A NOTE ON COORDINATED SPACES

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1. *Preliminary.* Preparing the previous paper [1][(1)], we were interested in some properties of coordinated spaces, that is, especially in relations between the coordinatedness and the existence of basis of linear convex topological spaces and in those between the coordinatedness of a linear convex topological space and that of its dual space. In the case of Banach spaces, one of the former relations is well-known ([2], pp. 110–111) and one of the latter is partially solved by Karlin [3]. In the present paper we shall generalize these relations to the case of linear convex topological spaces and moreover afford a more general positive answer to one of the three unsolved problems proposed by Karlin ([3], p. 984): "If the dual space E′ has a basis does that imply that the space E has a basis ?"

Here by a sequence space we mean an abstract linear space, whose each point x is represented uniquely, or more precisely one to one, by an infinite sequence of real or complex numbers $\{x_1, x_2, \cdots\cdots, x_n, \cdots\cdots\}$, or simply denoted by $\{x_n\}$, such that each mapping $x_n(x)$: $x \rightarrow x_n$ $(n=1, 2, \cdots\cdots)$ is linear, and by a coordinated space a sequence space topologized locally convex, such that each mapping $x_n(x)$: $x \rightarrow x_n$ $(n=1, 2, \cdots\cdots)$ is continuous. For each point x, the point which is constructed by equating to zero the coordinates with indices greater than n is denoted by $x^{[n]}$. A coordinated space is called to have the property "Abschnittkonvergenz", or simply (AK) [4], if for every point x of the space, the sequence $\{x^{[n]}\}$ converges to x as n tends to ∞.

2. *Coordinatedness and Basis.* Coordinated spaces have not always a basis, as it is seen in the space (m) and (c).

Theorem 1. Let E be a linear convex topological space. If E is coordinated and has the property (AK), then E has a basis.

Proof. For $i = 1, 2, \cdots\cdots, n, \cdots\cdots$, let e^i be a point represented by the sequence $\{\delta_{in};\ n=1, 2, \cdots\cdots\}$, where δ_{in}'s are Kroncker's deltas. For each point x of E represented by the sequence $\{x_n\}$, $x^{[n]}$ can be expressed in the following form:

$$x^{[n]} = \sum_{i=1}^{n} x_i e^i.$$

Since the property (AK) implies $x^{[n]} \rightarrow x$ as $n \rightarrow \infty$, x can be expressed in the form:

$$x = \sum_{i=1}^{\infty} x_i e^i. \qquad \cdots\cdots (1)$$

The uniquencess of the representation by coordinates implies that of the representation (1), and therefore the set of points $\{e^1, e^2, \cdots\cdots, e^n, \cdots\cdots\}$ is a basis of the space E, which

(1) Numbers in brackets refer to the bibliography at the end of the paper.

proves the theorem.

The converse of this theorem can be given in the form of the following

Theorem 2. If E is an F-space (i. e. a complete, metrizable, linear, convex topological space) and has a basis, then E is a coordinated space with the property (AK).

Proof. Let $\{e^n\}$ be the basis, and then for each x of E we have

$$x=\sum_{i=1}^{\infty} x_i e^i,$$

$\{x_i\}$ being uniquely, or more precisely one to one, determined, or

$$\sum_{i=1}^{n} x_i e^i \rightarrow x \quad \text{as} \quad n\rightarrow\infty.$$

We consider a sequence space $\tilde{E}$, consisting of the sequences $\tilde{x}$:

$$\tilde{x}=\{x_1, x_2, \cdots\cdots, x_n, \cdots\cdots\}.$$

Thus we have a mapping Φ of E onto $\tilde{E}$, i. e. Φ: $x \rightarrow \tilde{x}$. This mapping Φ is linear and one-to-one.

Let the topology of the space E be defined by a family of semi-norms:

$$p_\lambda(x)=\|x\|_\lambda \qquad (\lambda \varepsilon \Lambda)$$

Since $\sum_{i=1}^{\infty} x_i e^i$ is convergent, $\|\sum_{i=1}^{n} x_i e^i\|_\lambda$ is bounded for each λ and we define semi-norms $\|\tilde{x}\|_\lambda$ of the space $\tilde{E}$ as follows:

$$\|\tilde{x}\|_\lambda = \sup_{1\leq n<\infty} \|\sum_{i=1}^{n} x_i e^i\|_\lambda .$$

The space $\tilde{E}$ thus semi-normed is an F-space.

The inverse mapping $x = \Phi^{-1}(\tilde{x})$ of the mapping Φ is linear, one-to-one and continuous. In fact, by the definition of $\|\tilde{x}\|_\lambda$, we have

$$\|x\|_\lambda = \|\Phi^{-1}(\tilde{x})\|_\lambda \leq \|\tilde{x}\|_\lambda,$$

([5] II, § 5, p. 100, Prop. 9). Thus the mapping Φ^{-1} of the F-space $\tilde{E}$ onto the F-space E is bicontinuous ([2], p. 41, The. 5, or [5] Chap. I, § 3, p. 36, Cor. 1), and therefore Φ is an isomorphism.

The functional $\tilde{x}'$ defined on $\tilde{E}$, such that $<\tilde{x}, \tilde{x}'>=x_i$ where $\tilde{x}=\{x_n\}$, is evidently linear, and also continuous. In fact, for each $\lambda \varepsilon \Lambda$, $\|x_i e^i\|_\lambda \leq 2\ \|\tilde{x}\|_\lambda$, or $|x_i| \leq \frac{2\|\tilde{x}\|_\lambda}{\|e^i\|_\lambda}$ for semi-norms $\|\quad\|_\lambda$, such that $\|e^i\|_\lambda > 0$. Hence the space $\tilde{E}$ is coordinated, and so is the space E, since the mapping $x_n(x)$: $x\rightarrow x_n$ is continuous as composed mapping of two continuous ones

$$x \to \tilde{x} \quad \text{and} \quad \tilde{x} \to x_n .$$

The property (AK) is obvious. Thus the proof is completed.

We can conclude the equivalence of the coordinatedness with the property (AK) and the existence of basis for F-spaces.

3. Coordinatedness of a space E and that of its dual space E'.

*Theorem 3. If E is coordinated with the property (AK), then its dual space E' is also coordinated with the property *-weak-(AK) (that is, (AK) for the topology $\sigma(E', E)$).*

Proof. For each point x of E represented by the sequence $\{x_n\}$, we denote $x_n e^n$ by $x_{(n)}$, where each e^i is a point represented by the sequence $\{\delta_{in}\}$ as before, and $x_{(n)} = x_{(n)}(x)$, where $x_{(n)}$ on the right-hand side is the mapping: $x \to x_{(n)}$. The space $E_{(n)}$ of the points $x_{(n)}$ is a subspace of E isomorphic to R^1 or C, the space of real or complex nwmbers.

The dual space E' is a sequence space. In fact, for every point $x' \varepsilon E'$, since the continuous linear functional $x'_{(n)}$ defined by

$$<x, x'_{(n)}> = <x_{(n)}(x), x'> = <x_{(n)}, x'>$$

can be considered as a linear continuous functional defined on the 1-dimensional linear subspace $E_{(n)}$, we may put

$$x'_{(n)} = x'_n e'^{(n)}.$$

Hence x' can be represented by the sequence $\{x'_n\}$. This representation is unique, because if, for all n, $x'_n = 0$, then $x'_{(n)} = 0$ for all n, which implies that for all n and for every $x \varepsilon E$, $<x, x'_{(n)}> = 0$, or $<x_{(n)}, x'> = 0$, or furthermore $<x^{[n]}, x'> = 0$, and from the property (AK) and the continuity of the functional x', it follows that, for every point $x \varepsilon E$, $<x, x'> = 0$, which proves that $x' = 0$.

Each mapping $x'_n(x')$: $x' \to x'_n$ is continuous. In fact, a fundamental system of neighbourhoods of the origin in the dual space E' consists of the polars of the bounded subsets of E, *i. e.* of the subsets $\{x' ; |<x, x'>| \leq 1,\ x \varepsilon B\}$ where B is a bounded subset of E, and a fundamental system of neighbourhoods of the origin in the space $E'_{(n)}$ of the points $x'_{(n)}$ consists of the polars of the bounded subsets of the subspace $E_{(n)}$. Bounded subsets in the space $E_{(n)}$ is also bounded in the space E. Thus the mapping $x'_{(n)}(x')$: $x' \to x'_{(n)}$, accordingly $x'_n(x')$: $x' \to x'_n$ is continuous.

The representation $\{x'_n\}$ is *-weak-(AK). In fact, since

$$<x, x'^{[n]}> = <x, \sum_{i=1}^{n} x'_{(i)}> = \sum_{i=1}^{n} <x, x'_{(i)}>$$

$$= \sum_{i=1}^{n} <x_{(i)}, x'> = <\sum_{i=1}^{n} x_{(i)}, x'> = <x^{[n]}, x'>,$$

the property (AK) for the space E implies that, for all $x \varepsilon E$, $<x, x'^{[n]}>=<x^{[n]}, x'>$ converges to $<x, x'>$ as n tends to ∞, and therefore $x'^{[n]}$ converges $*$-weakly to x'. The theorem is proved.

The converse of this theorem is given for F-spaces by the following

Theorem 4. Let E be an F-space. If the dual space E' is coordinated with the property $*$-weak-(AK), then the space E itself is also coordinated with the property (AK).

Proof. For each point x' of E' represented by the sequence $\{x'_n\}$, we denote the sequence $\{\delta_{in} x'_n;\ i=1, 2, \cdots\cdots\}$ by $x'_{(n)}$ and $\{\delta_{in};\ i=1, 2, \cdots\cdots\}$ by $e'^{(n)}$, and then we have $x'_{(n)}=x'_n e'^{(n)}$. For each point x of E, we have

$$<x, x'_{(n)}>=x'_n<x, e'^{(n)}>.$$

Here if we put $<x, e'^{(n)}>=x_n$, then we have $<x, x'_{(n)}>=x_n x'_n$. Therefore the point x is represented by $\{x_n\}$.

This representation $\{x_n\}$ is linear and unique. The linearity is evident. If, for all n, $x_n=0$, *i. e.* $<x, e'^{(n)}>=0$, hence $<x, x'_{(n)}>=0$, then $<x, x'^{[n]}>=0$. The property $*$-weak (AK) implies that, for every $x' \varepsilon E'$, $<x, x'>=0$, and therefore $x=0$. Thus the uniqueness of the representation is verified.

The mapping $x_n(x)$: $x \rightarrow x_n$ is continuous. Since $x_n=<x, e'^{(n)}>$ is a continuous functional of x, the mapping is continuous.

The property $*$-weak-(AK) implies that for every $x \varepsilon E$ and every $x' \varepsilon E'$, since

$$\begin{aligned}<x, x'>&=\lim_{n\to\infty} <x, x'^{[n]}>=\lim_{n\to\infty} \sum_{i=1}^{n}<x, x'_{(i)}>\\ &=\lim_{n\to\infty} \sum_{i=1}^{n} x'_i<x, e'^{(i)}>=\lim_{n\to\infty} \sum_{i=1}^{n} x_i x'_i,\end{aligned}$$

the series $\sum_{n=1}^{\infty} x_n x'_n$ is convergent, and also that $<e^m, e'^{(n)}>=\delta_{mn}$, where $e^{(m)}=\{\delta_{mn}; n=1, 2, \cdots\cdots\}$.

The representation has the property weak-(AK). In fact, for each $x' \varepsilon E'$ we have

$$\begin{aligned}<x^{[n]}, x'>&=\sum_{i=1}^{n}<x_i e^{(i)}, x'>=\sum_{i=1}^{n} x_i<e^{(i)}, x'>\\ &=\sum_{i=1}^{n} x_i \sum_{m=1}^{\infty} \delta_{mi} x'_m=\sum_{i=1}^{n} x_i x'_i,\end{aligned}$$

and since $\sum_{i=1}^{n} x_i x'_i$ is convergent to $<x, x'>$, $<x^{[n]}, x'>$ converges to $<x, x'>$ as n tends to ∞ for each $x' \varepsilon E'$, and hence $x^{[n]}$ converge to x weakly as n tends to ∞.

Thus the completion of the proof will be reduced to the following

Lemma. If E is a coordinated F-space and its dual space E' is a sequence space, then the property weak-(AK) implies the property (strong-) (AK) (that is, the property (AK) is equivalent for both the weak topology and the original one).

Proof. Now let E_1 denote the closed linear subspace generated by the countable set $\{e^1, e^2, \cdots\cdots, e^n, \cdots\cdots\cdots\}$ of points of the basis of E.

We first show that $E_1 = E$. If we suppose the contrary, then there exists such a point x^0 of E, as does not belong to E_1, hence by virtue of Hahn-Banach theorem, there exists a linear continuous functional x' defined on E, such that

$$<x^0, x'> = 1 \text{ and } <x, x'> = 0 \text{ for each } x \varepsilon E_1.$$

Hence we have $<e^n, x'> = 0$ or $\sum_{i=1}^{\infty} \delta_{in} x_i' = x_n' = 0$ for each n, accordingly $x' = 0$, which contradicts to the propepeety $<x^0, x'> = 1$.

In order to prove that each point $x \varepsilon E_1$ or E satisfies the condition (AK), *i. e.* the point

$$x^{[n]} = \sum_{j=1}^{n} x_{(j)} = \sum_{j=1}^{n} x_j e^j = \sum_{j=1}^{n} <x, e'^{(j)}> e^j$$

converges (strongly) to the point x as n tends to ∞, we must previously prove that the countable subset $\{\sum_{i=1}^{n} x_i e^i; n = 1, 2, \cdots\cdots\}$ is bounded. Since $\sum_{i=1}^{\infty} x_i x_i'$ is convergent, the countable set of numbers $\{\sum_{i=1}^{n} x_i x_i'; n = 1, 2, \cdots\cdots\}$ is bounded, and let $\sup_{1 \leq n < \infty} |\sum_{i=1}^{n} x_i x_i'|$ be denoted by (x, x'). Let U be an arbitrary weak neighbourhood of the origin in the space E, *i. e.* $U = \{x; |<x, x_k'>| \leqq 1, k = 1, 2, \cdots\cdots, l\}$, and $\max_{1 \leq k \leq l} [(x, x'^k)]$ be denoted by M_x. Then we have $|<\sum_{i=1}^{n} x_i e^i, x'^k>| \leqq (x, x'^k) \leqq M_x$, and hence $\sum_{i=1}^{n} x_i e^i \varepsilon M_x U$ for $n = 1, 2, \cdots\cdots$. This shows that the contable set $\{\sum_{i=1}^{n} x_i e^i: n = 1, 2, \cdots\cdots\}$ is weakly bounded, and accordingly (strongly) bounded.

Putting $s_n(x) = \sum_{i=1}^{n} x_i e^i$, a sequence of linear countinuous applications $s_n(x)$ on E into itself: $x \rightarrow \sum_{i=1}^{n} x_i e^i$ is defined. Secondly we must prove that the set of applications $\{s_n: n = 1, 2, \cdots\cdots\cdots\cdots\cdots\}$ is equicontinuous. A neighbourhood of the origin in the space $\mathfrak{L}(E, E)$, with the topology of simple convergence, of all linear continuous applications on E into itself is defined by the set of linear continuous applications $u(x)$ such that $u(M) \subset V$, where M is a finite subset of E and V a neighbourhood of the origin in E, and is denoted by $T(M, V)$. For each pair of a finite subset M and a neighbourhood V of the origin in E, since the subset $\{s_n(M): n = 1, 2, \cdots\cdots\} = \bigcup_{x \varepsilon M} \{s_n(x): n = 1, 2, \cdots\cdots\}$ is bounded as a finite sum of bounded subsets, there exists a positive number λ such that

$$s_n(M) \subset \lambda V,$$

and accordingly

$$\{s_n; n = 1, 2, \cdots\cdots\} \subset \lambda T(M, V).$$

Thus the set of applications $\{s_n: n = 1, 2, \cdots\cdots\}$ is bounded in the space $\mathfrak{L}(E, E)$ with the topology of simple convergence, and accordingly equicontinuous ([6] Chap. IV, § 3, p. 27, Th. 2).

We have now to prove that $s_n(x)$ converges to x as n tends to ∞. In case $x = \sum_{i=1}^{m} x_i e^i$, the proof is evident, since $s_n(x) = \sum_{i=1}^{m \leq n} x_i e^i$ converges to $\sum_{i=1}^{m} x_i e^i = x$. In case the point x is a limit point of the sequence of points $\{x^m;\ m = 1, 2, \cdots\cdots\}$, where $x^m = \sum_{i=1}^{m} x_i^m e^i$, we proceed as follows. Because of the equicontinuity of the set of applications $\{s_n:\ n = 1, 2, \cdots\cdots\}$, for an arbitrary neighbourhood V of the origin there exists a neighbourhood U of the origin (independent of n), such that $s_n(U) \subset V$ for $n = 1, 2, \cdots\cdots$. $V' = U \frown V$ being a neighbourhood of the origin, there exists a positive integer m_0, such that

$$x^{m_0} - x \in V' = U \frown V \subset V,$$

and accordingly

$$s_n(x^{m_0} - x) = s_n(x^{m_0}) - s_n(x) \in V.$$

From the first case, it follows that there exists a positive integer n_0 (depending on m_0), such that $n \geqq n_0$ implies the relation:

$$s_n(x^{m_0}) - x^{m_0} \in V.$$

These three relations implies that

$$s_n(x) - x \in 3V \qquad \text{for } n \geqq n_0,$$

which proves that $s_n(x)$ converges x as n tends to ∞. Thus the proof of the lemma is completed.

Remark to the lemma. In the case of the dual space E', the result of the lemma does not hold, that is, the property *-weak-(AK) does not always imply the property strong-(AK), as is seen in an example: $E = (l)$, the space of the absolutely convergent sequences and its dual space $E' = (m)$, the space of the bounded sequences.

References

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