	0.9103	3356	8333	1622	8054	1946	1.3541	6213	4589
1.3	1.69	0.9340	3170	7692	1352	8242	1758	1.4669	6423	4378
1.4	1.96	0.9523	3002	7143	1139	8406	1594	1.5798	6613	4186
1.5	2.25	0.9661	2851	6667	0965	8553	1447	1.6926	6783	4007
1.6	2.56	0.9764	2717	6250	0816	8694	1306	1.8054	6934	3841
1.7	2.89	0.9838	2584	5882	0714	8786	1214	1.9183	7084	3693
1.8	3.24	0.9891	2468	5556	0619	8885	1115	2.0311	7215	3552
1.9	3.61	0.9928	2360	5263	0543	8968	1032	2.1440	7337	3422
2.0	4.00	0.9953	2266	5000	0468	9064	0936	2.2568	7443	3298
2.1	4.41	0.9970	2172	4762	0418	9122	0878	2.3696	7549	3186
2.2	4.84	0.9981	2083	4545	0380	9165	0835	2.4825	7650	3081
2.3	5.29	0.9989	2012	4348	0324	9255	0745	2.5953	7730	2978
2.4	5.76	0.9993	1946	4167	0275	9341	0659	2.7082	7804	2879
2.5	6.25	0.9996	1873	4000	0254	9365	0635	2.8210	7887	2796
2.6	6.76	0.9998	1814	3846	0218	9433	0567	2.9338	7953	2711
2.7	7.29	0.9999	1766	3704	0172	9536	0464	3.0469	8007	2628
2.8	7.84	0.9999	1711	3571	0149	9582	0418	3.1595	8069	2554

Note.   suffix o refers to continuous system

no suffix refers to batch system.

$\Phi(\mu)$ = Gauß's error function.

Fig. 1.

$$\eta(\mu)=2\mu e^{\mu^2}\int_{\mu}^{\infty}e^{-\xi^2}d\xi.$$

$$\kappa(\mu)=\frac{1}{\mu}\left[\frac{\sqrt{\pi}}{2}-e^{\mu^2}\int_{\mu}^{\infty}e^{-\xi^2}d\xi\right]$$
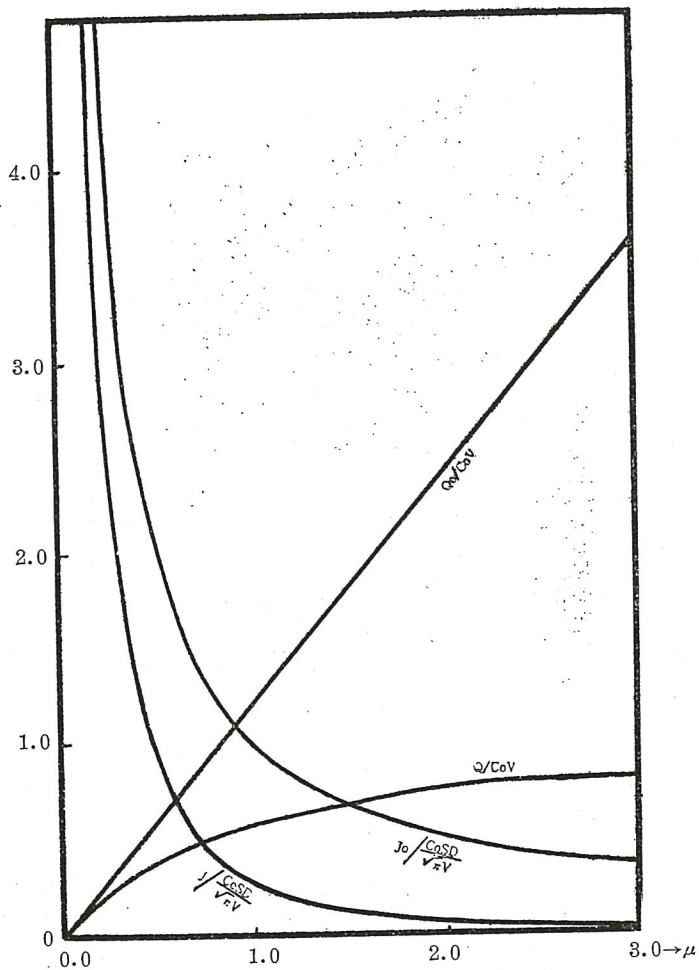
$$\mu=\frac{S}{V}\sqrt{D}\,t.$$

5) Application for the counter-flow system.

The preceding equations provide a sample of interesting applications also for the case of washing by the counterflow system. In this system, n equal tanks are placed in series as shown in the Figure 3. The substances which contain the solute in it self (solid, liquid or a mixture of both) come into the n-th tank first, and then pass to the $(n-1)$-th tank, then $(n-2)$-th tank and so on, the interval of staying in each tank being equal to τ. Concentrations of the solute substance (which is to be washed out) are denoted by C_r when it enters into the r-th tank, and C_{r-1} when it leaves the r-th and passes to the $(r-1)$-th. The fresh water comes from the opposite side and first enters into the first tank, passes to the second and so on, and finally leaves the n-th tank, receiving in each tank an amount of solute substances. The concentration of the solute in the water in the r-th tank is denoted by C'_r when it enters into the r-th tank, and C'_{r+1} when it leaves. More-over we denote by C_0 the concentration of the solute when it is taken out finally from the first tank, and by C'_f the concentration of the water when it is taken out from the last tank. In these circumstances, if the assumptions made in the preceding chapters are valid, it can be confirmed easily that the following equations will be valid in this case,

$$\frac{Q_r}{V}=C'_{r+1}-C'_r=C_r-C_{r-1} \quad\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \quad (5,\ 1)$$
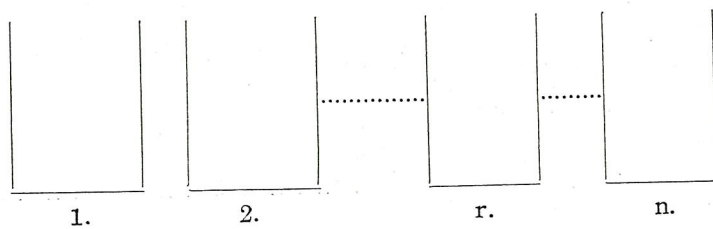
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Fig. 2.

$$\frac{Q_0}{C_0 V} = \frac{2}{\sqrt{\pi}} \mu. \qquad \frac{Q}{C_0 V} = 1 - \frac{2}{\sqrt{\pi}} e^{\mu^2} \int_{\mu}^{\infty} e^{-\xi^2} d\xi$$

$$\frac{\dfrac{J_0}{C_0 SD}}{\sqrt{\pi} V} = \frac{1}{\mu} \qquad \frac{\dfrac{J}{C_0 SD}}{\sqrt{\pi} V} = \frac{1}{\mu} - 2 e^{\mu^2} \int_{\mu}^{\infty} e^{-\xi^2} d\xi.$$

Fig. 3

where Q is the amount of solute substance passed into the water in the r-th tank.
Now, we use the Q from (4, 8) for Q_r in (5, 1), replacing simply C_0 by $C_r - C_r'$ and get

$$
\left.
\begin{aligned}
\frac{Q_r}{V} &= (C_r - C_r')\, F(\mu) \\
F(\mu) &= 1 - \frac{2}{\sqrt{\pi}}\, \frac{1 - \Phi(\mu)}{\Phi'(\mu)} \\
\mu &= \frac{S}{V}\sqrt{D\tau}
\end{aligned}
\right\} \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (5,\,2)
$$

where τ is time interval of staying of the solute and the water in the r-th tank, and is considered to be constant. Consequently, μ and $F(\mu)$ are considered to be equal for all the tanks.
(5, 2) will be rewritten in the forms

$$
\left.
\begin{aligned}
(C_1 - C_1')\, F &= C_2' - C_1' = C_1 - C_0 \\
(C_2 - C_2')\, F &= C_3' - C_2' = C_2 - C_1 \\
&\cdots\cdots\cdots\cdots\cdots\cdots \\
(C_n - C_n')\, F &= C_f' - C_n' = C_n - C_{n-1}
\end{aligned}
\right\} \quad \cdots\cdots\cdots\cdots\cdots\cdots (5,\,3)
$$

from which we can get easily

$$
\left.
\begin{aligned}
C_0 &= \frac{C_n}{1 + nG(\mu)} + \frac{nG(\mu)C_1'}{1 + nG(\mu)} \\
C_f' &= \frac{nG(\mu)C_n}{1 + nG(\mu)} + \frac{C_1'}{1 + nG(\mu)} = C_1' + C_n - C_0 \\
Q_r &= V\, \frac{G(\mu)}{1 + nG(\mu)}\,(C_n - C_1') \quad (r = 1.2 \cdots\cdots n)
\end{aligned}
\right\} \quad \cdots\cdots\cdots\cdots (5,\,4)
$$

where

$$
G(\mu) \equiv \frac{F(\mu)}{1 - F(\mu)} = \frac{1 - \frac{2}{\sqrt{\pi}}\frac{1 - \Phi(\mu)}{\Phi'(\mu)}}{\frac{2}{\sqrt{\pi}}\frac{1 - \Phi(\mu)}{\Phi'(\mu)}} \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots (5,\,5)
$$

Some conclusions from these results are, $\cdots\cdots\cdots\cdots$

i) In almost all the practical cases, while C_n and C_1' are supposed to be known primarily, C_0 must take a value determined from other external condition. In this case, we should determine μ, and the number of tanks, n, from the equation

$$
\frac{1}{n}\left(\frac{C_n - C_0}{C_0 - C_1'}\right) = G(\mu) \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (5,\,6)
$$

The table for this purpose is added at the end of this article (Table. 2)

ii) C_f' will be determined from C_1', C_n and C_0 only. If $C_1' = 0$ as in the usual case, this is simply the difference of the concentrations of the solute at the end and the beginning of the washing.

iii) Amounts of the solute extracted in each tank are all equal, and hence, if we want to make the washing as perfect as possible, we must increase the number of tanks n, or

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increase the parametre μ, which is proportional to $\sqrt{\tau}$

iv) Total time necessary for the washing may be found as constant times $n\mu^2$. The following numerical example shows how this total time changes with n.

Numerical example.

		$C_0 = 1\%$	$C_1 = 0$
$C_n = 35\%$			
n	$G(\mu)$	μ	$n\mu^2$
7	4.86	3.3	76
8	4.25	2.9	67
9	3.78	2.6	59
10	3.40	2.3	53
11	3.09	2.1	49
12	2.83	1.9	45
13	2.62	1.8	43
14	2.43	1.7	41
15	2.27	1.6	39
16	2.13	1.5	37
17	2.00	1.4	35

Table. 2

μ	$G(\mu)$	μ	$G(\mu)$	μ	$G(\mu)$
0.0	0.000				
1	0.116	1.1	1.489	2.1	3.080
2	0.236	1.2	1.641	2.2	3.255
3	0.361	1.3	1.796	2.3	3.405
4	0.491	1.4	1.953	2.4	3.554
5	0.624	1.5	2.109	2.5	3.732
6	0.761	1.6	2.262	2.6	3.885
7	0.902	1.7	2.429	2.7	4.018
8	1.045	1.8	2.591	2.8	4.179
9	1.190	1.9	2.755		
1.0	1.339	2.0	2.911		

μ	$G(\mu)$	μ	$G(\mu)$	μ	$G(\mu)$	μ	$G(\mu)$
3.0	4.587						
3.2	4.927	4.2	6.645	5.2	8.381	6.2	10.129
3.4	5.268	4.4	6.991	5.4	8.730	6.4	10.479
3.6	5.611	4.6	7.338	5.6	9.082	6.6	10.830
3.8	5.955	4.8	7.685	5.8	9.429	6.8	11.180
4.0	6.299	5.0	8.033	6.0	9.779	7.0	11.531
						7.2	11.883
						7.4	12.234

Asymptotic form of function $G(\mu)$ for large value of μ is obviously $\sqrt{\pi}\,\mu - 1$.