ON SIMILARITIES AND TRANSITIVE ABELIAN GROUPS OF MOTIONS IN FINSLER SPACES

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Introduction. In the paper we deal with a complete Finsler space of class C^1 which admits a non-isometric similarity or a transitive abelian group of motions. We showed in [1][1] that, if a complete Finsler space of class C^1 admits a non-isometric and global similarity, the space is Minkowskian or isometric to Minkowskian according as its indicatrices, which are unit spheres of tangent Minkowskian spaces, are convex or not necessarily convex. But we did not deal with the case where a Finsler space admits a non-isometric and local similarity. Hence we discuss first the properties of such a Finsler space and second the properties of a Finsler space which admits a transitive abelian group of motions.

H. Busemann showed in [3] that, if a metric space called a G-space admits a transitive abelian group of motions, the space is locally Minkowskian and homeomorphic to the topological product of the finite number of real lines and circles. Under a suitable assumption of differentiability the above also holds for a complete Finsler space. But, if the space is of class C^1, the above is not obvious. Hence we show the above under a reasonable assumption.

§ 1. Let R be an n-dimensional complete Finsler space of class C^1 with integrand $F(x, \dot{x})$ $(n\geqq 2)$. In a coordinate neighborhood U a point with coordinates $(x^1,\cdots\cdots,x^n)$ will be denoted by x and a contravariant vector with components $(\xi^1,\cdots\cdots,\xi^n)$ by ξ. Suppose that the function $F(x, \xi)$ is continuous in the variables x^i's and ξ^i's and satisfies the following conditions:

i) $F(x,\xi)>0$ *for any vector* $\xi\neq 0$.
ii) $F(x,c\xi)=cF(x,\xi)$ *for any positive number* c.

Suppose further that for any two points there exists an arc which connects these points. Then, by integral method, the length $l_F(C)$ of a curve C of class D' from a point p to a point q is defined, and the distance $\rho(p, q)$ is defined by the greatest lower bound of the lengths of those curves from p to q. By use of the distance ρ we can again define the length $l_\rho(E)$ of a continuous curve E. Obviously $l_\rho(C)\leqq l_F(C)$. The equal sign always holds when and only when the indicatrices are convex.

Let Ψ be a transformation of R on itself such that, if $p\Psi=p'$ and $q\Psi=q'$,

$$\rho(p', q')=k\rho(p, q),$$

where k is a positive constant not equal to 1. Then Ψ is said a non-isometric similarity or simply a global similarity of the space R on itself. If for a point p a positive

1) Numbers in brackets refer to the references cited at the end of the paper.

number δ_p exists such that

$$\rho(x\Psi, y\Psi) = k\rho(x, y) \ (k>0, \neq 1)$$

holds for any two points x and y in the sphere neighborhood $S(p, \delta_p)$ $(=\{x \mid \rho(p, x) < \delta_p\}, \delta_p>0)$, Ψ is said a local similarity of the space R on itself.

If the space R admits a global similarity Ψ on itself then the space is isometric to a Minkowskian space. If further the indicatrices: $F(x, \xi)=1$ are convex, the space is Minkowskian [1].

§ 2. Let $\tilde{R}$ be the universal covering space of R and Ω a covering transformation of $\tilde{R}$ onto R. Then we have

(2.1) *If the space R admits a local similarity Ψ on itself, then the space $\tilde{R}$ admits a global similarity $\tilde{\Psi}$ on itself such that $\tilde{\Psi}\Omega = \Omega\Psi$.*

Proof. Let p be a point of R and put $p\Psi = p'$. Suppose further that the universal covering space $\tilde{R}$ was constructed by choosing the point p as origin. Hence $\tilde{p}=p$ and we can put $\tilde{p}\Omega = p$.

Let x be a point of R and put $x\Psi = x'$, and further let C be an arc from p to x. Then $C\Psi$ is a continuous curve from p' to x'. We denote this curve by C'. Let D be an arc from p to p', $\tilde{D}$ the arc which lies over D and issues from $\tilde{p}$ and $\tilde{C}'$ the continuous curve which issues from the end point $\tilde{p}'$ of $\tilde{D}$ and lies over C'. Then the end-point $\tilde{x}'$ of $\tilde{C}'$ lies over x'. Let $\tilde{C}$ be an arc which lies over C and issues from $\tilde{p}$ and $\tilde{x}$ its end-point. The point $\tilde{x}$ lies over x.

By virtue of the assumption the sphere neighborhood $S(x, \delta_x)$ is mapped onto $S(x', \delta'_x)$ $(\delta'_x = k\delta_x)$ under the local similarity Ψ. On the other hand there exists a positive number ρ_x not greater than δ_x such that $S(\tilde{x}, \rho_x)$ and $S(\tilde{x}', \rho'_x)$ $(\rho'_x = k\rho_x)$ are under Ω isometrically mapped onto $S(x, \rho_x)$ and $S(x', \rho'_x)$ respectively. From this we see that there exists a similarity $\tilde{\Psi}$ of $S(\tilde{x}, \rho_x)$ onto $S(\tilde{x}', \rho'_x)$ such that

$$\rho(\tilde{y}\tilde{\Psi}, \tilde{z}\tilde{\Psi}) = k\rho(\tilde{y}, \tilde{z}) \text{ for two points } \tilde{y} \text{ and } \tilde{z} \text{ in } S(\tilde{x}, \rho_x)$$

and further

$$\rho(\tilde{y}\tilde{\Psi}\Omega, \tilde{z}\tilde{\Psi}\Omega) = \rho(\tilde{y}\Omega\Psi, z\Omega\Psi) = k\rho(y, z),$$

where $\tilde{y}\Omega = y$ and $\tilde{z}\Omega = z$. It is easy to see that the similarity $\tilde{\Psi}$ can be extended to the whole space $\tilde{R}$. Obviously $\tilde{\Psi}$ is a local similarity of $\tilde{R}$ on itself. Hence the proposition is proved by showing that $\tilde{\Psi}$ is global. To do this we prove the following

(2.2) *If the Finsler space R is simply connected, then the local similarity Ψ is global.*

Proof. Let G be a shortest connection from a point p to a point q and put

$$p' = p\Psi, \quad q' = q\Psi \text{ and } G\Psi = G'.$$

Then G' is a continuous curve from p' to q' and we see from the definition of local similarity

$$kl_\rho(G)=k\rho(p, q)=l_\rho(G').$$

If $l_\rho(G')=\rho(p', q')$, the proposition is clear. Hence we assume that there exists a shortest connection G'' from p' to q' such that

$$\rho(p', q')=l_\rho(G'')<l_\rho(G').$$

Let $U\times V$ be the topological product of the segments U: $0\leqq u\leqq 1$ and V: $0\leqq v\leqq 1$. Since R is simply connected, there exists a continuous mapping of $U\times V$ into R such that $f(0,0)=p'$, $f(1,1)=q'$ and $f(u, 0)$, $0\leqq u\leqq 1$, and $f(u, 1)$, $0\leqq u\leqq 1$, coincide with G' and G'' respectively.

Let $g_0(u)$, $0\leqq u\leqq 1$, be the parametrization of G such that

$$g_0(u)\Psi=f(u, 0) \text{ for } 0\leqq u\leqq 1.$$

To simplify the notation we put $\rho_{g_0(u)}=\rho_0(u)$. Then there exists a subdivision of the interval $[0, 1]$: $0=u_0<u_1<\cdots\cdots<u_m=1$ such that each $S(g_0(u_i), \rho_0(u_i))$ is mapped onto $S(f(u_i, 0), \rho'_0(u_i))$ $(\rho_0(u_i)k=\rho'_0(u_i))$ under Ψ and

$$S(g_0(u_i), \rho_0(u_i)) \cap S(g_0(u_{i+1}), \rho_0(u_{i+1}))\neq\phi$$
$$(i=0, 1,\cdots\cdots, m-1).$$

From this there exists a positive number v_1 such that

$$f(u, v_1)\subset S(f(u_i, 0), \rho'_0(u_i))$$

for $u_i\leqq u\leqq u_{i+1}$ $(i=0, 1,\cdots\cdots, m-1)$ and for $u_{i-1}\leqq u\leqq u_i (i=1,\cdots\cdots, m)$. By considering the contraction of Ψ to each $S(g_0(u_i), \rho_0(u_i))$ we have a continuous curve $g_1(u)$, $0\leqq u\leqq 1$, from p to q such that

$$g_1(u)\Psi=f(u, v_1),\ 0\leqq u\leqq 1, \text{ and}$$
$$g_1(u)\subset S(g_0(u_i), \rho_0(u_i))$$

for $u_i\leqq u\leqq u_{i+1} (i=0, 1,\cdots\cdots, m-1)$ and for $u_{i-1}\leqq u\leqq u_i (i=1,\cdots\cdots, m)$. Under the same consideration we have from $g_1(u)$, $0\leqq u\leqq 1$, and $f(u, v_1)$, $0\leqq u\leqq 1$, a continuous curve $g_2(u)$, $0\leqq u\leqq 1$, such that

$$g_2(u)\Psi=f(u, v_2) \text{ for } 0\leqq u\leqq 1,$$

where $0<v_1<v_2\leqq 1$. By repeating this process we have after finite steps a continuous curve $g_k(u)$, $0\leqq u\leqq 1$, from p to q such that

$$g_k(u)\Psi=f(u, 1) \text{ for } 0\leqq u\leqq 1.$$

We denote by H this continuous curve. It is easy to see from the above that the length of the curve H equals $l_\rho(G'')/k$. Thus we have

$$kl_\rho(H)=l_\rho(G'')<l_\rho(G')=kl_\rho(G).$$

We therefore have $l_\rho(H)<l_\rho(G)$. But this is a contradiction. From this it follows that Ψ is a global similarity.

If the space R admits a local similarity Ψ on itself, the universal covering space $\tilde{R}$ admits a global similarity $\tilde{\Psi}$ such that $\tilde{\Psi}\Omega=\Omega\tilde{\Psi}$. The space $\tilde{R}$ is Minkowskian [1].

If R is compact, then R does not admit a local similarity with dilation factor less than 1. If the dilation factor k is less than 1, the space is simply connected. Hence Ψ is global. Since there exists on $\tilde{R}$ only one fixed point under $\tilde{\Psi}$ there exists also on R only one fixed point. Thus we have from [1] the following

(2.3) **Theorem.** *Let R be a complete Finsler space of class C^1 with convex indicatrices. If R admits a local similarity Ψ, the space is Minkowskian or locally Minkowskian according as its dilation factor k is less than 1 or greater than 1. Then the space R has only one fixed point p under Ψ and is isometric or locally isometric to the tangent Minkowskian space at the point p according as the above two cases.*

The arguments apply to a Riemannian space which admits a local similarity, i. e., we have the following

(2.4) **Theorem.** *Let R be a complete Riemannian space of class C^1. If R admits a local similarity, then the space is Euclidean or locally Euclidean according as its dilation factor is less than 1 or greater than 1.*

At the end we consider the case where the indicatrices of the space R are not necessarily convex. Let us denote by γ_x the indicatrix: $F(x, \xi)=1$ at a point x. Let $\bar{\gamma}_x$ be the convex closure of γ_x. Then $\bar{\gamma}_x$ is represented as $\bar{F}(x, \xi)=1$ by choosing a continuous function in the variables $x^{i\prime}s$ and $\xi^{i\prime}s$ such that $\bar{F}(x, \xi)$ is positive for $\xi\neq 0$ and positively homogeneous of first order in the variables $\xi^{i\prime}s$. Let $\bar{R}$ be the Finsler space with the indicatrices $\bar{\gamma}_x$ instead of γ_x. Then the space $\bar{R}$ is isometric to R [2]. Hence, if R admits a local similarity Ψ, the space $\bar{R}$ also admits the local similarity Ψ. From this we have the following

(2.5) **Theorem.** *Let R be a complete Finsler space of class C^1 with not necessarily convex indicatrices. If R admits a local similarity Ψ on itself, the space is Minkowskian or locally Minkowskian according as its dilation factor is less than 1 or greater than 1. Then the space has only one fixed point p under Ψ and is isometric or locally isometric to the tangent Minkowskian space T_p at the point p according as the above two cases.*

§ 3. In this paragraph we deal with a Finsler space R of class C^1 which admits a transitive abelian group of motions Γ. For two points x and y there exists a motion $\Psi(\in\Gamma)$ such that $y=x\Psi$. If $\Phi\in\Gamma$, we have then

$$\rho(x, x\Phi)=\rho(x\Psi, x\Phi\Psi)=\rho(x\Psi, x\Psi\Phi)=\rho(y, y\Phi).$$

It follows from this that every element of Γ has no fixed point unless it is identity E. Hence Γ is simply transitive.

We topologize the group Γ by defining the distance of two elements Ψ and Φ such that $\sigma(\Psi, \Phi)=\rho(x\Psi, x\Phi)$ $(x\in R)$. Let $\{\Phi_n\}$ be a sequence of elements of Γ. If for a positive number ε a positive integer N exists such that $\sigma(\Phi_m, \Phi_n)<\varepsilon$ for $m, n\geqq N$, there exists a motion Φ of Γ such that $lim_{n\to\infty}\sigma(\Phi_n, \Phi)=0$. Hence the group Γ is closed. We first prove the following

(3.1) **Theorem.** *Let R be a complete Finsler space of class C^1 with not necessarily* *convex indicatrices. If R admits a transitive abelian group of motions Γ and $(x, x\Phi, x\Phi^2)$*[2] *for a point x and for a motion $\Phi(\neq E)$ of Γ, then the space is isometric to Minkowskian.*

Proof. Let p be a point of R and U a coordinate neighborhood of p. Let δ be a positive number such that $\overline{S(p, \delta)} \subset U$. Further let $\mathfrak{l}$ be a Euclidean half straight line issuing from p and x the first point on $\mathfrak{l}$ such that $\rho(p, x)=\delta$. The totality of such points x coincides with the boundary $K(p, \delta)$ of $S(p, \delta)$. Thus we see that $K(p, \delta)$ is homeomorphic to an $(n-1)$-dimensional Euclidean sphere.

As we said at the begining of this paragraph, the group Γ is simply transitive. Hence for a point x of $K(p, \delta/2)$ there exists a motion Φ of Γ such that $p\Phi=x$. The point $x\Phi$ clearly lies on $K(p, \delta)$, since $(p, p\Phi, p\Phi^2)$. The totality $\{x\Phi\}$ of such points $x\Phi$ is homeomorphic to $K(p, \delta/2)$ and therefore coincides with $K(p, \delta)$, We put $x\Phi=x_0$, $x=x_1$. Under the same consideration there exists a point x_2 of $K(p, \delta/2^2)$ and a motion Ψ of Γ such that

$$p\Psi=x_2,\ p\Psi^2=x_2\Psi=x_1 \text{ and } \Psi^2=\Phi.$$

If we put $x_1\Psi=x_3$, then we have (p, x_2, x_1), (x_2, x_1, x_3), (x_1, x_3, x_0) and (p, x_1, x_0). We denote by L_1 the Euclidean polygon $\{p, x_1, x_0\}$ and by L_2 the Euclidean polygon $\{p, x_2, x_1, x_3, x_0\}$. Repeating this process, we have a sequence of Euclidean polygons $\{L_n\}$. The sequence $\{L_n\}$ clearly converges to a shortest connection G from p to x_0 which is called a geodesic arc [3].

Obvoiusly the geodesic arc G is reversible and its prolongation is locally possible and unique under motions of Γ. We denote by T the whole prolongation. T is called a geodesic and denote by $\mathfrak{F}_p$ the totality of such geodesics. Two geodesics of $\mathfrak{F}_p$ have no common points except the point p. Next we prove this.

Suppose that there exist two geodesics T_1 and T_2 of $\mathfrak{F}_p$ which intersect at a point q different from p. Let G_1 and G_2 be the subarcs of T_1 and T_2 from p to q respectively and suppose further that q is the first common point of T_1 and T_2 from p. Then $l_\rho(G_1)=l_\rho(G_2)$ and there exist a positive integer $m(\geqq 2)$ and the two points p_1, p_2 on G_1, G_2 such that $l_\rho(G_1)/2^m=s$ and $\rho(p, p_1)=\rho(p, p_2)=s$. By the assumption the points p_1 and p_2 do not coincide. If we put $p\Psi=p_1$ and $p\Phi=p_2$, then it is easy to see

$$p\Psi^{2m}=p\Phi^{2m}=q,\ (p, p\Psi^{2m-1}, p\Psi^{2m}) \text{ and } (p, p\Phi^{2m-1}, p\Phi^{2m}).$$

We have from this

$$\Psi^{2m}\Phi^{-2m}=E.$$

On the other hand, since $\Psi\Phi^{-1}\neq E$, we also have $\Psi^{2m}\Phi^{-2m}\neq E$, which contradicts the above.

Let $\{a_n\}$ and $\{b_n\}$ be sequences of points which converges to a point a and a point b respectively and for each n G_n the geodesic of $\mathfrak{F}_{a_n}$ through the point b_n. Then the closed limit of the sequence of geodesics $\{G_n\}$ coincides with the geodesic of $\mathfrak{F}_a$ through the point b. The set of all points on the geodesics of $\mathfrak{F}_p$ forms an open and

2) If for three points x, y and z $\rho(x, y)+\rho(y, z)=\rho(x, z)$, then we denote this by (x, y, z).

closed set and therefore coincides with the space R.

From the above we see that the geodesics of $\mathfrak{F}_p$ simply covers the whole space R except p. Let $\mathfrak{F}$ be the totality of such systems $\mathfrak{F}_x(x \in R)$. If $p\Omega = x$, $\Omega \in \Gamma$, then we have $\mathfrak{F}_p\Omega = \mathfrak{F}_x$. To prove the theorem we show the following

(3.2) *Let a, b and c be three points which do not lie on a geodesic of $\mathfrak{F}$ and b' and c' the points on the geodesic arcs which connect a, b and a, c respectively and such that $2\rho(a, b') = \rho(a, b)$ and $2\rho(a, c') = \rho(a, c)$. Then $\rho(b, c) = 2\rho(b', c')$.*

If in the proposition $2\rho(a, b') = \rho(a, b)$ and $2\rho(a, c') = \rho(a, c)$, then $2\rho(b, b') =$ $\rho(b, a)$ and $2\rho(c, c') = \rho(c, a)$. This is clear from the above proof. If further the proposition is proved, it is also clear that $2\rho(c', b') = \rho(c, b)$. Next we prove the pro- position.

Proof. We put $a\Psi = b'$, $a\Phi = b'$ and $b'\Omega = c'$. Then the following is clear:

$$\Psi\Omega = \Phi, \ a\Psi^2 = b \text{ and } a\Phi^2 = c.$$

If we further put $d = a\Psi\Phi\Psi$, then

$$b\Omega = a\Psi^2\Omega = a\Psi\Phi\Psi = d \text{ and}$$
$$b\Omega^2 = a\Psi^2\Omega^2 = a(\Phi\Omega^{-1})^2\Omega^2 = a\Phi^2 = c.$$

Hence the point d lies on the geodesic arc of $\mathfrak{F}$ connecting the points b and c and is the mid-point of these points. It is easy to see

$$c'\Psi = a\Phi\Psi = a\Psi\Omega\Psi = d \text{ and } b'\Psi = b.$$

Hence $\rho(b', c') = \rho(b'\Psi, c'\Psi) = \rho(b, d) = \rho(b, c)/2$, from which the proposition follows.

Let x be a point of R and Ψ the motion of Γ such that $p\Psi = x$. We put $x\Psi =$ x' and denote by $x' = x\Theta$ this correspondence: $x \to x'$. In such a way we define a transformation Θ of R on itself. Obviously Θ is a non-isometric similarity with dilation factor 2. The space admits the similarity Θ and the point p is fixed under Θ. Hence the space is isometric to Minkowskian.

Remarks. If in a Minkowskian space the spheres are convex, all straight lines are geodesics. Even if the spheres are not convex, all straight lines are geodesics in the sense of l_ρ-length but not in the sense of l_F-length. It is to be noticed that in the theorem the assumption of the convexity of indicatrices is not necessary. If in the theorem the indicatrices are convex, the space is clearly a Minkowskian space.

Let Ψ be a motion of R on itself and a point x be carried into a point y under Ψ. Let U and V be coordinate neighborhoods of x and y and $(x^1, \cdots\cdots, x^n)$ and $(y^1,$ $\cdots\cdots, y^n)$ their coordinates respectively. Then the relation: $y = x\Psi$ can be represented in the following form:

$$y^i = f^i(x^1, \cdots\cdots, x^n)\ (i = 1, \cdots\cdots, n).$$

Obviously the functions $f^i(x^1, \cdots\cdots, x^n)$ are continuous in the variables $x^{i\prime}s$. If each $f^i(x^1, \cdots\cdots, x^n)$ has continuous derivatives of first order with respect to the variables $x^{i\prime}s$. The motion Ψ is said to be of class C^1.

(3.3) **Theorem.** *Let R be a complete Finsler space of class C^1 with convex indicatrices. If R admits a transitive abelian group of motions Γ such that each of Γ is of class C^1, the space is locally Minkowskian and homeomorphic to the topological product of the finite number of real lines and circles.*

If in the above theorem the space is of 2-dimensions, then the space is either a plane, a cylinder or a torus with Minkowskian metric. Next we prove the theorem.

Proof. As we said at the begining of this paragraph, the group Γ is simply transitive. Let p be a point of R, $\mathfrak{l}$ the half straight line issuing from p and $\lambda\,(=(\lambda^1, \cdots\cdots, \lambda^n))$ is the Euclidean unit vector with the direction of $\mathfrak{l}$. If a point q on $\mathfrak{l}$ tends to p, we have by virtue of the convexity of indicatrices

$$lim_{q\to p}\frac{\rho(p,q)}{F(p,\lambda)\,e(p,q)}=1.$$

Let x be a point of R and $p\Psi=x$, $\Psi\in\Gamma$. Further let U_x be a coordinate neighborhood of x and y a point of U_x such that the Euclidean segment $E(x,y)$ is contained in U_x. Then the motion Ψ carries the Euclidean segment $E(p,q)$, whose length equals 1 and direction is λ, into an arc L of class C^1. Let $\{r_n\}$ be a sequence of points of L which converges to the point x and λ_n the Euclidean unit vector of each $E(x,r_n)$. Then the sequence of Euclidean unit vectors $\{\lambda_n\}$ converges to a vector μ_x, and we have by putting $q_n=r_n\Psi^{-1}(n=1,2,\cdots\cdots)$

$$lim_{n\to\infty}\frac{F'(x,\lambda_n)\,e(x,r_n)}{F(p,\lambda)\,e(p,q_n)}=1, \tag{3.4}$$

where $e(x,y)$ is the Euclidean distance between two points x and y and $F'(x,\dot{x})$ denotes the integrand in the coordinate neighborhood U_x. From this it follows that the correspondence: $\lambda\to\mu_x$ is one-to-one and bicontinuous and the tangent Minkowskian spaces T_p and T_x are isometric in the above directions λ and μ_x.

Let z be a point of $E(x,y)$ and μ_z the Eulcidean unit vector at z which corresponds to λ as in the above. We show that μ_x and μ_y are parallel.

Let $E(x,x')$ be the Euclidean segment with length 1. There exist motions Φ and Σ of Γ such that $x'=x\Phi, y=x\Sigma$; $\Phi, \Sigma\in\Gamma$. Then the images of $E(x,x')$ under Σ and of $E(x,y)$ under Φ are arcs with the common end-point, since the group Γ is abelian. Let x'' be the point of $E(x,x')$ and Φ' the motion of Γ such that $x''=x\Phi'$. Then the images of $E(x,y)$ under Φ' and of $E(x,x'')$ under Σ have also the common end-point. From this we see that there exists a 2-dimensional surface S of class C^1 represented as $f(u,v)$, $0\leqq u\leqq 1$, $0\leqq v\leqq 1$, which satisfies the following conditions:

$$E(x,x'):\ f(u,0),\ 0\leqq u\leqq 1,$$
$$E(x,y):\ f(0,v),\ 0\leqq v\leqq 1,$$

for fixed u $f(u,v)$, $0\leqq v\leqq 1$, is the image of $E(x,y)$ under the motion Φ'' of Γ such that $x\Phi''=f(u,0)$ and for a fixed v $f(u,v)$, $0\leqq u\leqq 1$, the image of $E(x,x')$ under the motion Σ' such that $x\Sigma'=f(0,v)$. Then it is easy to see that

$$(3.5) \qquad \rho(f(0,v),\ f(u,v)) = \rho(f(0,v'),\ f(u,v')),$$
$$0 \leqq u \leqq 1,\ 0 \leqq v \leqq 1.$$

Let $\lambda_{u,v}$ be the Euclidean unit vector with the same direction as a Euclidean segment $E(f(0,v),\ f(u,v))$. We have then

$$lim_{u\to 0} \frac{\rho(f(0,v),\ f(u,v))}{F'(f(0,v),\ \lambda_{u,v})\, e(f(0,v),\ f(u,v))} = 1.$$

The group Γ is abelian and $lim_{u\to 0}\ \lambda_{u,v} = \mu_v\ (=\mu_{f(0,v)})$. Hence by (3.5) we get

$$(3.6) \qquad lim_{u\to 0} \frac{F'(f(0,v),\ \lambda_{u,v})\, e(f(0,v),\ f(u,v))}{F'(f(0,v'),\ \lambda_{u,v'})\, e(f(0,v'),\ f(u,v'))} = 1$$

for any v and v'. The tangent Minkowskian spaces at points $f(0,v)$ and $f(0,v')$ are isometric in two directions μ_v and $\mu_{v'}$. Hence we have

$$F'(f(0,v),\ \mu_v) = F'(f(0,v'),\ \mu_{v'}).$$

Therefore we get by (3.6)

$$lim_{u\to 0} \frac{e(f(0,v),\ f(u,v))}{e(f(0,v'),\ f(u,v'))} = 1.$$

We denote by L_u the arc $f(u,v)$, $0 \leqq v \leqq 1$, and by M_v the arc $f(u,v)$, $0 \leqq u \leqq 1$. Then L_0 and M_0 are identical with $E(x,y)$ and $E(x,x')$ respectively. Let α is a positive number and u, v and v' positive numbers such that $0 \leqq u \leqq 1$, $0 < v < v' \leqq 1$ and $\alpha = \rho(f(0,v),\ f(0,v'))/\rho(f(0,v),\ f(u,v))$. If under these conditions v' tends to v and u tends to zero, the Euclidean quadrangle $f(0,v) f(u,v) f(u,v') f(0,v')$ tends to the point $f(0,v)$ so as to be similar to a parallelogram.

Let P be the 2-plane determined by the segments L_0 and M_0. Then P is the tangent plane of S at $f(0,0)$. The Euclidean unit vectors μ_v $(0 \leqq v \leqq 1)$ lie on P and $l_\rho(L_0) = l_\rho(L_u)$ for each u. Hence a consecutive arc L_{0+du} of L_0 lies on P and its length equals that of L_0. From this it follows that the arc L_{0+du} is a Euclidean segment parallel to L_0 and hence the Euclidean unit vectors $\mu_0 (=\mu_x)$ and $\mu_1 (=\mu_y)$ are parallel. Thus we see that in U_x the function $F(x,\xi)$ does not depend on the variables x^i's and the space is locally Minkowskian. Since the group Γ is locally compact, connected, locally connected and commutative, the last part of the theorem is clear [4]. Thus the theorem is proved.

It is also clear that, if in the theorem the indicatrices are not necessarily convex, the space is locally isometric to Minkowskian. This follows directly from the same consideration as in § 2.

References

[1]. Y. Nasu: On similarities in a Finsler space, Tensor, Vol. 9 (1959), 175–189.
[2]. H. Busemann and W. Mayer: On the foundations of culculus of variations, Amer. Math. Soc., Vol. 49 (1941), 173–198.
[3]. H. Busemann: Geometry of geodesics, Acad. Press INC., (1955).
[4]. L. Pontrjagin: Topological groups (translated from the Russian by E. Lehmer, 1939).