ON A COMPLETELY HARMONIC SPACE WITH AN ALMOST COMPLEX STRUCTURE

Takeo OHKUBO

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§1. Consider a Riemannian n-space V_n for which

$$ds^2 = g_{ij} dx d^i x^j. \tag{1.1}$$

If $P_0(x_0^i)$ is a fixed point of the V_n and if $s = s(x_0^i, x^i)$ is the length of arc of the geodesic joining it and a variable point $P(x^i)$, then the V_n is called (centrally) harmonic with respect to the basepoint $P_0(x_0)$ if

$$\Delta_2 s \equiv \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i}\left(\sqrt{g}\, g^{ij} \frac{\partial s}{\partial x^j}\right) \tag{1.2}$$

is a function of s alone; it does not involve the coordinates x^i of the variable point explicitly, but the coordinates x_0^i of the base-point may be involved as parameters. If this holds for all choices of the base-point, the space will be called completely harmonic. If V_n is completely harmonic and $\Delta_2 s$ has the special form $(n-1)/s$, then V_n is called simply harmonic. E. T. Copson and H. S. Ruse(1) showed how to obtain the conditions, in term of the curvature tensor, that a V_n should be completely harmonic. These conditions were infinite in number and involved the covariant derivatives of the curvature tensor.

A. G. Walker(2) (1942) has found another method of obtaining them, using Ruse's invariant ρ. The Ruse's invariant ρ is defined by

$$\rho^2 = \frac{|g_{ij}|_{P_0} |g_{ij}|_P}{J^2}, \qquad J = \left| \frac{\partial^2 \frac{s^2}{2}}{\partial x^i \partial x^j} \right|. \tag{1.3}$$

Walker has given the expansion of log ρ:

$$\log \rho = \frac{W_1}{2!} s + \frac{W_2}{3!} s^2 + \frac{W_3}{4!} s^3 + \cdots\cdots + \frac{W_m}{(m+1)!} s^m + \cdots \tag{1.4}$$

Each of the coefficients is the trace of a certain polynomial in matrices Γ, $\dot{\Gamma}$, $\ddot{\Gamma}$, etc. (of degree $\frac{m-1}{2}$ or $\frac{m}{2}$, whichever is an integer), where

$$\left.\begin{aligned} &\Gamma = (R^i_{hjk} \lambda^h \lambda^j), \qquad \dot{\Gamma} = (R^i_{hik;\,l} \lambda^h \lambda^j \lambda^l), \\ &\ddot{\Gamma} = (R^i_{hjk;\,lm} \lambda^h \lambda^j \lambda^l \lambda^m), \text{ etc.} \end{aligned}\right\} \tag{1.5}$$

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(1) See the Bibliography at the end of the paper [1]

(2) See [2]

and λ^i is the unit tangent vector of the geodesic P_0P at P_0. These coefficients must all be evaluated at P_0.

When and only when the space V_n is centrally harmonic with respect to P_0, ρ is dependent only upon s, i. e., independent of λ^i and completely harmonic when it holds about every point of the space. These conditions can be derived from the equation (1.4), i. e., we require that each W_r shall be independent of λ^i. It is easily seen that W_r is of the form

$$W_r = W_{i_1 i_2 \cdots i_r} \lambda^{i_1} \lambda^{i_2} \cdots \lambda^{i_r}$$

where $W_{i_1 i_2 \cdots\cdots i_r}$ assumed symmetric in the suffices, are functions of the foundamental and curvature tensors and their derivatives. As the λ^i satisfy $g_{ij}\lambda^i\lambda^j =$ constant, the required conditions are

$$\left.\begin{aligned} &W_{i_1 i_2 \cdots i_r} = 0, && \text{r; odd number,} \\ &W_{i_1 i_2 \cdots i_r} = k_r \Sigma g_{i_1 i_2} g_{i_3 i_4} \cdots g_{i_{r-1} i_r}, && \text{r: even number,} \end{aligned}\right\} \tag{1.6}$$

where the k's are scalars, the sum being taken to give an expression symmetric in suffices. When these conditions are satisfied at P_0, it appears for any point P that ρ is a function of s, and is expressed as a power series in s^2, strictly in es^2 where $e = g_{ij}\lambda^i\lambda^j = \pm 1$ or 0.

When a space is completely harmonic, conditions (1.6) are satisfied at every point of the spasce. We see easily that, in general, the form of ρ as a function of es^2 must be the same for all choices of base-points. It follows that the sealars k_r in (1.6) are all constants.

It is well known that another special class of spaces, that is to say, simply harmonic spaces, consists of those spaces which are completely harmonic and in which $\rho = 1$ for every pair of points. (1.4) implies that $\rho = 1$ only when $W_r = 0$ for all λ's. Therefore the conditions for a simply harmonic space are that the equations: $W_r = 0$, $r = 1, 2, 3, \cdots\cdots$ must be satisfied at all points of the space and for all direction λ^i.

Let λ^i be any direction and put $\Lambda = g_{ij}\lambda^i\lambda^j$. As the matirces $\dot{\Gamma}$, $\ddot{\Gamma}$, $\dddot{\Gamma}$ etc. are all zero matrices for a symmetric space we have the following theorems.

Theorem A. *A symmetric space is harmonic if, for all N, the trace of Γ^N is of the form $k_N\Lambda^N$, where Γ^N is the N-th power of Γ and k_N is independent of the λ's.*

Theorem B. *The necessary and sufficient conditions for a symmetric V_n to be harmonic are that the latent roots of Γ should be of the form*

$$0,\ \alpha_1\Lambda,\ \alpha_2\Lambda,\ \cdots\cdots\cdots\cdots,\ \alpha_{N-1}\Lambda$$

where the α_i is independent of the λ's.

Theorem C. *The necessary and sufficient conditions for a symmetric space to be simply harmonic are that all the latent roots of Γ should be zero for all λ's.*

A problem of characterizing all harmonic spaces H_n has been only partially solved. Harmonic 2- and 3-spaces are trivial, and for real harmonic 4-spaces the following theorems have already been established.

(1) Every harmonic 4-space H_4 with signature 2 is a space of constant curvature, and the proposition also holds for $n>4$.

(2) There exist H_4's which are symmetric, but are not spaces of constant curvature. All such spaces are known; some have definite metrics and others have zero signature.

(3) There are H_4's which are recurrent, but are not symmetric. All such spaces are known and have zero signature.

(4) Every real H_4 with definite metric is symmetric.

(5) Every simply harmonic 4-space with definite metric is a flat space and the proposition also holds for $n>4$.

(6) Every simply harmonic recurrent 4-space has the metric of a form

$$ds^2=\alpha(x,y)dx^2+2\gamma(x,y)dxdy+\beta(x,y)dy^2+2dxdz+2dydt.$$

Only a few properties of harmonic spaces H_n with $n>4$ are known. Some of them are following:

(7) A harmonic space H_n conformal to a flat space is a space of constant curvature.

(8) A Riemannian space V_{2p} admitting a null strictly parallel p-plane is simply harmonic.

(9) The product of two simply harmonic K*-spaces is also a simply harmonic K*-space.

(10) If a harmonic K*-space is decomposable, it is a simply harmonic K*-space which is flat-extension of a certain simply harmonic K*-space.

(11) If a homogeneous space E of a Lie group G has a locally spherically transitive isotropy subgroup H, and a positive definite Riemannian metric invariant under G, then it is completely harmonic.

(12) The following spaces, (1) spheres, (2) real projective spaces, (3) complex projective spaces, (4) quaternionic projective spaces, and (5) the Cayley projective plane admit positive definite Riemannian metric which are harmonic.

In following sections we observe an even dimensional Riemannian manifold which admits an integrable real-analytic almost complex structure a^i_j and also is exact. We shall show that a space of constant holomorphic curvature(1) in such a space is completely harmonic.

§2. We observe in a manifold of dimension $n=2m$ a coordinate neighborhood U_x of which points are written by n real coordinates $x^1, x^2, \cdots\cdots, x^n$. Under an almost complex structure J of class C^h in a differentiable manifold M^n, we understand a tensorfield a^i_j of class C^h with respect to x^i, satisfying $a^i_j\, a^j_k=-\delta^i_k$. Then we have linear mapping $b=Ja$ which is represented by $a^i=a^i_j\, b^j$, where a^i and b^i are the components of vectors a and b respectively. We put

(1) See [3] and [4]

$$x^{\mu}+ix^{\bar{\mu}}=z^{\mu},\ \mu, \nu, \cdots=1,2,\cdots\cdots, m;\ \bar{\mu}=\mu+m. \tag{2.1}$$

A contravariant vectorfield in the manifold M^n is written by the components $a^1(x)$, $\cdots\cdots$, $a^n(x)$ in a local coordinate system (x^i), and by $a^1_{(y)}, \cdots\cdots, a^n_{(y)}$ in a y-coodinate system (y^i). Then we have in $U_x \cap U_y$, $a^k_{(y)} = \frac{\partial y^k}{\partial x^l} a^l_{(x)}$ (k, $l, \cdots\cdots=1,2,\cdots\cdots, n$). We call the complex masses

$$A^{\mu}_{(z)}= a^{\mu}_{(x)}+ ia^{\bar{\mu}}_{(x)} \tag{2.2}$$

the complex components of the vectorfield with respect to z-coordinate system, where (z^{μ}) is the complex coordinate system belonging to x^k. To a vector a with the complex components $A^{\mu}_{(z)}$ we make correspond the vector $b=J_{(z)}a$ with the components $B^{\mu}_{(z)}$ by means of

$$B^{\mu}_{(z)}= iA^{\mu}_{(z)}. \tag{2.3}$$

In the above we have shown the mapping by $J_{(z)}$, where the index (z) shows that the mapping is defined by using the coordinate system (z). Such a mapping of vectors should be independent of a choice of local coordinate system. $J_{(z)}$ and $J_{(w)}$ are equal, if and only if Cauchy-Riemann's differential equations holds between z^{μ} and w^{ν}. We say that a complex structure is given on M^n, if M^n is covered with such complex coordinate system that all coordinates transformations $w^{\nu}= w^{\nu}(z)$ are complex-analytic, where (w^{ν}) is the complex coordinate system belonging to y^k. Then we have[(1)];

For a coordinate system belonging to one complex structure, the multiplication of the complex vector components with $i(=\sqrt{-1})$ *is a linear mapping admitting* $A^2=-I$ *independent of the coordinate system.* And also we have:

If the multiplication of the complex vector components with i *in z-coordinate system and in w-coordinate system respectively is the same mapping, these both coordinate systems (in their intersection) are connected by a complex-analytic transformation.*

Therefore a complex structure is well determined by the belonging mapping. In the other words, to one complex structure belongs a well determined almost complex structure. Such almost complex structures derived from a complex structure are called *integrable.*

From the above, we see that an almost complex structure belongs to at most one complex structure, while an integrable almost complex structure does to just one precisely.

When an almost complex structure J is defined by a^i_j and the torsion tensor by $t^i_{kl}=a^i_{hk}a^h_l-a^i_{hl}a^h_k$ where $a^j_{kl} = \frac{\partial a^j_k}{\partial x^l} - \frac{\partial a^j_l}{\partial x^k}$, we have:

Theorem. A necessary and sufficient condition that a real-analytic almost complex structure is integrable, is vanishing of the belonging torsion tensor.

We now consider the manifold M^n with a positive definite metric and assume the almost complex structure on M^n is real-analytic and integrable.

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(1) See [5]

Furthermore we assume the almost complex structure a_j^i is orthogonal, or the length of any vector is invariant. Then we have

$$g_{ik}\,a_j^k + g_{jk}\,a_i^k = 0. \tag{2.4}$$

Then we assume that a differential form $\omega = a_{ij}\,dx^i \Lambda dx^j$ where $a_{ij} = g_{ik}\,a_j^k$, is exact, namely

$$d\omega = 0 \text{ or } D_l\,a_{ij} + D_i\,a_{jl} + D_j\,a_{li} = 0, \tag{2.5}$$

where D_i represents the covariant derivative with respect to g_{kl}.

§ 3. Now we assume that in a 2m dimensional real analytic manifold which has an almost complex analytic structure defined by a_j^i of class C^ω a positive definite metric

$$ds^2 = g_{ij}\,dx^i\,dx^j$$

is given and that the structure a_j^i is orthogonal and without torsion. Furthermore we assume that the differential form ω is exact. By these assumptions we have:

a) $a_{ij} = g_{ik}\,a_j^k = -\,a_{ji}$

b) $d\omega = 0$ or $D_l\,a_{ij} + D_i\,a_{jl} + D_j\,a_{li} = 0$

c) $t_{jk}^i = 0$

c) implies that $(a_{j.k}^i - a_{k.j}^i)a_l^j - (a_{j.l}^i - a_{l.j}^i)a_k^j = 0$ or $(D_k a_j^i - D_j a_k^i)a_l^i - (D_l a_j^i - D_j a_l^i)a_k^j = 0$. After some calculations we get the relation:

$$D_i\,a_k^j = 0. \tag{3.1}$$

If we take an unit vector u^i, as well readily seen, the vector $v^i = a_j^i u^j$ is also unit and orthogonal to u^i. Then we have a *holomorphic sectional curvature* with respect to u^i defined by

$$k = -\,R_{ijkl}\,u^i\,v^j\,u^k\,v^l$$

where R_{ijkl} is the curvature tensor. If the holomorphic sectional curvature is always constant with respect to any vector at every point, the curvature tensor has the form:

$$R_{ijkl} = \frac{-k}{2}\,(g_{[i|k|}\,g_{j]l} + a_{[i|k|}\,a_{j]l} + a_{ij}\,a_{kl}). \tag{3.2}$$

This space is symmetric in Cartan's sense.

Now we shall prove that it is a completely harmonic. In our space, the matrix Γ defined in § 1 has the following form:

$$\Gamma_l^i = R_{jkl}^i\,\lambda^j\,\lambda^k = \frac{k}{4}\,(\Lambda\delta_l^i - \lambda^i\,\lambda_l + 3l^i\,l_l), \tag{3.3}$$

where we put

$$l^i = a_k^i\,\lambda^k \text{ or equivalently } l_i = a_{ij}\,\lambda^j, \tag{3.4}$$

λ^i being any direction and $\lambda_i = g_{ij}\lambda^j$. Then we have

$$\Lambda = l_i l^i = g_{ij}\lambda^i\lambda^j \text{ and } l_i\lambda^i = \lambda_i l^i = 0 . \quad (3.5)$$

From the equations (3.3), (3.4) and (3.5) we get after some calculations:

$$\Gamma^i_k l_i = \Lambda k l_k \text{ and } \Gamma^i_k \lambda^k = 0 \quad (3.6)$$

and

$$\left.\begin{aligned} \Gamma^i_l \Gamma^l_j &= \frac{k}{4}\Lambda\Gamma^i_j + \frac{3k^2}{4}\Lambda l^i l_j, \\ \Gamma^i_l \Gamma^l_j \Gamma^j_k &= \left(\frac{k}{4}\right)^2 \Lambda^2 \Gamma^i_k + \frac{15k^3}{4^2}\Lambda^2 l^i l_k, \quad \text{etc.} \end{aligned}\right\} \quad (3.7)$$

Now by mathematical induction we shall prove that the N-th power of the matrix Γ has the following form:

$$\Gamma^N = \Lambda^{N-1}\{\alpha(\Gamma^i_j) + \beta(l^i l_j)\}, \quad (3.8)$$

where α and β are certain scalars.

In fact when $r = 2$, we have $(3.7)_1$. If we assume that (3.8) holds good, we have

$$\begin{aligned} \Gamma^{N+1} &= (\Lambda^{N-1}\{\alpha(\Gamma^i_k) + \beta(l^i l_k)\}\Gamma^k_j) \\ &= \Lambda^{N-1}\left\{\frac{k}{4}\Lambda\alpha(\Gamma^i_j) + \frac{3k^2}{4}\Lambda\alpha(l^i l_j) + \beta k\Lambda(l^i l_j)\right\} \\ &= \Lambda^N\{\alpha'(\Gamma^i_j) + \beta'(l^i l_j)\}, \end{aligned}$$

where we put $\alpha' = \frac{\alpha k}{4}$ and $\beta' = \frac{3k^2}{4}\alpha + \beta k$.

In the other hand we see that all the matrices $\dot{\Gamma}, \ddot{\Gamma}, \dddot{\Gamma}, \cdots$ are zero matrix, since the space is symmetric. Consequently the trace of the matrix $\Gamma^N (N=1, 2, 3, \cdots)$ is the form $k_N\Lambda^N$, k_N being independent of λ^i, and from the Theorem A in §1 we conclude that our space is completely harmonic. Thus we have

Theorem. If a 2 m dimensional real analytic manifold which has an almost analytic structure defined by a^i_j of class C^ω and a positive definite metric $ds^2 = g_{ij}dx^i dx^j$ satisfying $d\omega = 0$ and (2.4), has the constant holomorphic sectional curvature, then the manifold is completely harmonic.

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