# A NOTE ON THE SEMIRADICAL OF A SEMIRING

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S. BOURNE has introduced a concept of a Jacobson radical of a semiring[1], and recently S. BOURNE and H. ZASSENHAUS have given a definition of a semiradical[2]. It is shown, that the semiradical contains the Jacobson radical of a semiring and both radicals do not coincide in general; [4].

But we shall note here that the semiradical coincides with the Jacobson radical. In this note we shall adopt the same terminologies and notations as in [4], but we shall recite them here for the sake of completeness.

DEFINITION 1. $S$ is called a *semiring* if and only if

(i) A composition $+$ is defined in $S$ such that $(S, +)$ is a commutative semigroup having an identity 0;

$$0 + a = a \qquad (a \in S).$$

(ii) A composition $\cdot$ is defined in $S$ such that $(S, \cdot)$ is a semigroup.

(iii) Distributive laws hold;

$$(a+b)c=ac+bc, \quad a(b+c)=ab+ac.$$

DEFINITION 2. A subset $I$ of $S$ is called a *left* (*right*) *ideal* if and only if

(i) $i_1, i_2 \in I$ then $i_1+i_2 \in I$.

(ii) $i \in I$ then $si$ $(is) \in I$ for all $s \in S$.

(iii) $0 \in I$.

We use the term a *two-sided ideal* of $S$ as usual.

DEFINITION 3. If $i_1+x=i_2+x$ is solvable in $S$, $i_1$, $i_2$ are said to be *equivalent* and denoted as $i_1 \sim i_2$.

It is known that $\sim$ is an equivalence relation and the equivalence classes $i^*$ represented by $i \in S$ form a semiring $S^*$ according to the operations defined by

$$i_1^* + i_2^*=(i_1+ i_2)^*, \quad i_1^* \cdot i_2^*=(i_1 i_2)^*.$$

In the semiring $S^*$ the cancellation law of addition holds. Therefore $S^*$ can be imbedded into a ring $\tilde{S}$ generated by $S^*$ by the method given in [4].

DEFINITION 4. The right ideal $I$ is said to be *right semiregular* if and only if for any $i_1$, $i_2$ of $I$ there exist $j_1$, $j_2$ in $I$ such that

$$(1) \qquad i_1+j_1+i_1j_1+i_2j_2=i_2+j_2+i_1j_2+i_2j_1.$$

DEFINITION 5. The *Jacobson radical* $R(S)$ of $S$ is the union of all the right

1) Cf. [2].
2) Cf. [4].

semiregular ideals of $S$.

It has been proved that $R(S)$ is a two-sided ideal of $S$ and is characterized as the maximal right ideal in which for every $i_1$, $i_2$ of $R(S)$ there can be found elements $j_1$, $j_2$ in $R(S)$ such that the equality (1) holds.

DEFINITION 6. The *semiradical* $\sigma(S)$ of a semiring $S$ is the set of all elements $i$ of $S$ for which $i^*$ is contained in the Jacobson radical $R(S^*)$ of $S^*$.

From the above definition, it follows immediately that $\sigma(S)$ is a two-sided ideal of $S$ and $\sigma(S)\supseteqq R(S)$.

The following results are the known facts; [4].

(a) The semiradical of a semiring $S$ is the maximal right ideal $I$ of $S$ in which for every pair of elements $i_1$, $i_2$ of $I$ there exist $j_1$, $j_2$ in $I$ and $j$ in $S$ such that

(2) $$i_1+j_1+i_1j_1+i_2j_2+j=i_2+j_2+i_1j_2+i_2j_1+j.$$

From (2) we obtain readily

(3) $$i_1+(j_1+j)+i_1(j_1+j)+i_2(j_2+j)=i_2+(j_2+j)+i_2(j_1+j)+i_1(j_2+j).$$

(b) The semiradical of a semiring $S$ is the maximal right ideal $I$ of $S$ in which for any pair of $i_1$, $i_2$ of $I$ a pair of elements $j_1$, $j_2$ of $S$ can be found such that the equality (1) holds.

THEOREM. *The semiradical $\sigma(S)$ of a semiring $S$ coincides with the Jacobson radical $R(S)$.*

PROOF. Put $I=\sigma(S)$. For any pair $i_1$, $i_2$ of $I$ there exist $j_1$, $j_2$ in $S$ such that

(1) $$i_1+j_1+i_1j_1+i_2j_2=i_2+j_2+i_1j_2+i_2j_1.$$

We multiply each term of the above equality by $i_2$ on the right,

$$i_1i_2+j_1i_2+i_1j_1i_2+i_2j_2i_2=i_2^2+j_2i_2+i_1j_2i_2+i_2j_1i_2.$$

Exchanging the sides of (1), we multiply each term of the resulting equality by $i_1$ on the right,

$$i_2i_1+j_2i_1+i_1j_2i_1+i_2j_1i_1=i_1^2+j_1i_1+i_1j_1i_1+i_2j_2i_1.$$

Adding the last two equalities and addint $i_1+i_2$ both sides of the resulting equality we obtain after rearrangement that

$$i_1+j_1'+i_1j_1'+i_2j_2'=i_2+j_2'+i_1j_2'+i_2j_1'$$

where $j_1'=i_2+j_2i_1+j_1i_2$, $j_2'=i_1+j_1i_1+j_2i_2$. As $I$ is a two-sided ideal, we see that $j_1'$, $j_2'$ are in $I$. Hence from the definition of a Jacobson radical, we get $I\subseteqq R(S)$. Therefore we obtain $\sigma(S)=R(S)$.

We shall consider an example which is essentially the same as that of BOURNE and ZASSENHAUS; [4].

Let $T_i$ be the semiring of all polynomials in the indeterminates $x_i$ $(i=1, 2)$ with non-negative rational integral coefficients.

Let $S$ be the semiring formed by the pairs $(t_1, t_2)$ with $t_i\in T_i$ and the rules:

$(t_1, t_2)=(t'_1, t'_2)$ if and only if

(i) $t_2=t'_2$

and

(ii) $t_1=t'_1=0$ or $t_1t'_1\neq 0$,

addition and multiplication in $S$ are defined as follows

$$(u_1, u_2)+(v_1, v_2)=(u_1+v_1, u_2+v_2),$$
$$(u_1, u_2)(v_1, v_2)=(u_1v_1, u_2v_2).$$

It is evident that all the elements $(t_1, 0)$ form an ideal $A$ of $S$ which consists of only two different elements $(0, 0)$, $(1, 0)$, and therefore $A$ is not isomorphic to $T_1$. In this example $\sigma(S)=R(S)=A$.

## Refferences


[1] N. JACOBSON, The Radical and Semi-simplicity for Arbitrary Rings, Amer. Jour. Math. Soc., vol. 67(1945), pp. 300–319.

[2] S. BOURNE, The Jacobson Radical of a Semirings, Proc. Nat. Acad., vol. 37(1951), pp. 163–173.

[3] ———, On Multiplicative Idempotents of a Ponent Semiring, Proc. Nat. Acad., vol. 42 (1956), pp. 632–636.

[4] S. BOURNE and H. ZASSENHAUS, On the Semiradical of a Semiring, Proc. Nat. Acad., vol. 44 (1958), pp. 907–914.