ON n-DIMENSIONAL FINSLERIAN MANIFOLDS ADMITTING HOMOTHETIC TRANSFORMATION GROUPS OF DIMENSION $>\frac{1}{2}n(n-1)+1$

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On groups of motions of Finslerian manifolds, H. C. Wang [7][1)] showed that if a connected Finslerian manifold of dimension $n\neq 4$ admits an effective and connected group of motions of dimension $>\frac{1}{2}n(n-1)+1$ then the manifold is Riemannian and of constant curvature, and later this fact was proved by N. H. Kuiper and K. Yano [3] by a different method.

Recently, Gy Soós [6] treated homothetic transformations of Finslerian manifolds and showed that a Finslerian manifold with constant curvature $R\neq 0$ admits no homothetic transformation which is not a motion. Y. Nasu [5] also showed that a complete and connected Finslerian manifold of class C^1 admits a homothetic transformation which is not a motion then the manifold is Minkowskian. A similar theorem for a Riemannian manifold was already proved by S. Ishihara and M. Obata [2].

In the present paper, we shall study the structures of n-dimensional Finslerian manifolds admitting homothetic transformation groups of dimension $>\frac{1}{2}n(n-1)+1$. The results will appear in theorems of the paper. In the special cases where the manifolds are Riemannian our theorems coincide with already known theorems which were obtained by K. Yano [8] and the present author [1].

Let $\mathfrak{M}$ be an n-dimensional *connected* differentiable Finslerian manifold with fundamental metric function L. For each coordinate neighborhood U of $\mathfrak{M}$, L has the expression $L(u^i, \xi^j)$[2)] which is defined on $u(U)\times R^n$, R^n being n-dimensional real number space and u the coordinate system of U. In the following we shall denote by ∂_i and $\dot{\partial}_i$ operators giving the derivatives of a function with respect to u^i and ξ^i respectively. We put

$$g_{jk}\equiv\dot{\partial}_j\dot{\partial}_k F,\quad C_{jkl}\equiv\frac{1}{2}\dot{\partial}_j\dot{\partial}_k\dot{\partial}_l F,\quad C^i_{jk}\equiv g^{ia}C_{ajk}\ (F\equiv\frac{1}{2}L^2),$$

$$\Gamma^{*i}_{jk}\equiv\left\{ {i \atop jk} \right\}-C^i_{ja}G^a_k-C^i_{ka}G^a_j+C_{jka}G^a_b g^{bi},$$

$$\left\{ {i \atop jk} \right\}\equiv\frac{1}{2}g^{ia}(\partial_j g_{ka}+\partial_k g_{ja}-\partial_a g_{jk}),\quad G^i_j\equiv\frac{1}{2}\dot{\partial}_j(\left\{ {i \atop kl} \right\}\xi^k\xi^l)$$

and

$$R^{*i}_{jkl}\equiv\partial_l\Gamma^{*i}_{jk}-\partial_k\Gamma^{*i}_{jl}-(\dot{\partial}_a\Gamma^{*i}_{jk})\Gamma^{*a}_{bl}\xi^b+(\dot{\partial}_a\Gamma^{*i}_{jl})\Gamma^{*a}_{bk}\xi^b$$
$$+\Gamma^{*a}_{jk}\Gamma^{*i}_{al}-\Gamma^{*a}_{jl}\Gamma^{*i}_{ak},$$

1) See the Bibliography at the end of the paper.
2) Throughout the paper the indices take values $1,\cdots\cdots, n$.

where g^{ij} are elements of the inverse of the matrix $\|g_{jk}\|$[3], and we denote by a semi-colon an operator giving covariant derivatives of scalar and tensor fields with respect to Γ^{*i}_{jk}, for example,

$$T^i_{jk;l} \equiv \partial_l T^i_{jk} - (\dot{\partial}_a T^i_{jk}) \Gamma^{*a}_{bl} \xi^b + T^a_{jk} \Gamma^{*i}_{al} - T^i_{ak} \Gamma^{*a}_{jl} - T^i_{ja} \Gamma^{*a}_{ak}.$$

We notice that, with respect to the coordinate neighborhood U, g_{jk}, C_{jkl}, $\dot{\partial}_l \Gamma^{*i}_{jk}$, R^{*i}_{jkl} and $T^i_{jk;l}$ are respectively components of tensor fields of $\mathfrak{M}$ and Γ^{*i}_{jk} is an expression of a linear connection of $\mathfrak{M}$.

A differentiable homeomorphism on $\mathfrak{M}$ is called a transformation on $\mathfrak{M}$ if its inverse is also differentiable. Given a transformation φ on $\mathfrak{M}$, take any coordinate neighborhoods U and U' such that $U \subset \varphi^{-1}(U')$. Then the φ maps U on $\varphi(U)$ $(\subset U')$ and is expressed in terms of local coordinates as $u'^i = f^i(u^1, \cdots\cdots, u^n)$. If there exists a positive number α determined by φ only and

$$L'(u'^i, \xi'^j) = \alpha L(u^i, \xi^j) \quad (\xi'^i \equiv \xi^a \partial_a f^i)$$

holds, then φ is called to be a homothetic transformation of $\mathfrak{M}$ and α the *associated number* of φ. If $\alpha = 1$, φ is called to be a motion or isometry.

Let G be a homothetic transformation group of $\mathfrak{M}$, that is, a Lie transformation group each element of which is a homothetic transformation of $\mathfrak{M}$. In the following we assume that G is *connected* and *effective*. To each φ of G there corresponds the associated number $\alpha(\varphi)$ and the correspondence α is a homomorphism of G into the multiplicative group of real positive numbers. If we denote by M the kernel of α then *M is a group of motions and*

$$\dim M = \dim G - 1$$

holds. We can also prove that *if G is compact G is necessarily a group of motions.*

We take a point p of $\mathfrak{M}$ and denote by G_p the isotropy group of p. Each φ of G_p induces a linear transformation $\tilde{\varphi}_p$ on the tangent space of p to $\mathfrak{M}$, where $\tilde{\varphi}_p$ is the differential of φ at p. The correspondence $\varphi \rightarrow \tilde{\varphi}_p$ is a linear representation of G_p onto the so-called linear isotropy group $\tilde{G}_p$ and, from the following Lemma 1, this linear representation is faithful. If we denote by M_p the kernel of the contraction of α to G_p and by $\tilde{M}_p$ the image of M_p under the linear representation then we have

$$\dim \tilde{M}_p = \dim G_p \quad \text{or} \quad \dim \tilde{M}_p = \dim G_p - 1$$

according as the identity component of G_p is a group of motions or not.

H. C. Wang [7] proved that, *at any point p of $\mathfrak{M}$, a set of all the matrices $\|a^j_k\|$ satisfying the relation*

$$L(u^i_o, a^j_k \xi^k) = L(u^i_o, \xi^i)$$

for all ξ^i makes an orthogonal group up to a conjugation, (u^i_o) being the coordinates of p.

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3) We assume that the matrix $\|g_{jk}\|$ has rank n.

Therefore the $\tilde{M}_p$ is necessarily a group of orthogonal transformations under a suitable base.

Lemma 1. The linear representation $G_p \to \tilde{G}_p$ defined by $\varphi \to \tilde{\varphi}_p$ is faithful.

Proof. Let φ be an element of G_p such that $\tilde{\varphi}_p$ is the identity element of $\tilde{G}_p$. Taking a cubic coordinate neighborhood V of the point p covered by local coordinates $(u^i)(|u^i|<a)$, since φ leaves invariant the point there exists for the V a suitable small neighborhood $W(\subset V)$ covered by local coordinates $(u^i)(|u^i|<b, b<a)$ such that $\varphi(W)\subset V$. Denoting by $u'^i=f^i(u^1,\cdots,u^n)$ the expression of φ in terms of local coordinates, we have

$$f(0,\cdots,0)=0 \quad \text{and} \quad (\partial_j f^i)_{u=0}=\delta^i_j, \tag{1}$$

$(0,\cdots,0)$ being the coordinate of p. For arbitrary but given values ξ^i, we have

$$\partial_j\partial_k f^i=\underset{\xi}{H}{}^i_{jk}\Big(u,\frac{\partial f}{\partial u}\Big)\equiv\Gamma^{*a}_{jk}(u,\xi)\partial_a f^i-\Gamma^{*i}_{bc}(u',\xi')\partial_j f^b\partial_k f^c \tag{2}$$
$$(\xi'^i\equiv\xi^a\partial_a f^i)$$

which holds on the demain: $|u^i|<b$.

We consider a system of partial differential equations

$$\partial_j\partial_k u'^i=\underset{\xi}{H}{}^i_{jk}\Big(u,\frac{\partial f}{\partial u}\Big) \tag{3}$$

with n unknown functions $u'^1,\cdots,u'^n$. (2) shows that (3) has the solutions $f^i(u^1,\cdots,u^n)$ with the initial conditions (1). On the other hand, functions $u'^i=u^i$ are clearly solutions of (3) which have the same initial conditions. Therefore these solutions must coincide, more precisely, for a suitable positive number $c(<b)$ we have $f^i(u)\equiv u^i(|u^i|<c)$ and consequently $\partial_j f^i=\delta^i_j$. Thus we see that there exists a neighborhood $U(\subset W)$ of p such that, for any point q of U, $\varphi\in G_q$ and $\tilde{\varphi}_q$ is the identity element of $\tilde{G}_q$.

We consider a set N of points q of $\mathfrak{M}$ such that $\varphi\in G_q$ and $\tilde{\varphi}_q$ is the identity element of $\tilde{G}_q$. It follows immediately from the above argument that N is open in $\mathfrak{M}$. On the other hand, we can easily prove that N is closed in $\mathfrak{M}$. From the assumption that $\mathfrak{M}$ is connected we have $N=\mathfrak{M}$. Therefore φ leaves invariant each point of $\mathfrak{M}$ and is the identity element of G_p because G is assumed to be effective.

A tensor field T of $\mathfrak{M}$ of any type, for instance, of type (1, 2) is said to be invariant under a homothetic transformation φ if the relation

$$T'^i_{ab}(u',\xi')\partial_j f^a\partial_k f^b=T^a_{jk}(u,\xi)\partial_a f^i \ (\xi'^i\equiv\xi^a\partial_a f^i)$$

holds with respect to the local coordinates of any coordinate neighborhoods U and U' such that $U\subset\varphi^{-1}(U')$.

Here we notice the following. For a homothetic transformation φ we have easily

$$R'^*(u',\xi')=\alpha(\varphi)^{-2}R^*(u,\xi)\ (\xi'^i\equiv\xi^a\partial_a f^i)$$
$$(R^*\equiv R^{*a}{}_{jka}\,g^{jk}),$$

from which we have a fact obtained by Gy Soós [6]: *if a scalar field of $\mathfrak{M}$ whose local* *expression is R^* is constant and not equal to zero then there exists no homothetic transformation of $\mathfrak{M}$ which is not a motion.*

Lemma 2. Let φ be a homothetic transformation of $\mathfrak{M}$ which is not a motion and leaves invariant a point p of $\mathfrak{M}$ and T a tensor of type (r, s) at the point. If the components $T^{i_1\cdots i_r}{}_{j_1\cdots j_s}(u, \xi)$ of T are homogeneous of order t with respect to the variables ξ^i and $r-s-t\neq 0$ holds, (u_0^i) being the coordinates of p, then the T is a zero-tensor, that is,

$$T^{i_1\cdots i_r}{}_{j_1\cdots j_s}(u_0, \xi)=0$$

for all ξ^j.

Proof. Since φ is a homothetic transformation which is not a motion, the associated number $\alpha(\varphi)$ is not equal to 1. Therefore we assume without loss of generality that $\alpha(\varphi)<1$. For any given positive integer m, the associated number of the homothetic transformation φ^m is $\alpha(\varphi)^m$ and

$$L(u_0^i, \underset{(m)}{\alpha}{}_k^j \xi^k)=L(u_0^i, \xi^j)$$

holds for all ξ^j, where we have put

$$\underset{(m)}{\alpha}{}_k^j\equiv\alpha(\varphi)^{-m}(\partial_k \underset{(m)}{f}{}^i)(u_0)$$

and $\underset{(m)}{f}{}^i(u)$ is the expression of φ^m around p. The T being invariant under φ^m we have for any given values ξ^j

$$\begin{aligned}(4)\qquad &\underset{(m)}{\alpha}{}_{j_1}^{a_1}\cdots\underset{(m)}{\alpha}{}_{j_s}^{a_s} T^{i_1\cdots i_r}{}_{a_1\cdots a_s}(u_0^i, \underset{(m)}{\alpha}{}_k^j\xi^k)\\ &=\alpha(\varphi)^{m(r-s-t)}\underset{(m)}{\alpha}{}_{a_1}^{i_1}\cdots\underset{(m)}{\alpha}{}_{a_r}^{i_r} T^{a_1\cdots a_r}{}_{j_1\cdots j_s}(u_0^i, \xi^j)\end{aligned}$$

Now we can assume without loss of generality that the matrix $\|\underset{(m)}{\alpha}{}_j^i\|$ is orthogonal. Therefore the matrices $\|\underset{(m)}{\alpha}{}_j^i\|(m=1,2,\cdots)$ make a sequence in the full orthogonal group $O(n)$ and there exists a subsequence $\|\underset{ml}{\alpha}{}_j^i\|(l=1,2,\cdots)$ which converges to an orthogonal matrix $\|\beta_j^i\|$ because $O(n)$ is compact.

When l tends to infinity, the left hand side and the second factor of the right hand side of the relations obtained from (4) by replacing m by m_l converge to

$$\beta_{j_1}^{a_1}\cdots\beta_{j_s}^{a_s}T^{i_1\cdots i_r}{}_{a_1\cdots a_s}(u_0^i, \beta_k^j\xi^k)$$

and

$$\beta_{a_1}^{i_1}\cdots\beta_{a_r}^{i_r}T^{a_1\cdots a_r}{}_{j_1\cdots j_s}(u_0^i, \xi^j)$$

respectively. If $r-s-t$ is positive, $\alpha(\varphi)^{m_l(r-s-t)}$ tends to zero and we have

$$T^{i_1\cdots i_r}{}_{a_1\cdots a_s}(u_0^i, \beta_k^j\xi^k)=0,$$

from which

$$T^{i_1\cdots i_r}{}_{j_1\cdots j_s}(u_0^i, \xi^j)=0$$

by virtue of the arbitrariness of ξ^j. If $r-s-t$ is negative, $\alpha(\varphi)^{m_l(r-s-t)}$ tends to infinity and we have

$$T^{i_1\cdots i_r}{}_{j_1\cdots j_s}(u^i_0,\ \xi^j)=0.$$

We notice that tensor fields whose components are respectively $g^{ij}F_{;j}$, $C^i_{jkm;l}$, $\dot{\partial}_l\Gamma^{*i}_{jk}$ and $R^{*i}{}_{jkl}$ are invariant under any homothetic transformation. Hence we have

Theorem 1. If a Finslerian manifold admits a homothetic transformation which is not a motion and fixes a point then the above cited tensor fields of $\mathfrak{M}$ are zero at the point.

Lemma 3. If $n\neq 4$ and

$$dim\ G>\frac{1}{2}n(n-1)+1$$

then G is transitive.

Proof. Take any point p of $\mathfrak{M}$ such that

$$dim\ G_p> dim\ G-n.$$

Then we have

$$dim\ \tilde{M}_p\geqq dim\ G_p-1>dim\ G-n-1\geqq\frac{1}{2}(n-1)(n-2).$$

According to a theorem of D. Montgomery and H. Samelson [4], in an Euclidean space of dimension $n\neq 4$, the full rotation group cannot have a subgroup K such that

$$\frac{1}{2}(n-1)(n-2)<dim\ K<\frac{1}{2}n(n-1).$$

Therefore we have

$$dim\ \tilde{M}_p=\frac{1}{2}n(n-1)$$

because $\tilde{M}_p$ is an orthogonal transformation group under a suitable base, and consequently $dim\ G>dim\ G_p$. From $dim\ G>dim\ G_p$ and

$$dim\ \tilde{M}_p=\frac{1}{2}n(n-1),$$

it follows that G is locally transitive at the point, that is, the orbit of G passing through p contains a suitable small neighborhood of p.

On the other hand, G is clearly locally transitive at any point q of $\mathfrak{M}$ such that

$$dim\ G_q=dim\ G-n.$$

If we take an arbitrary but fixed point p_0 of $\mathfrak{M}$ and denote by N the orbit of G passing through p_0, then N is open and closed in $\mathfrak{M}$. Consequently, from our assumption that $\mathfrak{M}$ is connected, we have $N=\mathfrak{M}$ which proves our lemma.

Theorem 2. Let $\mathfrak{M}$ be a connected Finslerian manifold of dimension n. Then the maximum dimension of effective and connected homothetic transformation groups of $\mathfrak{M}$ is $\frac{1}{2}n(n+1)+1$.

In fact, let G be an effective and connective homothetic transformation group of $\mathfrak{M}$ and assume that

$$dim\, G > \frac{1}{2} n(n+1) + 1.$$

Then we have at a point p of $\mathfrak{M}$

$$dim\, \tilde{M}_p \geqq dim\, G_p - 1 \geqq dim\, G - n - 1 > \frac{1}{2} n(n-1)$$

which is a contradiction. Hence

$$dim\, G \leqq \frac{1}{2} n(n+1) + 1.$$

It follows from the following example that $\frac{1}{2} n(n+1)+1$ is the maximum: a group of all homothetic transformations in an Euclidean space. This example is trivial, but the appositeness of this will be understood from the following Theorem 3.

H. C. Wang [7] showed that in a connected Finslerian manifold of dimension $n \neq 4$ there exists no effective and connected group H of motions such that

$$\frac{1}{2} n(n-1) + 1 < dim\, H < \frac{1}{2} n(n+1).$$

By using this fact we have easily

Theorem 3. Let $\mathfrak{M}$ be a connected Finslerian manifold of dimension $n \neq 4$. Then $\mathfrak{M}$ admits no effective and connected homothetic transformation group G which is not a group of motions, such that

$$\frac{1}{2} n(n-1) + 2 < dim\, G < \frac{1}{2} n(n+1) + 1.$$

Theorem 4. Let $\mathfrak{M}$ be a connected Finslerian manifold of dimension n and G an effective and connected homothetic transformation group of $\mathfrak{M}$. (a) if

$$dim\, G = \frac{1}{2} n(n+1) + 1$$

then $\mathfrak{M}$ is locally Euclidean and (b) if $n \neq 4$ and

$$dim\, G = \frac{1}{2} n(n-1) + 2$$

then $\mathfrak{M}$ is Minkowskian.

Proof. First we prove (a). The group G is not a group of motions. In fact, if it is true, we have at a point p of $\mathfrak{M}$

$$dim\, \tilde{M}_p = dim\, G_p \geqq dim\, G - n = \frac{1}{2} n(n-1) + 1$$

which is a contradiction. Therefore G contains a subgroup of motions whose dimension is equal to $\frac{1}{2} n(n+1)$. From a theorem of H. C. Wang [3], [7], $\mathfrak{M}$ is necessarily Riemannian. It is clear that G_p of each point p of $\mathfrak{M}$ is not a group of motions. Therefore, from Theorem 1, the tensor field of $\mathfrak{M}$ whose components are $R^{*i}{}_{jkl}$ is zero on $\mathfrak{M}$ and hence $\mathfrak{M}$ is locally Euclidean.

Next we prove (b). At each point p of $\mathfrak{M}$, G_p is not a group of motions. In fact, if it is true, we have

$$dim\ \tilde{M}_p = dim\ G_p \geqq dim\ G - n = \frac{1}{2}(n-1)(n-2)+1,$$

from which

$$dim\ \tilde{M}_p = \frac{1}{2}n(n-1)$$

because $n \neq 4$. Since G is transitive from Lemma 3, we have

$$dim\ G = \frac{1}{2}n(n+1)$$

which is a contradiction. By using Theorem 1, the tensor fields whose components are respectively $g^{jk}F_{;k}$, $\dot{\partial}_j \Gamma^{*i}_{kl}$ and $R^{*i}{}_{jkl}$ are zero on $\mathfrak{M}$. From the fact the second and third tensor fields are zero on $\mathfrak{M}$ it follows that each point of $\mathfrak{M}$ has a suitable coordinate neighborhood U of the point on which the connection parameters Γ^{*i}_{jk} are zero. Therefore, from $g^{jk}F_{;k}=0$, we have $L_{;k}=\partial_k L=0$ on U and consequently L do not contain the variables u^i. This proves that $\mathfrak{M}$ is Minkowskian.

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