THE POINT ESTIMATION OF THE VARIANCE COMPONENTS IN RANDOM EFFECT MODEL

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1. Introduction.

In this paper we shall be concerned with the estimation of the variance components of the r-way layout of random effect model. Concerning the theory of estimation in the design of experiments the author should like at first to mention the work of R. C. Bose [2][1], where the estimation problem was fully discussed under the general linear model, which is applicable to the estimation of the treatment effects under the fixed effect model. In his work the normality of the distribution was not assumed and the arguments were solely based on the Markov Theorem.

On the other hand concerning the estimate of the variance of the error term and that of the variance components in both the fixed effect and random effect model, there have not been known so far except for the unbiasedness.

Neverthless in view of the developments of the theory of estimation as a part of the current statistical inference theory (Lehmann-Scheffé [6], Lehmann [5]), it has been felt to be needed to develop the theory in the model of design of experiments from the standpoint of the current statistical inference theory. In this connection we mention the work of Y. Washio [7], where he proved that the ordinary estimates of the parameters in the fixed effect model are the best unbiased estimates in the sense that the estimates are of uniformly minimum variance as are based on the complete sufficient statistics. Thus the problem concerning the r-way layout of the fixed effect model has been solved, and also he treated the same problem concerning the random effect model. His result, however, is restricted to the 1-way layout only.

The purpose of this paper is to treat the problem of estimation in the r-way layout of random effect model. The main difficulty in this problem lies in deriving the joint density function, and for this purpose we have to prepare with some complicated notation system in handling the variance matrix, its determinant and inverse (Theorem 4.1 and 4.2). After such cumbersome calculations, we shall come to the derivation of the joint density function (Theorem 4.3), and then we shall observe that, as is pointed out by Washio, the sufficient statistics of the family of the distribution in our concern can not be proved to be complete by the usual method appealing to the unicity of the Laplace transform. Therefore we shall prove, instead of following the line of Washio, that the estimates of the variance components ordinary used in the practice of statistical analysis are the minimum variance estimates in the sense of Bhattacharyya (Theorem 4.4). In proving it we shall appeal to the result due to Bhattacharyya [1], which enables us to prove it without verifying the lower bound of

1) Numbers in brackets refer to the references of the end of the paper.

Cramér-Rao [3] or its generalization due to Bhattacharyya is attained by the variance of the estimate.

As the arguments and the notation system are very much complicated we shall treat the special case of the 2-way layout (Section 3) as a preparatory exposition to the general case.

After treating the random effect model, there naturally arises the corresponding problem for the case of the mixed effect model, which the author wishes to discuss on another occasion.

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2. Preliminaries.

In this paper we shall be concerned with the r-way layout of random effect model whose model equation is given by the following

$$x_{t_0t_1\cdots t_r}=\mu+\sum_{k=1}^{r}\sum_{T_{ik}\subset T_r} a_{t_{i1}\cdots t_{ik}}+e_{t_0t_1\cdots t_r}\quad (t_{ij}=1,2,\cdots\cdots,n_{ij}, j=1,\cdots,k) \tag{2.1}$$

where μ denotes the general mean, $a_{t_{i1}\cdots t_{ik}}$ denotes the interraction between i_1-*th*, i_2-*th*, $\cdots\cdots$, i_k-*th* factors with the level t_{i_1}, t_{i_2}, $\cdots\cdots$, t_{i_k}, and $e_{t_0t_1\cdots t_r}$ denotes the error term. In the above equation T_r denotes the set of suffixes, $t_1, t_2, \cdots\cdots, t_r$ and T_{ik} denotes a subset $(t_{i_1}, t_{i_2}, \cdots\cdots, t_{i_k})$ of $T_r=(t_1, t_2, \cdots\cdots, t_r)$, with the relation $i_1<i_2<\cdots\cdots<i_k$. $\sum_{T_{ik}\subset T_r}$ denotes the summation for all subsets $T_{i_k}=(t_{i_1},\cdots\cdots,t_{i_k})$ of size k in $T_r=(t_1,\cdots\cdots,t_r)$, or in other words for all subsets of integers $(i_1,\cdots\cdots,i_k)$ in $(1,2,\cdots\cdots,r)$.

We assume that μ is a constant, all $a_{t_{i1}\cdots t_{ik}}$ and $e_{t_0t_1\cdots t_r}$ are distributed independently to each other as normal with mean all equal to 0 and variance of $a_{t_{i1}\cdots t_{ik}}$ equal to $\sigma_{i_1\cdots i_k}$, the variance of $e_{t_0t_1\cdots t_r}$ all equal to σ_0.

The Kronecker product of two or any number of matrixes are defined in this paper in the way reverse to the usual ones for the convenience in handling the cumbersome notation systems, which will become clear in the course of the developments of the arguments in this paper.

Let $A=(a_{ij})$, $B=(b_{ij})$, the Kronecker product denoted by $A\otimes B$ is defined as the matrix with the (i,j)-*th* submatrix Ab_{ij} instead of $a_{ij}B$, in the usual manner. The Kronecker product of any number of matrixes is defined as the natural generalization of two matrixes, we shall write the Kronecker product of n matrixes A_1, A_2, $\cdots\cdots$, A_n, as $\prod_{i=1}^{n}\otimes A_i$.

In this paper we shall make use of the well-known relations concerning the Kronecker products of two matrixes such as $(A\otimes B)(C\otimes D)=AC\otimes BD$, $(A\otimes B)^{-1}=A^{-1}\otimes B^{-1}$, $(A\otimes B)'=A'\otimes B'$, and their generalizations to the products of any number of matrixes without mentioning explicitly. Throughout this paper we shall write $n\times n$ unit matrix as I_n, E_n denotes the $n\times n$ matrix with the elements all equal to 1. Let H_n be the $n\times n$ matrix with the elements all equal to zero except for the element of the first row in the first column equal to 1, and let $K_n=I_n-H_n$; namely,

$$(2.2)\qquad I_n=\begin{pmatrix}1 & & & \\ & 1 & & 0\\ & & \ddots & \\ 0 & & & 1\end{pmatrix},\quad E_n=\begin{pmatrix}1 & 1 & \cdots & 1\\ \vdots & & & \vdots\\ 1 & \cdots & \cdots & 1\end{pmatrix},\quad H_n=\begin{pmatrix}1 & 0 & \cdots & 0\\ 0 & 0 & & \vdots\\ \vdots & & \ddots & \vdots\\ 0 & \cdots & \cdots & 0\end{pmatrix},\quad K_n=\begin{pmatrix}0 & & & 0\\ & 1 & & \\ & & 1 & \\ 0 & & & \ddots \ 1\end{pmatrix}.$$

Further let T_n be defined as the orthogonal matrix with the elements of the first column all equal to $\frac{1}{\sqrt{n}}$, namely

$$(2.3)\qquad T_n=\begin{pmatrix}\frac{1}{\sqrt{n}} & \times & \times\cdots\times\\ \frac{1}{\sqrt{n}} & \times & \times\cdots\times\\ \vdots & \vdots & \vdots \qquad \vdots\\ \frac{1}{\sqrt{n}} & \times & \times\cdots\times\end{pmatrix}.$$

Then we have easily

$$(2.4)\qquad T_n' E_n T_n = nH_n.$$

3. The case of the 2-way layout

This section is devoted to the case of the 2-way layout. Although there is no essential difference between the case of the 2-way layout and the case of the general r-way layout the notation system we need in the latter case is so cumbersome and complicated that it would be, the author feels, necessary to treat this special case for the preparatory exposition of the basic techniques in the developments of the arguments in the general case.

In this section we shall be concerned with the model equation,

$$(3.1)\qquad x_{t_0t_1t_2}=\mu+a_{t_1}+a_{t_2}+a_{t_1t_2}+e_{t_0t_1t_2}\qquad \begin{matrix}t_0=1,2,\cdots,n_0,\\ t_1=1,2,\cdots,n_1,\\ t_2=1,2,\cdots,n_2,\end{matrix}$$

where μ is a constant denoting the general mean, and a_{t_1}, a_{t_2}, $a_{t_1t_2}$ and $e_{t_0t_1t_2}$, are distributed normally with mean 0 and the variance σ_1, σ_2, σ_{12} and σ_0, respectively, and further they are all independent to each other. Then the variance matrix of these $n_0 n_1 n_2$ variables $x_{t_0t_1t_2}$ are given by

$$(3.2)\qquad V=\underbrace{\overbrace{\begin{pmatrix}L & M & \cdots & M\\ M & L & & \vdots\\ \vdots & & \ddots & M\\ M & \cdots & M & L\end{pmatrix}}^{n_2}}_{n_2}$$

$$=\begin{pmatrix}M & \cdots & \cdots & M\\ \vdots & & & \vdots\\ \vdots & & & \vdots\\ M & \cdots & \cdots & M\end{pmatrix}+\begin{pmatrix}N & 0 & \cdots & 0\\ 0 & N & & \vdots\\ \vdots & & \ddots & 0\\ 0 & \cdots & 0 & N\end{pmatrix}+\begin{pmatrix}P & 0 & \cdots & 0\\ 0 & P & & \vdots\\ \vdots & & \ddots & 0\\ 0 & \cdots & 0 & P\end{pmatrix}+\begin{pmatrix}Q & 0 & \cdots & 0\\ 0 & Q & & \vdots\\ \vdots & & \ddots & 0\\ 0 & \cdots & 0 & Q\end{pmatrix}$$

where

(3.3)
$$L=\overbrace{\left(\begin{array}{c|c|c|c} \overbrace{\begin{matrix}A & & B\\ & A\ddots & \\ B & & A\end{matrix}}^{n_0} & C & \cdots\cdots & C \\ \hline C & \begin{matrix}A & & B\\ & A\ddots & \\ B & & A\end{matrix} & & \vdots \\ \hline \vdots & & \ddots & C \\ \hline C & \cdots\cdots & C & \begin{matrix}A & & B\\ & A\ddots & \\ B & & A\end{matrix}\end{array}\right)}^{n_1}\begin{matrix}\}n_0\\ \\ \}n_1\end{matrix}, \qquad M=\left(\begin{array}{c|c|c|c} D & 0 & \cdots\cdots & 0 \\ \hline 0 & D & & \vdots \\ \hline \vdots & & \ddots & 0 \\ \hline 0 & \cdots\cdots & 0 & D\end{array}\right),$$

$$N=\left(\begin{array}{c|c|c} C & \cdots\cdots & C \\ \hline \vdots & & \vdots \\ \hline C & \cdots\cdots & C\end{array}\right), \qquad P=\left(\begin{array}{c|c|c|c} B-C-D & 0 & \cdots\cdots & 0 \\ \hline 0 & B-C-D & & \vdots \\ \hline \vdots & & \ddots & 0 \\ \hline 0 & \cdots\cdots & 0 & B-C-D\end{array}\right),$$

$$Q=\left(\begin{array}{c|c|c|c} \begin{matrix}A-B & 0\\ 0\ddots & A-B\end{matrix} & 0 & \cdots\cdots & 0 \\ \hline 0 & \begin{matrix}A-B & 0\\ 0\ddots & A-B\end{matrix} & & \vdots \\ \hline \vdots & & \ddots & 0 \\ \hline 0 & \cdots\cdots & 0 & \begin{matrix}A-B & 0\\ 0\ddots & A-B\end{matrix}\end{array}\right),$$

and

(3.4)
$$A=\sigma_1+\sigma_2+\sigma_{12}+\sigma_0,$$
$$B=\sigma_1+\sigma_2+\sigma_{12},$$
$$C=\sigma_1,$$
$$D=\sigma_2.$$

This can be expressed simply in terms of the Kronecker product of the matrixes as follows.

(3.5) $V=\sigma_1 E_{n_0}\otimes I_{n_1}\otimes E_{n_2}+\sigma_2 E_{n_0}\otimes E_{n_1}\otimes I_{n_2}+\sigma_{12}E_{n_0}\otimes I_{n_1}\otimes I_{n_2}+\sigma_0 I_{n_0}\otimes I_{n_1}\otimes I_{n_2}.$

At first we shall evaluate the determinant of this matrix, which is equal to the determinat of the following matrix.

(3.6) $(T_{n_0}\otimes T_{n_1}\otimes T_{n_2})'\ V(T_{n_0}\otimes T_{n_1}\otimes T_{n_2}).$

In view of (2.4), this is equal to

(3.7)
$$\begin{aligned}&n_0n_2\sigma_1 H_{n_0}\otimes I_{n_1}\otimes H_{n_2}+n_0n_1\sigma_2 H_{n_0}\otimes H_{n_1}\otimes H_{n_2}+n_0\sigma_{12}H_{n_0}\otimes I_{n_1}\otimes I_{n_2}+\sigma_0 I_{n_0}\otimes I_{n_1}\otimes I_{n_2}\\ &=n_0n_2\sigma_1 H_{n_0}\otimes(H_{n_1}+K_{n_1})\otimes H_{n_2}+n_0n_1\sigma_2 H_{n_0}\otimes H_{n_1}\otimes(H_{n_2}+K_{n_2})\\ &\quad+n_0\sigma_{12}H_{n_0}\otimes(H_{n_1}+K_{n_1})\otimes(H_{n_2}+K_{n_2})+\sigma_0(H_{n_0}+K_{n_0})\otimes(H_{n_1}+K_{n_1})\otimes(H_{n_2}+K_{n_2})\\ &=(n_0n_2\sigma_1+n_0n_1\sigma_2+n_0\sigma_{12}+\sigma_0)H_{n_0}\otimes H_{n_1}\otimes H_{n_2}+(n_0n_2\sigma_1+n_0\sigma_{12}+\sigma_0)H_{n_0}\otimes K_{n_1}\otimes H_{n_2}\\ &\quad+(n_0n_1\sigma_2+n_0\sigma_{12}+\sigma_0)H_{n_0}\otimes H_{n_1}\otimes K_{n_2}+(n_0\sigma_{12}+\sigma_0)H_{n_0}\otimes K_{n_1}\otimes K_{n_2}\\ &\quad+\sigma_0\{K_{n_0}\otimes H_{n_1}\otimes H_{n_2}+K_{n_0}\otimes K_{n_1}\otimes H_{n_2}+K_{n_0}\otimes H_{n_1}\otimes K_{n_2}+K_{n_0}\otimes K_{n_1}\otimes K_{n_2}\}.\end{aligned}$$

Thus the matrix (3.6) is expressed as the linear form of eight matrixes, and as all of them are diagonal, this matrix is also diagonal, and any two matrixes have no non-zero element in common. This fact leads to the evaluation of the determinant as follows,

(3.8)
$$\begin{aligned}|V|=&(n_0n_2\sigma_1+n_0n_1\sigma_2+n_0\sigma_{12}+\sigma_0)\,(n_0n_2\sigma_1+n_0\sigma_{12}+\sigma_0)^{(n_1-1)}(n_0n_1\sigma_2+n_0\sigma_{12}+\sigma_0)^{(n_2-1)}\\ &\cdot(n_0\sigma_{12}+\sigma_0)^{(n_1-1)(n_2-1)}\sigma_0^{n_1n_2(n_0-1)},\end{aligned}$$

or by writing

(3.9)
$$\begin{aligned}&\theta_0=\sigma_0,\\ &\theta_{12}=n_0\sigma_{12}+\sigma_0,\\ &\theta_1=n_0n_2\sigma_1+n_0\sigma_{12}+\sigma_0,\\ &\theta_2=n_0n_1\sigma_2+n_0\sigma_{12}+\sigma_0,\\ &\theta_E=n_0n_2\sigma_1+n_0n_1\sigma_2+n_0\sigma_{12}+\sigma_0,\end{aligned}$$

we have finally

(3.10) $|V|=\theta_E\cdot\theta_1^{(n_1-1)}\theta_2^{(n_2-1)}\theta_{12}^{(n_1-1)(n_2-1)}\theta_0^{n_1n_2(n_0-1)}.$

Now let us find out the inverse matrix of the variance matrix (3.5). The variance matrix is given as the linear form of four matrixes, and anticipating its inverse to be a linear form of these four and of $E_{n_0}\otimes E_{n_1}\otimes E_{n_2}$, we have, after a simple calculations,

(3.11)
$$\begin{aligned}&[\sigma_1 E_{n_0}\otimes I_{n_1}\otimes E_{n_2}+\sigma_2 E_{n_0}\otimes E_{n_1}\otimes I_{n_2}+\sigma_{12}E_{n_0}\otimes I_{n_1}\otimes I_{n_2}+\sigma_0 I_{n_0}\otimes I_{n_1}\otimes I_{n_2}]\\ &\quad\cdot[X_E E_{n_0}\otimes E_{n_1}\otimes E_{n_2}+X_1 E_{n_0}\otimes I_{n_1}\otimes E_{n_2}+X_2 E_{n_0}\otimes E_{n_1}\otimes I_{n_2}\\ &\quad+X_{12}E_{n_0}\otimes I_{n_1}\otimes I_{n_2}+X_0 I_{n_0}\otimes I_{n_1}\otimes I_{n_2}]\\ &=E_{n_0}\otimes E_{n_1}\otimes E_{n_2}[(n_0n_2\sigma_1+n_0n_1\sigma_2+n_0\sigma_{12}+\sigma_0)X_E+n_0\sigma_2X_1+n_0\sigma_1X_2]\\ &\quad+E_{n_0}\otimes I_{n_1}\otimes E_{n_2}[(n_0n_2\sigma_1+n_0\sigma_{12}+\sigma_0)X_1+n_0\sigma_1X_{12}+\sigma_1X_0]\\ &\quad+E_{n_0}\otimes E_{n_1}\otimes I_{n_2}[(n_0n_1\sigma_2+n_0\sigma_{12}+\sigma_0)X_2+n_0\sigma_2X_{12}+\sigma_2X_0]\end{aligned}$$

$$+E_{n_0}\otimes I_{n_1}\otimes I_{n_2}[(n_0\sigma_{12}+\sigma_0)X_{12}+\sigma_{12}X_0]$$
$$+I_{n_0}\otimes I_{n_1}\otimes I_{n_2}X_0\sigma_0.$$

In order to have the second matrix which is a linear form of five matrixes to be the inverse of the first, the product of these two should be the unit matrix, for which we should have

$$\begin{aligned}
&(n_0n_2\sigma_1+n_0n_1\sigma_2+n_0\sigma_{12}+\sigma_0)X_E+n_0\sigma_2X_1+n_0\sigma_1X_2=0,\\
&(n_0n_2\sigma_1+n_0\sigma_{12}+\sigma_0)X_1+n_0\sigma_1X_{12}+\sigma_1X_0=0,\\
(3.12)\quad &(n_0n_1\sigma_2+n_0\sigma_{12}+\sigma_0)X_2+n_0\sigma_2X_{12}+\sigma_2X_0=0,\\
&(n_0\sigma_{12}+\sigma_0)X_{12}+\sigma_{12}X_0=0,\\
&\sigma_0X_0=1.
\end{aligned}$$

The solution of these linear equations is given by

$$\begin{aligned}
&X_0=\frac{1}{\theta_0},\\
&X_{12}=\frac{1}{n_0}\left(\frac{1}{\theta_{12}}-\frac{1}{\theta_0}\right),\\
(3.13)\quad &X_1=\frac{1}{n_0n_2}\left(\frac{1}{\theta_1}-\frac{1}{\theta_{12}}\right),\\
&X_2=\frac{1}{n_0n_1}\left(\frac{1}{\theta_2}-\frac{1}{\theta_{12}}\right),\\
&X_E=\frac{1}{n_0n_1n_2}\left(\frac{1}{\theta_{12}}-\frac{1}{\theta_1}-\frac{1}{\theta_2}+\frac{1}{\theta_E}\right).
\end{aligned}$$

Thus we have obtained the determinant and the inverse of the variance matrix, which enables us to give the joint density function of all the $n_0n_1n_2$ variables in our concern. By noting the relations

$$(3.14)\quad \mathfrak{X}_n'E_n\mathfrak{X}_n=\left(\sum_{i=1}^{n}x_i\right)^2,\quad \mathfrak{X}_n'I_n\mathfrak{X}_n=\sum_{i=1}^{n}x_i^2$$

where $\mathfrak{X}_n$ is any n-dimensional vector $\mathfrak{X}_n'=(x_1, x_2,\cdots\cdots, x_n)$, and by writing

$$(3.15)\quad u_{t_0t_1t_2}=x_{t_0t_1t_2}-\mu,$$

our joint density function is given by the following

$$(3.16)\quad f(\mathbf{X})=\left(\frac{1}{\sqrt{2\pi}}\right)^{n_0n_1n_2}(|V|)^{-1/2}\,exp\left(-\frac{1}{2}S\right),$$

where

$$\begin{aligned}
(3.17)\quad S=&X_E(\sum_{t_0}\sum_{t_1}\sum_{t_2}u_{t_0t_1t_2})^2+X_1\sum_{t_1}(\sum_{t_0}\sum_{t_2}u_{t_0t_1t_2})^2+X_2\sum_{t_2}(\sum_{t_0}\sum_{t_1}u_{t_0t_1t_2})^2\\
&+X_{12}\sum_{t_1}\sum_{t_2}(\sum_{t_0}u_{t_0t_1t_2})^2+X_0\sum_{t_0}\sum_{t_1}\sum_{t_2}u^2_{t_0t_1t_2},
\end{aligned}$$

where X_E, X_1, X_2, X_{12}, and X_0 are given by (3.12).

After a simple modification, we have finally

$$(3.18)\quad f(\mathrm{X})=\mathrm{K}\,\theta_E^{-1/2}\,\theta_1^{-(n_1-1)/2}\,\theta_2^{-(n_2-1)/2}\,\theta_{12}^{-(n_1-1)(n_2-1)/2}\,\theta_2^{-n_1n_2(n_0-1)/2}$$

$$\cdot exp\Big[-\frac{1}{2}\Big\{\frac{1}{\theta_0}\sum_{t_0}\sum_{t_1}\sum_{t_2}(x_{t_0t_1t_2}-\bar{x}_{.t_1t_2})^2+\frac{n_1n_2}{\theta_1}\sum_{t_1}(\bar{x}_{.t_1.}-\bar{x}_{...})^2$$

$$+\frac{n_0n_1}{\theta_2}\sum_{t_2}(\bar{x}_{..t_2}-\bar{x}_{...})^2+\frac{n_0}{\theta_{12}}\sum_{t_1}\sum_{t_2}(\bar{x}_{.t_1t_2}-\bar{x}_{.t_1.}-\bar{x}_{..t_2}+\bar{x}_{...})^2+\frac{n_0n_1n_2}{\theta_E}(\bar{x}_{...}-\mu)^2\Big\}\Big]$$

where K is a constant independent of the parameters in our concern.

We have the family of distributions whose parameter space is written explicitly as

$$(3.19)\quad \Omega=\begin{pmatrix} 0\leqq\theta_0<\infty, & \theta_E=\theta_1+\theta_2-\theta_{12}, \\ \theta_0\leqq\theta_{12}<\infty, & \\ \theta_{12}\leqq\theta_1<\infty, & -\infty<\mu<\infty, \\ \theta_{12}\leqq\theta_2<\infty, & \end{pmatrix},$$

and whose sufficient statistics are given by the following five statistics

$$(3.20)\quad \begin{aligned} S_0 &= \sum_{t_0}\sum_{t_1}\sum_{t_2}(x_{t_0t_1t_2}-\bar{x}_{.t_1t_2})^2, \\ S_1 &= n_0n_2\sum_{t_1}(\bar{x}_{.t_1.}-\bar{x}_{...})^2, \\ S_2 &= n_0n_1\sum_{t_2}(\bar{x}_{..t_2}-\bar{x}_{...})^2, \\ S_{12} &= n_0\sum_{t_1}\sum_{t_2}(\bar{x}_{.t_1t_2}-\bar{x}_{.t_1.}-\bar{x}_{..t_2}+\bar{x}_{...})^2, \\ &\bar{x}_{...}. \end{aligned}$$

If the family of distribution of these sufficient statistics is complete, the theory of estimation tells us that the usual estimates are the unique unbiased minimum variance estimates of the variance components σ_0, σ_1, σ_2, σ_{12} and the general mean μ (Lehmann [5]). As already pointed out by Washio, we have not so far been able to conclude whether it is complete or not, and we shall appeal to the notion of the minimum variance estimate due to Bhattacharyya, and in this connection we shall make use of the result due to him [1] (c. f. section 6 of Chapter I in his paper).

In view of (3.18), (3.20) we have

$$(3.21)\quad \ln f=\mathrm{K}'-\frac{1}{2}\Big\{\ln\theta_E+(n_1-1)\ln\theta_1+(n_2-1)\ln\theta_2+(n_1-1)(n_2-1)\ln\theta_{12}+$$

$$+n_1n_2(n_0-1)\ln\theta_0\Big\}-\frac{1}{2}\Big\{\frac{S_0}{\theta_0}+\frac{S_1}{\theta_1}+\frac{S_2}{\theta_2}+\frac{S_{12}}{\theta_{12}}+\frac{n_0n_1n_2(\bar{x}_{...}-\mu)^2}{\theta_E}\Big\}$$

Hence we have

$$(3.22)\quad \begin{aligned} &\frac{2\theta_0^2}{n_1n_2(n_0-1)}\frac{\partial f}{\partial\theta_0}=\Big(\frac{S_0}{n_1n_2(n_0-1)}-\theta_0\Big)f, \\ &\frac{2\theta_1^2}{(n_1-1)}\frac{\partial f}{\partial\theta_1}-\frac{\theta_1^2}{n_0n_1n_2(n_1-1)}\frac{\partial^2 f}{\partial\mu^2}=\Big(\frac{S_1}{(n_1-1)}-\theta_1\Big)f, \\ &\frac{2\theta_1^2}{(n_2-1)}\frac{\partial f}{\partial\theta_2}-\frac{\theta_2^2}{n_0n_1n_2(n_2-1)}\frac{\partial^2 f}{\partial\mu^2}=\Big(\frac{S_2}{(n_2-1)}-\theta_2\Big)f, \end{aligned}$$

$$\frac{2\theta_{12}^2}{(n_1-1)(n_2-1)}\frac{\partial f}{\partial\theta_{12}}+\frac{\theta_{12}^2}{n_0n_1n_2(n_1-1)(n_2-1)}\frac{\partial^2 f}{\partial\mu^2}=\left(\frac{S_{12}}{(n_1-1)(n_2-1)}-\theta_{12}\right)f,$$

$$\frac{\theta_E}{n_0n_1n_2}\frac{\partial f}{\partial\mu}=(\bar{x}\ldots-\mu)f.$$

This fact and the result in Bhattacharyya [1] yield us that the minimum variance estimates of $\sigma_0, \sigma_1, \sigma_2, \sigma_{12}$ and μ is given by $\frac{S_0}{n_1n_2(n_0-1)}$, $\frac{S_1}{n_0n_2(n_1-1)}-\frac{S_{12}}{n_0n_2(n_1-1)(n_2-1)}$, $\frac{S_2}{n_0n_1(n_2-1)}-\frac{S_{12}}{n_0n_1(n_1-1)(n_2-1)}$, $\frac{S_{12}}{n_0(n_1-1)(n_2-1)}-\frac{S_0}{n_0n_1n_2(n_0-1)}$, $\bar{x}\ldots$.

4. The case of the r-way layout.

4.1 *The determinant of the variance matrix.* In this section we shall give the results and the proofs in the case of the r-way layout with the model given in (2.1) under the assumptions stated in the beginning of section 2.

Corresponding to (3.4), we have the expression of the variance matrix in terms of the Kronecker products as follows

$$V=\sum_{k=1}^{r}\sum_{I_k\subset R}\sigma_{i_1\ldots\ldots i_k}E_{n_0}\otimes\prod_{j=1}^{r}\otimes\left(E_{n_j}^{1-\delta^j_{i_1\ldots i_k}}\times I_{n_j}^{\delta^j_{i_1\ldots i_k}}\right)+\sigma_0 I_{n_0}\otimes I_{n_1}\otimes\cdots\cdots\otimes I_{n_r}, \tag{4.1}$$

where $\delta^j_{i_1\ldots\ldots i_k}$ is a sort of generalization of the Kronecker's delta which is

$$\delta^j_{i_1\ldots\ldots i_k}=\begin{cases}1 & \text{if } j \text{ is equal to either of } (i_1, i_2,\cdots\cdots, i_k)\\ 0 & \text{otherwise.}\end{cases} \tag{4.2}$$

and E^0 of a matrix E is defined to be the unit matrix I. The reason for this expression is clear.

Throughout this paper the notations such as T_α, S_β, I_k etc. mean a set of integers $(t_1, t_2,\cdots\cdots, t_\alpha)$, $(s_1, s_2,\cdots\cdots, s_\beta)$, $(i_1, i_2,\cdots\cdots, i_k)$ etc. and $R=(1, 2, 3, \cdots\cdots, r)$, and the summations such as $\sum_{A\subset B}a_A$, $\sum_{\substack{A\subset B\\ A\supset C}}a_A$, where A, B, C are such sets of integers as stated above, mean the sum of all numbers a_A's having A as the suffixes which are included in B, or included in B and including C, respectively.

For the developments of the arguments in this section we have to prepare with a number of notations as follows.

DEFINITION 4.1.

$$A_{(t_1\ldots\ldots t_\alpha)}=\sum_{k=\alpha}^{r}\sum_{\substack{I_k\supset T_\alpha\\ I_k\subset R}}\sigma_{i_1\ldots\ldots i_k}\prod_{j=0}^{r}n^{1-\delta^j_{i_1\ldots\ldots i_k}}, \tag{4.3}$$

$$A^{(s_1\ldots\ldots s_\beta)}_{(t_1\ldots\ldots t_\alpha)}=\sum_{k=\alpha}^{r-\beta}\sum_{\substack{I_k\supset T_\alpha\\ I_k\subset R-S_\beta}}\sigma_{i_1\ldots\ldots i_k}\prod_{j=0}^{r}n_j^{1-\delta^j_{i_1\ldots\ldots i_k s_1\ldots\ldots s_\beta}}, \tag{4.4}$$

$$A=\sum_{k=1}^{r}\sum_{I_k\subset R}\sigma_{i_1\ldots\ldots i_k}\prod_{j=0}^{r}n_j^{1-\delta^j_{i_1\ldots\ldots i_k}}, \tag{4.5}$$

(4.6) $A_{(t_1\cdots\cdots t_\alpha)}+\sigma_0=B_{(t_1\cdots\cdots t_\alpha)}$,

(4.7) $A+\sigma_0=B.$

Now we shall evalute at first the determinant of the variance matrix (4.1)

THEOREM 4.1. *The determinant* $|V|$ *of the variance matrix* V *of* (4.1) *is given in the notation of* (4.6) *and* (4.7) *as follows*

$$(4.8)\quad |V|=B\cdot\prod_{k=1}^{r}\prod_{I_k\subset R}\left\{B_{(i_1\cdots\cdots i_k)}\right\}^{(n_{i_1}-1)(n_{i_2}-1)\cdots\cdots(n_{i_k}-1)}\sigma_0^{(n_0-1)n_1\cdots\cdots n_r}.$$

PROOF. Let us at first transform this matrix by the orthogonal matrix which is the Kronecker product of the matrixes T_{n_i} defined in (2.3), and we have

$$(4.9)\quad (T_{n_0}\otimes T_{n_1}\otimes\cdots\cdots\otimes T_{n_r})'V(T_{n_0}\otimes T_{n_1}\otimes\cdots\cdots\otimes T_{n_r})$$

$$=\sum_{k=1}^{r}\sum_{I_k\subset R}\sigma_{i_1\cdots\cdots i_k}\prod_{j=0}^{r}n_j^{1-\delta^j_{i_1\cdots i_k}}H_{n_0}\otimes\prod_{t=1}^{r}\otimes\left(H_{n_t}^{1-\delta^t_{i_1\cdots i_k}}\times I_{n_t}^{\delta^t_{i_1\cdots i_k}}\right)+\sigma_0 I_{n_0}\otimes I_{n_1}\otimes\cdots\cdots\otimes I_{n_r}$$

$$=\sum_{k=1}^{r}\sum_{I_k\subset R}\sigma_{i_1\cdots\cdots i_k}\prod_{j=0}^{r}n_j^{1-\delta^j_{i_1\cdots i_k}}H_{n_0}\otimes\prod_{t=1}^{r}\otimes\left(H_{n_t}^{1-\delta^t_{i_1\cdots\cdots i_k}}\times(H_{n_t}+K_{n_t})^{\delta^t_{i_1\cdots i_k}}\right)$$

$$+\sigma_0\prod_{j=1}^{r}\otimes(H_{n_j}+K_{n_j})$$

$$=\sum_{k=1}^{r}\sum_{I_k\subset R}\sigma_{i_1\cdots\cdots i_k}\prod_{j=0}^{r}n_j^{1-\delta^j_{i_1\cdots i_k}}H_{n_0}\otimes\prod_{t=1}^{r}\otimes(H_{n_t}+\delta^t_{i_1\cdots i_k}K_{n_t})+\sigma_0\prod_{j=1}^{r}\otimes(H_{n_j}+K_{n_j}).$$

This matrix is also a linear form of the matrixes of the type

$$(4.10)\quad H_{n_0}\otimes A_1\otimes A_2\otimes\cdots\cdots\otimes A_r,$$

where

$$(4.11)\quad A_i=H_{n_i}\ \text{or}\ H_{n_i}+K_{n_i}\equiv I_{n_i}.$$

There are 2^r different matrixes of this type, which are all diagonal, and hence the matrix (4.9) is diagonal. The product of any two matrixes of this type is the null matrix, and the matrix (4.9) itself is a nonsingular matrix. Therefore the determinant is equal to the product of $n_0n_1\cdots\cdots n_r$ numbers each of which is equal to either of the coefficient of 2^r different matrixes in the linear form (4.9). In this product the coefficient of $H_{n_0}\otimes(H_{n_1}+K_{n_1})\otimes H_{n_2}\otimes\cdots\cdots\otimes H_{n_r}$, for instance, appears exactly n_1-1 times which is equal to the rank of this matrix, and this coefficient is equal to $B_{(1)}^{(n_1-1)}$.

Thus we have

$$(4.12)\quad |V|=\left[\sum_{k=1}^{r}\sum_{I_k\subset R}\sigma_{i_1\cdots\cdots i_k}\prod_{j=0}^{r}n_j^{1-\delta^j_{i_1\cdots i_k}}+\sigma_0\right]$$

$$\cdot\prod_{k=1}^{r}\prod_{I_k\subset R}\left\{\sum_{p=k}^{r}\sum_{\substack{L_p\supset I_k\\ L_p\subset R}}\sigma_{l_1\cdots\cdots l_p}\prod_{j=1}^{r}n_j^{1-\delta^j_{l_1\cdots l_p}}+\sigma_0\right\}^{(n_{i_1}-1)(n_{i_2}-1)\cdots(n_{i_k}-1)}\cdot\ \sigma_0^{(n_0-1)n_1\cdots n_r},$$

which is equal to (4.8) from the definition 4.1.

4.2 *The inverse of the variance matrix.* Before finding out the inverse of the variance matrix, we need to consider some relations between the notations defined in the definition 4.1. At first we observe the recurrence relation of $A_{(t_1\cdots t_\alpha)}^{(s_1\cdots s_\beta)}$.

LEMMA 4.1.

$$A_{(t_1\cdots t_\alpha)}^{(s_1\cdots s_\beta)}=\frac{1}{n_{s_\beta}}\left[A_{(t_1\cdots t_\alpha)}^{(s_1\cdots s_{\beta-1})}-A_{(t_1\cdots t_\alpha s_\beta)}^{(s_1\cdots s_{\beta-1})}\right]. \tag{4.13}$$

PROOF.

$$\begin{aligned} A_{(t_1\cdots t_\alpha)}^{(s_1\cdots s_\beta)} &= \sum_{k=\alpha}^{r-\beta}\sum_{\substack{I_k\supset T_\alpha\\ I_k\subset R-S_\beta}}\sigma_{i_1\cdots i_k}\prod_{j=0}^{r}n_j^{1-\delta^j_{i_1\cdots i_k s_1\cdots s_\beta}} \\ &= \sum_{k=\alpha}^{r-\beta+1}\sum_{\substack{I_k\supset T_\alpha\\ I_k\subset R-S_\beta}}\sigma_{i_1\cdots i_k}\prod_{j=0}^{r}n_j^{1-\delta^j_{i_1\cdots i_k s_1\cdots s_\beta}} - \sum_{k=\alpha+1}^{r-\beta+1}\sum_{\substack{I_k\supset (T_\alpha,S_\beta)\\ I_k\subset R-S_\beta}}\sigma_{i_1\cdots i_k}\prod_{j=0}^{r}n_j^{1-\delta^j_{i_1\cdots i_k s_1\cdots s_\beta}} \\ &= \frac{1}{n_{s_\beta}}\left[\sum_{k=\alpha}^{r-\beta+1}\sum_{\substack{I_k\supset T_\alpha\\ I_k\subset R-S_{\beta-1}}}\sigma_{i_1\cdots i_k}\prod_{j=0}^{r}n_j^{1-\delta^j_{i_1\cdots i_k s_1\cdots s_{\beta-1}}} - \sum_{k=\alpha+1}^{r-\beta+1}\sum_{\substack{I_k\supset (T_\alpha,S_\beta)\\ I_k\subset R-S_{\beta-1}}}\sigma_{i_1\cdots i_k}\prod_{j=0}^{r}n_j^{1-\delta^j_{i_1\cdots i_k s_1\cdots s_{\beta-1}}}\right] \\ &= \frac{1}{n_{s_\beta}}[A_{(t_1\cdots t_\alpha)}^{(s_1\cdots s_{\beta-1})}-A_{(t_1\cdots t_\alpha s_\beta)}^{(s_1\cdots s_{\beta-1})}]. \end{aligned} \tag{4.14}$$

This lemma enables us to express $A_{(t_1\cdots t_\alpha)}^{(s_1\cdots s_\beta)}$ in terms of $A_{(t_1\cdots t_\alpha l_1\cdots l_p)}$, which is given by

LEMMA 4.2.

$$A_{(t_1\cdots t_\alpha)}^{(s_1\cdots s_\beta)}=\frac{1}{n_{s_1}\cdots n_{s_\beta}}\sum_{p=0}^{\beta}\sum_{L_p\subset S_\beta}(-1)^p A_{(t_1\cdots t_\alpha l_1\cdots l_p)}. \tag{4.15}$$

PROOF.

We shall give the proof by making use of the mathematical induction in β. In case $\beta=1$, we have

$$\begin{aligned} A_{(t_1\cdots t_\alpha)}^{(s_1)} &= \frac{1}{n_{s_1}}[A_{(t_1\cdots t_\alpha)}-A_{(t_1\cdots t_\alpha s_1)}] \\ &= \frac{1}{n_{s_1}}\sum_{p=0}^{1}\sum_{L_p\subset S_1}(-1)^p A_{(t_1\cdots t_\alpha l_1\cdots l_p)}. \end{aligned} \tag{4.16}$$

Then assuming (4.15) to be valid in case $\beta=h$ i. e.,

$$A_{(t_1\cdots t_\alpha)}^{(s_1\cdots s_h)}=\frac{1}{n_{s_1}\cdots n_{s_h}}\sum_{p=0}^{h}\sum_{L_p\subset S_h}(-1)^p A_{(t_1\cdots t_\alpha l_1\cdots l_p)} \tag{4.17}$$

we shall prove this is also valid in case $\beta=h+1$, which is given by

$$\begin{aligned} A_{(t_1\cdots t_\alpha)}^{(s_1\cdots s_{h+1})} &= \frac{1}{n_{s_{h+1}}}\left[A_{(t_1\cdots t_\alpha)}^{(s_1\cdots s_h)}-A_{(t_1\cdots t_\alpha s_{h+1})}^{(s_1\cdots s_h)}\right] \\ &= \frac{1}{n_{s_{h+1}}}\left[\frac{1}{n_{s_1}\cdots n_{s_h}}\left\{\sum_{p=0}^{h}\sum_{L_p\subset S_h}(-1)^p A_{(t_1\cdots t_\alpha l_1\cdots l_p)} - \sum_{p=0}^{h}\sum_{L_p\subset S_h}(-1)^p A_{(t_1\cdots t_\alpha s_{h+1} l_1\cdots l_p)}\right\}\right] \\ &= \frac{1}{n_{s_1}\cdots n_{s_{h+1}}}\left[\sum_{p=0}^{h}\sum_{L_p\subset S_h}(-1)^p A_{(t_1\cdots t_\alpha l_1\cdots l_p)} + \sum_{p=0}^{h+1}\sum_{\substack{L_p\subset S_{h+1}\\ L_p\ni s_{h+1}}}(-1)^p A_{(t_1\cdots t_\alpha l_1\cdots l_p)}\right] \end{aligned} \tag{4.18}$$

$$= \frac{1}{n_{s_1}\cdots n_{s_{h+1}}} \sum_{p=0}^{h+1} \sum_{L_p \subset S_{h+1}} (-1)^p A_{(t_1\cdots t_\alpha l_1 \cdots l_p)}.$$

Now let us turn to the inversion of the variance matrix. The arguments follow the similar line to that of the 2-way layout.

THEOREM 4.2. *The inverse of the variance matrix* (2.1) *is given by*

$$X_E E_{n_0}\otimes E_{n_1}\otimes\cdots\cdots\otimes E_{n_r} + \sum_{k=1}^{r}\sum_{I_k\subset R} X_{i_1\cdots i_k} E_{n_0}\otimes \prod_{j=1}^{r}\otimes\left(E_{n_j}^{1-\delta^j_{i_1\cdots i_k}} \times I_{n_j}^{\delta^j_{i_1\cdots i_k}}\right) + X_0 I_{n_0}\otimes I_{n_1}\otimes\cdots\cdots\otimes I_{n_r}, \tag{4.19}$$

where

$$X_0 = \frac{1}{\sigma_0}, \tag{4.20}$$

$$X_{123\cdots r} = \frac{1}{n_0}\left[\frac{1}{B_{(12\cdots r)}} - \frac{1}{\sigma_0}\right], \tag{4.21}$$

$$X_{i_1 i_2\cdots i_k} = \frac{1}{\prod_{j=0}^{r} n_j^{1-\delta^j_{i_1\cdots i_k}}}\left[\frac{(-1)^{r-k}}{B_{(12\cdots r)}} + \sum_{\alpha=0}^{r-k-1}\sum_{S_\alpha\subset R-I_k}\frac{(-1)^\alpha}{B_{(i_1\cdots i_k s_1\cdots s_\alpha)}}\right] (I_k\subset R,\ k=1,2,\cdots,r-1), \tag{4.22}$$

$$X_E = \frac{1}{\prod_{j=0}^{r} n_j}\left[\frac{(-1)^r}{B_{(12\cdots r)}} + \frac{1}{B} + \sum_{c=1}^{r-1}\sum_{I_c\subset R}\frac{(-1)^c}{B_{(i_1\cdots i_c)}}\right]. \tag{4.23}$$

PROOF. As we have done in case of the 2-way layout, anticipating the inverse to be the form of (4.19), we seek for the condition that (4.19) is actually the inverse. The product of the variance matrix (4.1) and the matrix (4.19) is

$$\begin{aligned}
&E_{n_0}\otimes E_{n_1}\otimes\cdots\cdots\otimes E_{n_r}\Bigg[X_E\Big\{\sum_{k=1}^{r}\sum_{I_k\subset R}\sigma_{i_1\cdots i_k}\prod_{j=0}^{r} n_j^{1-\delta^j_{i_1\cdots i_k}} + \sigma_0\Big\} \\
&\quad + \sum_{k=1}^{r}\sum_{I_k\subset R} X_{i_1\cdots i_k}\Big\{\sum_{l=1}^{r-k}\sum_{T_l\subset R-I_k}\sigma_{t_1\cdots t_l}\prod_{j=0}^{r} n_j^{1-\delta^j_{i_1\cdots i_k t_1\cdots t_l}}\Big\}\Bigg] \\
&\quad + \sum_{k=1}^{r-1}\sum_{I_k\subset R} E_{n_0}\otimes\prod_{j=1}^{r}\otimes\left(E_{n_j}^{1-\delta^j_{i_1\cdots i_k}}\times I_{n_j}^{\delta^j_{i_1\cdots i_k}}\right) \\
&\quad\cdot\Bigg[\sum_{l=k}^{r}\sum_{T_l\supset I_k} X_{t_1\cdots t_l}\Big\{\sum_{m=k}^{r-l+k}\sum_{\substack{S_m\supset I_k\\ S_m\subset R-(T_l-I_k)}}\sigma_{s_1\cdots s_m}\prod_{j=0}^{r} n_j^{1-\delta^j_{t_1\cdots t_l s_1\cdots s_m}}\Big\} + X_0\sigma_{i_1\cdots i_k}\Bigg] \\
&\quad + E_{n_0}\otimes I_{n_1}\otimes\cdots\cdots\otimes I_{n_r}[X_{12\cdots r}(n_0\sigma_{12\cdots r}+\sigma_0) + X_0\sigma_{12\cdots r}] \\
&\quad + I_{n_0}\otimes I_{n_1}\otimes\cdots\cdots\otimes I_{n_r}\cdot X_0\sigma_0
\end{aligned} \tag{4.24}$$

$$\begin{aligned}
&= E_{n_0}\otimes E_{n_1}\otimes\cdots\cdots\otimes E_{n_r}\Big[X_E(A+\sigma_0) + \sum_{k=1}^{r}\sum_{I_k\subset R} X_{i_1\cdots i_k} A^{(i_1\cdots i_k)}\Big] \\
&\quad + \sum_{k=1}^{r-1}\sum_{I_k\subset R} E_{n_0}\otimes\prod_{j=1}^{r}\otimes\left(E_{n_j}^{1-\delta^j_{i_1\cdots i_k}}\times I_{n_j}^{\delta^j_{i_1\cdots i_k}}\right)\Bigg[X_{i_1\cdots i_k}\Big\{\sum_{m=k}^{r}\sum_{\substack{S_m\supset I_k\\ S_m\subset R}}\sigma_{s_1\cdots s_m}\prod_{j=0}^{r} n_j^{1-\delta^j_{i_1\cdots i_k s_1\cdots s_m}} + \sigma_0\Big\} \\
&\quad + \sum_{l=k+1}^{r}\sum_{T_l\supsetneqq I_k} X_{t_1\cdots t_l}\Big\{\sum_{m=k}^{r-l+k}\sum_{\substack{S_m\supset I_k\\ S_m\subset R-(T_l-I_k)}}\sigma_{s_1\cdots s_m}\prod_{j=0}^{r} n_j^{1-\delta^j_{t_1\cdots t_l s_1\cdots s_m}}\Big\} + X_0\sigma_{i_1\cdots i_k}\Bigg]
\end{aligned}$$

$$+ E_{n_0}\otimes I_{n_1}\otimes\cdots\cdots\otimes I_{n_r}[X_{12\cdots r}B_{(12\cdots r)}+X_0\sigma_{12\cdots r}]$$
$$+ I_{n_0}\otimes I_{n_1}\otimes\cdots\cdots\otimes I_{n_r}X_0\sigma_0$$

The first term is equal to

$$(4.25)\quad E_{n_0}\otimes E_{n_1}\otimes\cdots\cdots\otimes E_{n_r}\left[X_E(A+\sigma_0)+\sum_{k=1}^{r}\sum_{I_k\subset R}X_{i_1\cdots i_k}A^{(i_1\cdots i_k)}\right]$$
$$= E_{n_0}\otimes E_{n_1}\otimes\cdots\cdots\otimes E_{n_r}\left[X_E B+\sum_{k=1}^{r}\sum_{I_k\subset R}X_{i_1\cdots i_k}A^{(i_1\cdots i_k)}\right]$$

The second term is equal to

$$(4.26)\quad \sum_{k=1}^{r-1}\sum_{I_k\subset R}E_{n_0}\otimes\prod_{j=1}^{r}\otimes\left(E_{n_j}^{1-\delta^j_{i_1\cdots i_k}}\times I_{n_j}^{\delta^j_{i_1\cdots i_k}}\right)$$
$$\cdot\left[X_{i_1\cdots i_k}\left\{\sum_{m=k}^{r}\sum_{\substack{S_m\supset I_k\\ S_m\subset R}}\sigma_{s_1\cdots s_m}\prod_{j=0}^{r}n_j^{1-\delta^j_{i_1\cdots i_k s_1\cdots s_m}}+\sigma_0\right\}\right.$$
$$\left.+\sum_{l=k+1}^{r}\sum_{T_l\supsetneqq I_k}X_{t_1\cdots t_l}\left\{\sum_{m=k}^{r-l+k}\sum_{\substack{S_m\supset I_k\\ S_m\subset R-(T_l-I_k)}}\sigma_{s_1\cdots s_m}\prod_{j=0}^{r}n_j^{1-\delta^j_{t_1\cdots t_l s_1\cdots s_m}}\right\}+X_0\sigma_{i_1\cdots i_k}\right]$$
$$=\sum_{k=1}^{r-1}\sum_{I_k\subset R}E_{n_0}\otimes\prod_{j=1}^{r}\otimes\left(E_{n_j}^{1-\delta^j_{i_1\cdots i_k}}\times I_{n_j}^{\delta^j_{i_1\cdots i_k}}\right)$$
$$\cdot\left[X_{i_1\cdots i_k}\left\{\sum_{m=k}^{r}\sum_{\substack{S_m\supset I_k\\ S_m\subset R}}\sigma_{s_1\cdots s_m}\prod_{j=0}^{r}n_j^{1-\delta^j_{s_1\cdots s_m}}+\sigma_0\right\}\right.$$
$$\left.+\sum_{l=1}^{r-k}\sum_{T_l\subset R-I_k}X_{i_1\cdots i_k t_1\cdots t_l}\left\{\sum_{m=k}^{r-l}\sum_{\substack{S_m\supset I_k\\ S_m\subset R-T_l}}\sigma_{s_1\cdots s_m}\prod_{j=0}^{r}n_j^{1-\delta^j_{t_1\cdots t_l s_1\cdots s_m}}\right\}+X_0\sigma_{i_1\cdots i_k}\right]$$
$$=\sum_{k=1}^{r-1}\sum_{I_k\subset R}E_{n_0}\otimes\prod_{j=1}^{r}\otimes\left(E_{n_j}^{1-\delta^j_{i_1\cdots i_k}}\times I_{n_j}^{\delta^j_{i_1\cdots i_k}}\right)$$
$$\cdot\left[X_{i_1\cdots i_k}B_{(i_1\cdots i_k)}+\sum_{l=1}^{r-k}\sum_{T_l\subset R-I_k}X_{i_1\cdots i_k t_1\cdots t_l}A^{(t_1\cdots t_l)}_{(i_1\cdots i_k)}+X_0A^{(t_1\cdots t_{r-k})}_{(i_1\cdots i_k)}/n_0\right].$$

Thus the condition is expressed by the following equations, which are the generalization of (3.11)

$$(4.27)\quad X_0\sigma_0=1,$$

$$(4.28)\quad X_{12\cdots r}B_{(12\cdots r)}=-X_0\sigma_{12\cdots r},$$

$$(4.29)\quad X_{i_1\cdots i_k}B_{(i_1\cdots i_k)}=-\sum_{\beta=1}^{r-k}\sum_{T_\beta\subset R-I_k}X_{i_1\cdots i_k t_1\cdots t_\beta}A^{(t_1\cdots t_\beta)}_{(i_1\cdots i_k)}-X_0A^{(t_1\cdots t_{r-k})}_{(i_1\cdots i_k)}/n_0,$$

$$(4.30)\quad X_E B=-\sum_{k=1}^{r}\sum_{I_k R}X_{i_1\cdots i_k}A^{(i_1\cdots i_k)}.$$

The proof of this theorem is completed, it is obvious, by proving the following:

LEMMA 4.3. *The solutions of the equations* (4.27),……, (4.30) *are given by* (4.20),……, (4.23).

PROOF. (4.20) comes from (4.27) directly and (4.21) comes from (4.20) and (4.28). (4.22) is obtained by mathematical induction in k and (4.20) and (4.21), which is as follows. At the first stage, we shall prove (4.22) holds true for all $(i_1, i_2 \cdots\cdots, i_k) = I_k \subset R$ when $k=r-1$. Then we shall prove, assuming that this holds true for all $I_k \subset R$ when $k=r-q, r-q+1, \cdots\cdots, r-1$, this also holds true for all $I_k \subset R$ when $k=r-q-1$.

The equations to be solved in the first stage is

$$(4.31)\quad X_{i_1\cdots i_{r-1}} B_{(i_1\cdots i_{r-1})}$$
$$= -X_{12\cdots r} A^{(i_r)}_{(i_1\cdots i_{r-1})} - X_0 A^{(i_r)}_{(i_1\cdots i_{r-1})}/n_0$$
$$= -\left[X_{12\cdots r} + \frac{X_0}{n_0}\right] A^{(i_r)}_{(i_1\cdots i_{r-1})}$$
$$= -\left[\frac{1}{n_0}\left\{\frac{1}{B_{(12\cdots r)}} - \frac{1}{\sigma_0}\right\} + \frac{1}{n_0\sigma_0}\right] \frac{1}{n_{i_r}} \left\{A_{(i_1\cdots i_{r-1})} - A_{(12\cdots r)}\right\}$$
$$= -\frac{1}{n_0 n_{i_r}} \left\{\frac{B_{(i_1\cdots i_{r-1})} - B_{(12\cdots r)}}{B_{(12\cdots r)}}\right\} \qquad (I_{r-1} \subset R).$$

Hence we have

$$(4.32)\quad X_{i_1\cdots i_{r-1}} = \frac{1}{n_0 n_{i_r}} \left\{\frac{-1}{B_{(12\cdots r)}} + \frac{1}{B_{(i_1\cdots i_{r-1})}}\right\} \qquad (I_{r-1} \subset R).$$

which completes the first stage.

For the proof of the second stage, at first we observe in view of the assumptions for the mathematical induction,

$$(4.33)\quad X_{i_1\cdots i_{r-q-1}} B_{(i_1\cdots i_{r-q-1})}$$
$$= -\sum_{\beta=1}^{q+1} \sum_{T\beta \subset R - I_{r-q-1}} X_{i_1\cdots i_{r-q-1} t_1 \cdots t_\beta} A^{(t_1\cdots t_\beta)}_{(i_1\cdots i_{r-q-1})} - \frac{1}{n_0\sigma_0}\left[n_0\sigma_0 X_{12\cdots r} + 1\right] A^{(t_1\cdots t_{q+1})}_{(i_1\cdots i_{r-q-1})}$$
$$= \frac{1}{N_{r-q-1}} \left[-\sum_{\beta=1}^{q+1} \sum_{T\beta \subset R - I_{r-q-1}} \left\{ \frac{(-1)^{q+1-\beta}}{B_{(12\cdots r)}} + \sum_{\alpha=0}^{q-\beta} \sum_{S\alpha \subset R - (I_{r-q-1} \cup T\beta)} \frac{(-1)^\alpha}{B_{(i_1\cdots i_{r-q-1} t_1 \cdots t_\beta s_1 \cdots s_\alpha)}} \right\} \right.$$
$$\left. \cdot \sum_{p=0}^{\beta} \sum_{Lp \subset T\beta} (-1)^p A_{(i_1\cdots i_{r-q-1} l_1 \cdots l_p)} - \frac{1}{B_{(12\cdots r)}} \sum_{p=0}^{q+1} \sum_{Lp \subset R - I_{r-q-1}} (-1)^p A_{(i_1\cdots i_{r-q-1} l_1 \cdots l_p)} \right],$$

where

$$(4.34)\quad N_{r-q-1} = \prod_{j=0}^{r} n_j^{1-\delta^j_{i_1\cdots i_{r-q-1}}},$$

Now, putting $C_{(p)} = A_{(i_1\cdots i_{r-q-1} l_1 \cdots l_p)}$, the coefficient to $\frac{1}{B_{(12\cdots r)}}$ is given by

$$(4.35)\quad -\sum_{\beta=1}^{q} \sum_{T\beta \subset R - I_{r-q-1}} \left\{ (-1)^{q+1-\beta} \sum_{p=0}^{\beta} \sum_{Lp \subset T\beta} (-1)^p A_{(i_1\cdots i_{r-q-1} l_1 \cdots l_p)} \right\}$$
$$- \sum_{p=0}^{q+1} \sum_{Lp \subset R - I_{r-q-1}} (-1)^p A_{(i_1\cdots i_{r-q-1} l_1 \cdots l_p)}$$
$$= -\sum_{\beta=1}^{q+1} \sum_{T\beta \subset R - I_{r-q-1}} \left\{ (-1)^{q+1-\beta} \sum_{p=0}^{\beta} \sum_{Lp \subset T\beta} (-1)^p A_{(i_1\cdots i_{r-q-1} l_1 \cdots l_p)} \right\}$$
$$= -\left[\sum_{T_1 \subset R - I_{r-q-1}} (-1)^q \sum_{p=0}^{1} \sum_{Lp \subset T_1} C_{(p)} + \sum_{T_2 \subset R - I_{r-q-1}} (-1)^{q-1} \sum_{p=0}^{2} \sum_{Lp \supset T_2} C_{(p)} \right.$$

$$+\cdots\cdots+\sum_{T_q\subset R-I_{r-q-1}}(-1)\sum_{p=0}^{q}\sum_{L_p\subset T_q}C_{(p)}+\sum_{T_{q+1}\subset R-I_{r-q-1}}(-1)^0\sum_{p=0}^{q+1}\sum_{L_p\subset T_{q+1}}C_{(p)}\Big]$$

except for the coefficient $\frac{1}{N_{r-q-1}}$.

The coefficient of $C_{(0)}$ in (4.35) is

$$(4.36)\quad -\Big[(-1)^q{}_{q+1}C_1+(-1)^{q-1}{}_{q+1}C_2+\cdots\cdots+(-1)_{q+1}C_q+{}_{q+1}C_{q+1}\Big]C_{(0)}$$
$$=(-1)^{q+1}A_{(i_1\cdots i_{r-q-1})}$$

and the partial sum of (4.35) for $1\leqq h\leqq q$ is given by

$$(4.37)\quad -\Big[\sum_{T_h\subset R-I_{r-q-1}}(-1)^{q-h+1}\sum_{L_h\subset T_h}C_{(h)}+\sum_{T_{h+1}\supset R-I_{r-q-1}}(-1)^{q-h}\sum_{L_h\subset T_h}C_{(h)}$$
$$+\cdots\cdots+\sum_{T_q\subset R-I_{r-q-1}}(-1)\sum_{L_h\subset T_q}C_{(h)}+\sum_{T_{q+1}\subset R-I_{r-q-1}}(-1)^0\sum_{L_h\subset T_{q+1}}C_{(h)}\Big]$$
$$=-\Big[(-1)^{q-h+1}{}_{q-h+1}C_0+(-1)^{q-h}{}_{q-h+1}C_1+(-1)^{q-h-1}{}_{q-h+1}C_2$$
$$+\cdots\cdots+{}_{q-h+1}C_{q-h+1}\Big]\sum_{L_h\subset R-I_{r-q-1}}C_{(h)}=0$$

and finally that of $C_{(q+1)}$ is

$$(4.38)\quad -\sum_{L_{q+1}\subset R-I_{r-q-1}}(-1)^{q+1}A_{(i_1\cdots i_{r-q-1}l_1\cdots l_{q+1})}=-(-1)^{q+1}A_{(12\cdots r)}$$

and hence (4.33) is composed of

$$(4.39)\quad \frac{(-1)^{q+1}}{B_{(12\cdots r)}}\Big[A_{(i_1\cdots i_{r-q-1})}-A_{(12\cdots r)}\Big]$$

and other remaining terms, among which the partial sum for $\alpha+\beta=c$ is given by

$$(4.40)\quad -\sum_{\beta=1}^{c}\sum_{T_\beta\subset R-I_{r-q-1}}\Big\{\sum_{\alpha=0}^{c-\beta}\sum_{S_\alpha\subset R-(I_{r-q-1}\cup T_\beta)}\frac{(-1)^\alpha}{B_{(i_1\cdots i_{r-q-1}t_1\cdots t_\beta s_1\cdots s_\alpha)}}$$
$$\cdot\sum_{p=0}^{\beta}\sum_{L_p\subset T_\beta}(-1)^pA_{(i_1\cdots i_{r-q-1}l_1\cdots l_p)}\Big\}.$$

This is divided into three parts, the sum for $p=0$, the sum for $p=h$ $(1\leqq h\leqq c-1)$ and the sum for $p=c$. These are evaluated in (4.43) (4.44) and (4.45) respectively, where some cumbersome considerations about the number of combinations are needed in simplifying the notation of summation, and the notations

$$(4.41)\quad D_{(t_1\cdots t_\beta s_1\cdots s_{c-\beta})}=\frac{A_{(i_1\cdots i_{r-q-1})}}{B_{(i_1\cdots i_{r-q-1}t_1\cdots t_\beta s_1\cdots s_{c-\beta})}}$$

$$(4.42)\quad E_{(t_1\cdots t_h l_1\cdots l_h s_1\cdots s_{c-h})}=\frac{A_{(i_1\cdots i_{r-q-1}l_1\cdots l_h)}}{B_{(i_1\cdots i_{r-q-1}t_1\cdots t_h s_1\cdots s_{c-h})}}$$

are used.

$$(4.43)\quad -\sum_{\beta=1}^{c}\sum_{T\beta\subset R-I_{r-q-1}}\left\{\sum_{\alpha=0}^{c-\beta}\sum_{S\alpha\subset R-(I_{r-q-1}\cup T\beta)}\frac{(-1)^{\alpha}A_{(i_1\cdots i_{r-q-1})}}{B_{(i_1\cdots i_{r-q-1}t_1\cdots t_\beta s_1\cdots s_\alpha)}}\right\}$$

$$=-\sum_{\beta=1}^{c}\sum_{T\beta\subset R-I_{r-q-1}}\sum_{Sc\text{-}\beta\subset R-(I_{r-q-1}\cup T\beta)}(-1)^{c-\beta}D_{(t_1\cdots t_\beta s_1\cdots s_{c-\beta})}$$

$$=-\sum_{Tc\subset R-I_{r-q-1}}(-1)^0D_{(t_1\cdots t_c)}-\sum_{Tc\text{-}1\subset R-I_{r-q-1}}\sum_{S1\subset R-(I_{r-q-1}\cup Tc\text{-}1)}(-1)^1D_{(t_1\cdots t_{c-1}s_1)}$$

$$-\cdots\cdots\cdots-\sum_{T1\subset R-I_{r-q-1}}\sum_{Sc\text{-}1\subset R-(I_{r-q-1}\cup T1)}(-1)^{c-1}D_{(t_1s_1\cdots s_{c-1})}$$

$$=-\left[{}_cC_0(-1)^0+{}_cC_1(-1)^1+\cdots\cdots+{}_cC_{c-1}(-1)^{c-1}\right]\sum_{Tc\subset R-I_{r-q-1}}D_{(t_1\cdots t_c)}$$

$$=(-1)^c\sum_{Tc\subset R-I_{r-q-1}}D_{(t_1\cdots t_c)}$$

$$=(-1)^c\sum_{Tc\subset R-I_{r-q-1}}\frac{A_{(i_1\cdots i_{r-q-1})}}{B_{(i_1\cdots i_{r-q-1}t_1\cdots t_c)}}\,.$$

$$(4.44)\quad -\sum_{Th\subset R-I_{r-q-1}}\sum_{Sc\text{-}h\subset R-(I_{r-q-1}\cup Th)}\sum_{Lh\subset Th}(-1)^cE_{(t_1\cdots t_hl_1\cdots l_hs_1\cdots s_{c-h})}$$

$$-\sum_{Th+1\subset R-I_{r-q-1}}\sum_{Sc\text{-}h\subset R-(I_{r-q-1}\cup Th+1)}\sum_{Lh\subset Th+1}(-1)^{c-1}E_{(t_1\cdots t_{h+1}l_1\cdots l_hs_1\cdots s_{c-h-1})}$$

$$-\sum_{Th+2\subset R-I_{r-q-1}}\sum_{Sc\text{-}h\subset R-(I_{r-q-1}\subset Th+2)}\sum_{Lh\subset Th+2}(-1)^{c-2}E_{(t_1\cdots t_{h+2}l_1\cdots l_hs_1\cdots s_{c-h-2})}$$

$$-\cdots\cdots\cdots\cdots\cdots\cdots$$

$$-\sum_{Tc\subset R-I_{r-q-1}}\sum_{Lh\subset Tc}(-1)^hE_{(t_1\cdots t_cl_1\cdots l_h)}$$

$$=-\left[\sum_{j=0}^{h}(-1)^{c-j}{}_{c-h}C_j\right]\sum_{Th\subset R-I_{r-q-1}}\sum_{Sc-h\subset R-(I_{r-q-1}\cup Th)}\sum_{Lh\subset Th}E_{(t_1\cdots t_hl_1\cdots l_hs_1\cdots s_{c-h})}$$

$$=0.$$

$$(4.45)\quad -\sum_{Tc\subset R-I_{r-q-1}}\frac{(-1)^cA_{(i_1\cdots i_{r-q-1}t_1\cdots t_c)}}{B_{(i_1\cdots i_{r-q-1}t_1\cdots t_c)}}\,.$$

Now (4.33) is simplified to

$$(4.46)\quad X_{i_1\cdots i_{r-q-1}}B_{(i_1\cdots i_{r-q-1})}=\frac{1}{N_{r-q-1}}\left[\frac{(-1)^{q+1}}{B_{(12\cdots r)}}\left\{A_{(i_1\cdots i_{r-q-1})}-A_{(12\cdots r)}\right\}\right.$$
$$\left.+\sum_{c=1}^{q}\sum_{Tc\subset R-I_{r-q-1}}(-1)^c\left\{\frac{A_{(i_1\cdots i_{r-q-1})}-A_{(i_1\cdots i_{r-q-1}t_1\cdots t_c)}}{B_{(i_1\cdots i_{r-q-1}t_1\cdots t_c)}}\right\}\right]$$

and we have

$$(4.47)\quad X_{i_1\cdots i_{r-q-1}}=\frac{1}{N_{r-q-1}}\left[(-1)^{q+1}\left\{\frac{1}{B_{(12\cdots r)}}-\frac{1}{B_{(i_1\cdots i_{r-q-1})}}\right\}\right.$$
$$\left.+\sum_{c=1}^{q}\sum_{Tc\subset R-I_{r-q-1}}(-1)^c\left\{\frac{1}{B_{(i_1\cdots i_{r-q-1}t_1\cdots t_c)}}-\frac{1}{B_{(i_1\cdots i_{r-q-1})}}\right\}\right]$$
$$=\frac{1}{N_{r-q-1}}\left[\frac{(-1)^{q+1}}{B_{(12\cdots r)}}-\frac{(-1)^{q+1}}{B_{(i_1\cdots i_{r-q-1})}}+\sum_{c=1}^{q}\sum_{Tc\subset R-I_{r-q-1}}(-1)^c\frac{1}{B_{(i_1\cdots i_{r-q-1}t_1\cdots t_c)}}\right.$$
$$\left.-\sum_{c=1}^{q}\sum_{Tc\subset R-I_{r-q-1}}(-1)^c\frac{1}{B_{(i_1\cdots i_{r-q-1})}}\right]$$

$$=\frac{1}{N_{r-q-1}}\left[\frac{(-1)^{q+1}}{B_{(12\cdots r)}}+\frac{1}{B_{(i_1\cdots i_{r-q-1})}}+\sum_{c=1}^{q}\sum_{T_c\subset R-I_{r-q-1}}(-1)^c\frac{1}{B_{(i_1\cdots i_{r-q-1}t_1\cdots t_c)}}\right]$$

$$=\frac{1}{N_{r-q-1}}\left[\frac{(-1)^{q+1}}{B_{(12\cdots r)}}+\sum_{c=0}^{q}\sum_{T_c\subset R-I_{r-q-1}}\frac{(-1)^c}{B_{(i_1\cdots i_{r-q-1}t_1\cdots t_c)}}\right].$$

Thus we have proved that the solution of (4.29) is given by (4.22).

Finally (4.23) is obtained by inserting (4.20), (4.21) and (4.22) in (4.30) in the following way.

After inserting them in (4.30), we have

$$(4.48)\quad X_E B=-\sum_{k=1}^{r-1}\sum_{I_k\subset R}\left[\frac{1}{\prod_{j=0}^{r} n_j^{1-\delta^j_{i_1\cdots i_k}}}\left\{\frac{(-1)^{r-k}}{B_{(12\cdots r)}}+\sum_{\alpha=0}^{r-k-1}\sum_{S_\alpha\subset R-I_k}\frac{(-1)^\alpha}{B_{(i_1\cdots i_k s_1\cdots s_\alpha)}}\right\}\right.$$
$$\left.\cdot\frac{1}{n_{i_1}\cdots n_{i_k}}\sum_{p=0}^{k}\sum_{L_p\subset I_k}(-1)^p A_{(l_1\cdots l_p)}\right]$$
$$=-\frac{1}{\prod_{j=0}^{r} n_j}\left[\sum_{k=1}^{r-1}\sum_{I_k\subset R}\frac{(-1)^{r-k}}{B_{(12\cdots r)}}\sum_{p=0}^{k}\sum_{L_p\subset I_k}(-1)^p A_{(l_1\cdots l_p)}\right]$$
$$-\frac{1}{\prod_{j=0}^{r} n_j}\left[\sum_{k=1}^{r-1}\sum_{I_k\subset R}\sum_{\alpha=0}^{r-k-1}\sum_{S_\alpha\subset R-I_k}\frac{(-1)^\alpha}{B_{(i_1\cdots i_k s_1\cdots s_\alpha)}}\sum_{p=0}^{k}\sum_{L_p\subset I_k}(-1)^p A_{(l_1\cdots l_p)}\right].$$

By writing $G_{(p)}=A_{(l_1\cdots l_p)}/B_{(12\cdots r)}$ the first term is equal to

$$(4.49)\quad -\sum_{k=1}^{r-1}\sum_{I_k\subset R}\sum_{p=0}^{k}\sum_{L_p\subset I_k}\frac{(-1)^{r-k+p}A_{(l_1\cdots l_p)}}{B_{(12\cdots r)}}$$
$$=-\sum_{I_1\subset R}\sum_{p=0}^{1}\sum_{L_p\subset I_1}(-1)^{r-1+p}G_{(p)}-\sum_{I_2\subset R}\sum_{p=0}^{2}\sum_{L_p\subset I_2}(-1)^{r-2+p}G_{(p)}$$
$$-\cdots\cdots-\sum_{I_{r-1}\subset R}\sum_{p=0}^{r-1}\sum_{L_p\subset I_{r-1}}(-1)^{r-(r-1)+p}G_{(p)}.$$

The sum for $p=0$ and the sum for $p=h$ $(1\leqq h\leqq r-1)$ are given by (4.50) and (4.51) respectively.

$$(4.50)\quad -\sum_{j=1}^{r-1}{}_rC_j(-1)^{r-j}G_{(0)}=-\left[\sum_{j=0}^{r}{}_rC_{r-j}(-1)^{r-j}-{}_rC_r(-1)^r-{}_rC_0\right]G_{(0)}$$
$$=-\left[-(-1)^r-1\right]G_{(0)}=\left\{(-1)^r+1\right\}\frac{A}{B_{(12\cdots r)}}.$$

$$(4.51)\quad -\sum_{I_h\subset R}\sum_{L_h\subset I_h}(-1)^r G_{(h)}-\sum_{I_{h+1}\subset R}\sum_{L_h\subset I_{r+1}}(-1)^{r-1}G_{(h)}$$
$$-\cdots\cdots-\sum_{I_{r-1}\subset R}\sum_{L_h\subset I_{r-1}}(-1)^{h+1}G^{(h)}$$
$$=-\left[\sum_{j=0}^{r-h-1}(-1)^{r-j}{}_{r-h}C_j\right]\sum_{I_h\subset R}\sum_{L_h\subset I_h}G_{(h)}=(-1)^h\sum_{I_h\subset R}\frac{A_{(i_1\cdots i_h)}}{B_{(12\cdots r)}}.$$

On the other hand the second term in (4.48) is equal to

$$(4.52)\quad -\sum_{c=1}^{r-1}\sum_{k=1}^{c}\sum_{I_k\subset R}\sum_{S_{c-k}\subset R-I_k}\frac{(-1)^{c-k}}{B_{(i_1\cdots i_k s_1\cdots s_{c-k})}}\sum_{p=0}^{k}\sum_{L_p\subset I_k}(-1)^p A_{(l_1\cdots l_p)}$$

The sum for $p=0$, $p=h\ (1\leq h\leq c-1)$ and $p=c$ are given by

$$(4.53)\quad -\sum_{I_1\subset R}\sum_{S_{c-1}\subset R-I_1}\frac{(-1)^{c-1}A}{B_{(i_1s_1\cdots s_{c-1})}}-\sum_{I_2\subset R}\sum_{S_{c-2}\subset R-I_2}\frac{(-1)^{c-2}A}{B_{(i_1i_2s_1\cdots s_{c-2})}}$$

$$-\cdots\cdots-\sum_{I_c\subset R}\frac{(-1)^0A}{B_{(i_1\cdots i_c)}}$$

$$=(-1)^c\sum_{I_c\subset R}\frac{A}{B_{(i_1\cdots i_c)}}.$$

$$(4.54)\quad -\sum_{I_h\subset R}\sum_{S_{c-h}\subset R-I_h}\frac{(-1)^{c-h}}{B_{(i_1\cdots i_hs_1\cdots s_{c-h})}}\sum_{L_h\subset I_h}A_{(l_1\cdots l_h)}$$

$$-\sum_{I_{h+1}\subset R}\sum_{S_{c-h-1}\subset R-I_{h+1}}\frac{(-1)^{c-h-1}}{B_{(i_1\cdots i_{h+1}s_1\cdots s_{c-h-1})}}\sum_{L_h\subset I_{h+1}}A_{(l_1\cdots l_h)}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$-\sum_{I_c\subset R}\frac{(-1)^0}{B_{(i_1\cdots i_c)}}\sum_{L_h\subset I_c}A_{(l_1\cdots l_h)}$$

$$=-\left[\sum_{j=0}^{c-h}(-1)^{c-h-j}{}_{c-h}C_j\right]\sum_{L_h\subset R}\sum_{S_{c-h}\subset R-L_h}\frac{(-1)^{c-h}A_{(l_1\cdots l_h)}}{B_{(l_1\cdots l_hs_1\cdots s_{c-h})}}$$

$$=0.$$

$$(4.55)\quad -\left[\sum_{I_c\subset R}\frac{(-1)^0}{B_{(i_1\cdots i_c)}}\sum_{L_c\subset I_c}(-1)^cA_{(l_1\cdots l_c)}\right]$$

$$=-\sum_{I_c\subset R}\frac{(-1)^cA_{(i_1\cdots i_c)}}{B_{(i_1\cdots i_c)}}.$$

The combination of (4.48), ……, (4.55) yields

$$(4.56)\quad X_EB=\frac{1}{\prod_{j=0}^{r}n_j}\left[\frac{\{(-1)^r+1\}A}{B_{(12\cdots r)}}+\sum_{h=1}^{r-1}\sum_{I_h\subset R}\frac{(-1)^hA_{(i_1\cdots i_h)}}{B_{(12\cdots r)}}\right.$$

$$\left.+\sum_{c=1}^{r-1}\sum_{I_c\subset R}\frac{(-1)^cA}{B_{(i_1\cdots i_c)}}-\sum_{c=1}^{r-1}\sum_{I_c\subset R}\frac{(-1)^cA_{(i_1\cdots i_c)}}{B_{(i_1\cdots i_c)}}\right].$$

On the other hand $A^{(12\cdots r)}$ is the sum over the null index set and is equal to 0, and Lemma 4.2 should holds true even if $T_\alpha=(t_1,\cdots\cdots,t_\alpha)$ is the null set, and we have

$$(4.57)\quad A^{(12\cdots r)}=\frac{1}{\prod_{j=0}^{r}n_j}\sum_{h=0}^{r}\sum_{I_h\subset R}(-1)^hA_{(i_1\cdots i_h)}$$

$$=\frac{1}{\prod_{j=0}^{r}n_j}\left[A+\sum_{h=1}^{r-1}\sum_{I_h\subset R}(-1)^hA_{(i_1\cdots i_h)}+(-1)^rA_{(12\cdots r)}\right]=0$$

This is equivalent to

$$(4.58)\quad \sum_{h=1}^{r-1}\sum_{I_h\subset R}(-1)^hA_{(i_1\cdots i_h)}=-\{A+(-1)^rA_{(12\cdots r)}\}$$

Inserting (4.58) in (4.56) we have finally

$$(4.59)\quad X_E=\frac{1}{\prod_{j=0}^{r} n_jB}\Big[\frac{\{(-1)^r+1\}A}{B_{(12\cdots r)}}-\frac{A+(-1)^rA_{(12\cdots r)}}{B_{(12\cdots r)}}$$

$$+\sum_{c=1}^{r-1}\sum_{I_c\subset R}\frac{(-1)^cA}{B_{(i_1\cdots i_c)}}-\sum_{c=1}^{r-1}\sum_{I_c\subset R}\frac{(-1)^cA_{(i_1\cdots i_c)}}{B_{(i_1\cdots i_c)}}\Big]$$

$$=\frac{1}{\prod_{j=0}^{r} n_jB}\Big[\frac{(-1)^rA}{B_{(12\cdots r)}}-\frac{(-1)^rA_{(12\cdots r)}}{B_{(12\cdots r)}}$$

$$+\sum_{c=1}^{r-1}\sum_{I_c\subset R}\frac{(-1)^cA}{B_{(i_1\cdots i_c)}}-\sum_{c=1}^{r-1}\sum_{I_c\subset R}\frac{(-1)^cA_{(i_1\cdots i_c)}}{B_{(i_1\cdots i_c)}}$$

$$=\frac{1}{\prod_{j=0}^{r} n_j}\Big[\frac{(-1)^r}{B_{(12\cdots r)}}-\frac{(-1)^r}{B}+\sum_{c=1}^{r-1}\sum_{I_c\subset R}\frac{(-1)^c}{B_{(i_1\cdots i_c)}}-\sum_{c=1}^{r-1}\sum_{I_c\subset R}\frac{(-1)^c}{B}\Big]$$

$$=\frac{1}{\prod_{j=0}^{r} n_j}\Big[\frac{(-1)^r}{B_{(12\cdots r)}}+\frac{1}{B}+\sum_{c=1}^{r-1}\sum_{I_c\subset R}\frac{(-1)^c}{B_{(i_1\cdots i_c)}}\Big].$$

Thus we have completed the proof of this lemma and also Theorem 4.2.

4.3 *The joint density function.* We have derived the determinant and the inverse of the variance matrix, and what we have to do is to derive the joint density function as the generalization of (3.17), which is ennunciated in

THEOREM 4.3. The joint density function of $x_{t_0t_1\cdots t_r}$ is given by

$$(4.60)\quad f(\mathbf{X})=(2\pi)^{-n_0n_1\cdots n_r/2}B^{-1/2}\prod_{k=1}^{r}\prod_{I_k\subset R}\{B_{(i_1\cdots i_k)}\}^{-(n_{i_1}-1)(n_{i_2}-1)\cdots(n_{i_k}-1)/2}\sigma_0^{-(n_0-1)n_1\cdots n_r/2}$$

$$\cdot exp\Big[-\frac{1}{2}\Big\{\prod_{j=0}^{r}n_j(\bar{X}-\mu)^2\frac{1}{B}+\sum_{k=1}^{r}\sum_{I_k\subset R}\frac{S_{(i_1\cdots i_k)}}{B_{(i_1\cdots i_k)}}+\frac{S_0}{\sigma_0}\Big\}\Big],$$

where

$$(4.61)\quad \bar{X}_{t_{l_1}\cdots t_{l_\beta}}=\frac{1}{\prod_{j=0}^{r} n_j^{1-\delta^j_{i_1\cdots i_k}}}\sum_{\substack{t_{j_1},\cdots,t_{j_{r-\beta}}\\ J_{r-\beta}\subset R-L_\beta}}\sum_{t_0}x_{t_0t_1\cdots t_r}\qquad (L_\beta\subset R,\ \beta=1,\cdots\cdots,r),$$

$$(4.62)\quad \bar{X}=\frac{1}{\prod_{j=0}^{r} n_j}\sum_{t_0,t_1,\cdots,t_r}x_{t_0t_1\cdots t_r},$$

$$(4.63)\quad S_{(i_1\cdots i_k)}=\prod_{j=0}^{r}n_j^{1-\delta^j_{i_1\cdots i_k}}\sum_{t_{i_1},\cdots,t_{i_k}}\Big\{\sum_{\beta=0}^{k}\sum_{L_\beta\subset I_k}(-1)^{k-\beta}\bar{X}_{t_{l_1}\cdots t_{l_\beta}}\Big\}^2,$$

$$(4.64)\quad S_0=\sum_{t_0,t_1,\cdots,t_r}(x_{t_0t_1\cdots t_r}-\bar{X}_{t_1\cdots t_r})^2.$$

PROOF. In this proof we shall use the convention that if $(t_{l_1},\cdots\cdots,t_{l_\beta})$ is the null set $\bar{X}_{t_{l_1}\cdots t_{l_\beta}}=\bar{X}$. As the density function should be the multivariate normal density function, the constant factor in (4.60) is easily derived from Theorem 4.2, and there remains only to derive the quadratic form of $x_{t_0t_1\cdots t_r}$.

Now let us introduce new variables defined by

(4.65) $u_{t_0 t_1 \cdots t_r} = x_{t_0 t_1 \cdots t_r} - \mu,$

(4.66) $U_{t_{i_1} \cdots t_{i_k}} = \sum_{\substack{t_{j_1}, \cdots, t_{j_{r-k}} \\ J_{r-k} \subset R - I_k}} \sum_{t_0} u_{t_0 t_1 \cdots t_r},$

(4.67) $U = \sum_{t_0, t_1, \cdots, t_r} u_{t_0 t_1 \cdots t_r} = \prod_{j=0}^{r} n_j \bar{U},$

(4.68) $\bar{U}_{t_{i_1} \cdots t_{i_k}} = \frac{1}{\prod_{j=0}^{r} n_j^{1-\delta^j_{i_1 \cdots i_k}}} U_{t_{i_1} \cdots t_{i_k}},$

and we shall use the convention for $\bar{U}_{t_{l_1} \cdots t_{l_\beta}}$ same to that we have made for $\bar{X}_{t_{l_1} \cdots t_{l_\beta}}$ at the beginning of the proof.

As the inverse matrix has already been derived in Theorem 4.2., the density should be written as *exp* $[-S/2]$ except for the constant factor, where

(4.69)
$$S = X_E \left(\sum_{t_0, t_1, \cdots, t_r} u_{t_0 t_1 \cdots t_r} \right)^2 + \sum_{k=1}^{r-1} \sum_{I_k \subset R} X_{i_1 \cdots i_k} \left\{ \sum_{t_{i_1}, \cdots, t_{i_k}} \left(\sum_{\substack{t_{j_1}, \cdots, t_{j_{r-k}} \\ J_{r-k} \subset R - I_k}} \sum_{t_0} u_{t_0 t_1 \cdots t_r} \right)^2 \right\}$$
$$+ X_{12 \cdots r} \sum_{t_1, \cdots, t_r} \left(\sum_{t_0} u_{t_0 t_1 \cdots t_r} \right)^2 + X_0 \sum_{t_0, t_1, \cdots, t_r} u^2_{t_0 t_1 \cdots t_r}$$
$$= X_E U^2 + \sum_{k=1}^{r-1} \sum_{I_k \subset R} \left\{ X_{i_1 \cdots i_k} \sum_{t_{i_1}, \cdots, t_{i_k}} U^2_{t_{i_1} \cdots t_{i_k}} \right\}$$
$$+ X_{12 \cdots r} \sum_{t_1, \cdots, t_r} U^2_{t_1 \cdots t_r} + X_0 \sum_{t_0, t_1, \cdots, t_r} u^2_{t_0 t_1 \cdots t_r}$$
$$= U^2 \frac{1}{\prod_{j=0}^{r} n_j} \left[\frac{(-1)^r}{B_{(12 \cdots r)}} + \frac{1}{B} + \sum_{c=1}^{r-1} \sum_{I_c \subset R} \frac{(-1)^c}{B_{(i_1 \cdots i_c)}} \right]$$
$$+ \sum_{k=1}^{r-1} \sum_{I_k \subset R} \left(\sum_{t_{i_1}, \cdots, t_{i_k}} U^2_{t_{i_1} \cdots t_{i_k}} \right) \frac{1}{\prod_{j=0}^{r} n_j^{1-\delta^j_{i_1 \cdots i_k}}} \left[\frac{(-1)^{r-k}}{B_{(12 \cdots r)}} + \sum_{\alpha=0}^{r-k-1} \sum_{S_\alpha \subset R - I_k} \frac{(-1)^\alpha}{B_{(i_1 \cdots i_k s_1 \cdots s_\alpha)}} \right]$$
$$+ \sum_{t_1, \cdots, t_r} U^2_{t_1 \cdots t_r} \frac{1}{n_0} \left[\frac{1}{B_{(12 \cdots r)}} - \frac{1}{\sigma_0} \right] + \frac{1}{\sigma_0} \sum_{t_0, t_1, \cdots, t_r} u^2_{t_0 t_1 \cdots t_r}.$$

After evaluating the coefficient to $\frac{1}{B_{(i_1 \cdots i_c)}}$ and $\frac{1}{B_{(12 \cdots r)}}$ in (4.69) which are given by

(4.70)
$$D_{(i_1 \cdots i_c)} = U^2 \frac{1}{\prod_{j=0}^{r} n_j} (-1)^c$$
$$+ \sum_{\beta=1}^{c} \sum_{L_\beta \subset I_c} \left(\sum_{t_{l_1}, \cdots, t_{l_\beta}} U^2_{t_{l_1} \cdots t_{l_\beta}} \right) \frac{1}{\prod_{j=0}^{r} n_j^{1-\delta^j_{l_1 \cdots l_\beta}}} (-1)^{c-\beta}$$
$$= \sum_{\beta=0}^{c} \sum_{L_\beta \subset I_c} \frac{(-1)^{c-\beta}}{\prod_{j=0}^{r} n_j^{1-\delta^j_{l_1 \cdots l_\beta}}} \left(\sum_{t_{l_1}, \cdots, t_{l_\beta}} U^2_{t_{l_1} \cdots t_{l_\beta}} \right)$$

and

$$(4.71)\quad D_{(12\cdots r)}=U^2\frac{1}{\prod_{j=0}^{r} n_j}(-1)^r + \sum_{k=1}^{r-1}\sum_{I_k\subset R}\Big(\sum_{t_{i1},\cdots,t_{ik}} U^2_{t_{i1}\cdots t_{ik}}\Big)\frac{(-1)^{r-k}}{\prod_{j=0}^{r} n_j^{1-\delta^j_{i_1\cdots i_k}}} + \sum_{t_1,\cdots,t_r} U^2_{t_1\cdots t_r}\frac{1}{n_0}$$

$$=\sum_{k=0}^{r}\sum_{I_k\subset R}\frac{(-1)^{r-k}}{\prod_{j=0}^{r} n_j^{1-\delta^j_{i_1\cdots i_k}}}\Big(\sum_{t_{i1},\cdots,t_{ik}} U^2_{t_{i1}\cdots t_{ik}}\Big)$$

respectively, we have (4.69) is equal to

$$(4.72)\quad S=\sum_{c=1}^{r-1}\sum_{I_c\subset R}\frac{D_{(i_1\cdots i_c)}}{B_{(i_1\cdots i_c)}}+\frac{D_{(12\cdots r)}}{B_{(12\cdots r)}}+U^2\frac{1}{\prod_{j=0}^{r} n_j}\cdot\frac{1}{B} + \Big[\sum_{t_0,t_1,\cdots,t_r} u^2_{t_0t_1\cdots t_r}-\frac{1}{n_0}\sum_{t_1,\cdots,t_r} U^2_{t_1\cdots t_r}\Big]\frac{1}{\sigma_0}$$

$$=\sum_{k=1}^{r}\sum_{I_k\subset R}\frac{D_{(i_1\cdots i_k)}}{B_{(i_1\cdots i_k)}}+U^2\frac{1}{\prod_{j=0}^{r} n_j}\frac{1}{B} + \Big[\sum_{t_0,t_1,\cdots,t_r} u^2_{t_0t_1\cdots t_r}-\frac{1}{n_0}\sum_{t_1,\cdots,t_r} U^2_{t_1\cdots t_r}\Big]\frac{1}{\sigma_0}$$

$$=\sum_{k=1}^{r}\sum_{I_k\subset R}\sum_{\beta=0}^{k}\sum_{L_\beta\subset I_k}\frac{(-1)^{k-\beta}}{\prod_{j=0}^{r} n_j^{1-\delta^j_{l_1\cdots l_\beta}}}\Big(\sum_{t_{l1},\cdots,t_{l\beta}} U^2_{t_{l1}\cdots t_{l\beta}}\Big)\frac{1}{B_{(i_1\cdots i_k)}} + U^2\frac{1}{\prod_{j=0}^{r} n_j}\frac{1}{B}+\Big[\sum_{t_0,t_1,\cdots,t_r} u^2_{t_0t_1\cdots t_r}-\frac{1}{n_0}\sum_{t_1,\cdots,t_r} U^2_{t_1\cdots t_r}\Big]\frac{1}{\sigma_0}$$

$$=\sum_{k=1}^{r}\sum_{I_k\subset R}\sum_{\beta=0}^{k}\sum_{L_\beta\subset I_k}(-1)^{k-\beta}\prod_{j=0}^{r} n_j^{1-\delta^j_{l_1\cdots l_\beta}}\Big(\sum_{t_{l1},\cdots,t_{l\beta}} \bar U^2_{t_{l1}\cdots t_{l\beta}}\Big)\frac{1}{B_{(i_1\cdots i_k)}} + \prod_{j=0}^{r} n_j\bar U^2\frac{1}{B}+\Big[\sum_{t_0,t_1,\cdots,t_r} u^2_{t_0t_1\cdots t_r}-n_0\sum_{t_1,\cdots,t_r} \bar U^2_{t_1\cdots t_r}\Big]\frac{1}{\sigma_0}$$

$$=\sum_{k=1}^{r}\sum_{I_k\subset R}\prod_{j=0}^{r} n_j^{1-\delta^j_{i_1\cdots i_k}}\Big[\sum_{t_{i1},\cdots,t_{ik}}\sum_{\beta=0}^{k}\sum_{L_\beta\subset I_k}(-1)^{k-\beta}\bar U^2_{t_{l1}\cdots t_{l\beta}}\Big]\frac{1}{B_{(i_1\cdots i_k)}} + \prod_{j=0}^{r} n_j\bar U^2\frac{1}{B}+\Big[\sum_{t_0,t_1,\cdots t_r} u^2_{t_0t_1\cdots t_r}-n_0\sum_{t_1,\cdots,t_r} \bar U^2_{t_1\cdots t_r}\Big]\frac{1}{\sigma_0}.$$

Here we need to prove the following

LEMMA 4.4.

$$(4.73)\quad \sum_{t_{i1},\cdots,t_{ik}}\sum_{\beta=0}^{k}\sum_{L_\beta\subset I_k}(-1)^{k-\beta}\bar U^2_{t_{l1}\cdots t_{l\beta}}=\sum_{t_{i1},\cdots,t_{ik}}\Big[\sum_{\beta=0}^{k}\sum_{L_\beta\subset I_k}(-1)^{k-\beta}\bar U_{t_{l1}\cdots t_{l\beta}}\Big]^2$$

PROOF. Proof is given by making use of the mathematical induction. In case $k=1$, the proof is given by

$$(4.74)\quad \sum_{t_{i1}}\{(-1)\bar{U}^2+(-1)^0\bar{U}^2_{t_{i1}}\}=\sum_{t_{i1}}(\bar{U}^2_{t_{i1}}-\bar{U}^2)$$

$$=\sum_{t_{i1}}\bar{U}^2_{t_{i1}}-2\sum_{t_{i1}}\bar{U}^2+\sum_{t_{i1}}\bar{U}^2$$

$$=\sum_{t_{i1}}\bar{U}^2_{t_{i1}}-2\sum_{t_{i1}}\bar{U}_{t_{i1}}\bar{U}+\sum_{t_{i1}}\bar{U}^2$$

$$=\sum_{t_{i1}}(\bar{U}_{t_{i1}}-\bar{U})^2.$$

Further assuming (4.73) to be valid in case $k=h$ we have

$$(4.75)\quad \sum_{t_{i1},\cdots,t_{ih},t_{ih+1}}\sum_{\beta=0}^{h+1}\sum_{L_\beta\subset I_{h+1}}(-1)^{h+1-\beta}\bar{U}^2_{t_{l1}\cdots t_{l\beta}}$$

$$=\sum_{t_{i1}\cdots t_{ih+1}}\sum_{\beta=0}^{h}\sum_{L_\beta\subset I_h}(-1)^{h-\beta}\bar{U}^2_{t_{l1}\cdots t_{l\beta}t_{ih+1}}-\sum_{t_{i1},\cdots,t_{ih+1}}\sum_{\beta=0}^{h}\sum_{L_\beta\subset I_h}(-1)^{h-\beta}\bar{U}^2_{t_{l1}\cdots t_{l\beta}}$$

$$=\sum_{t_{ih+1}}\left[\sum_{t_{i1},\cdots,t_{ih}}\sum_{\beta=0}^{h}\sum_{L_\beta\subset I_h}(-1)^{h-\beta}\bar{U}^2_{t_{l1}\cdots t_{l\beta}t_{ih+1}}-\sum_{t_{i1},\cdots,t_{ih}}\sum_{\beta=0}^{h}\sum_{L_\beta\subset I_h}(-1)^{h-\beta}\bar{U}^2_{t_{l1}\cdots t_{l\beta}}\right]$$

$$=\sum_{t_{ih+1}}\left[\sum_{t_{i1},\cdots,t_{ih}}\left\{\sum_{\beta=0}^{h}\sum_{L_\beta\subset I_h}(-1)^{h-\beta}\bar{U}_{t_{l1}\cdots t_{l\beta}t_{ih+1}}\right\}^2-\sum_{t_{i1},\cdots,t_{ih}}\left\{\sum_{\beta=0}^{h}\sum_{L_\beta\subset I_h}(-1)^{h-\beta}\bar{U}_{t_{l1}\cdots t_{l\beta}}\right\}^2\right]$$

$$=\sum_{t_{i1},\cdots,t_{ih+1}}\left[\left\{\sum_{\beta=0}^{h}\sum_{L_\beta\subset I_h}(-1)^{h-\beta}\bar{U}_{t_{l1}\cdots t_{l\beta}t_{ih+1}}\right\}^2-\left\{\sum_{\beta=0}^{h}\sum_{L_\beta\subset I_h}(-1)^{h-\beta}\bar{U}_{t_{l1}\cdots t_{l\beta}}\right\}^2\right].$$

And by writing

$$(4.76)\quad \bar{Z}_{t_{i1}\cdots t_{ih}}=\sum_{\beta=0}^{h}\sum_{L_\beta\subset I_h}(-1)^{h-\beta}\bar{U}_{t_{l1}\cdots t_{l\beta}},$$

$$(4.77)\quad \bar{Z}_{t_{i1}\cdots t_{ih}(t_{ih+1})}=\sum_{\beta=0}^{h}\sum_{L_\beta\subset I_h}(-1)^{h-\beta}\bar{U}_{t_{l1}\cdots t_{l\beta}t_{ih+1}},$$

(4.75) is equal to

$$(4.78)\quad \sum_{t_{i1},\cdots,t_{ih+1}}\left[\bar{Z}^2_{t_{i1}\cdots t_{ih}(t_{ih+1})}-\bar{Z}^2_{t_{i1}\cdots t_{ih}}\right]$$

$$=\sum_{t_{i1},\cdots,t_{ih+1}}\left[\bar{Z}^2_{t_{i1}\cdots t_{ih}(t_{ih+1})}-2\bar{Z}^2_{t_{i1}\cdots t_{ih}}+\bar{Z}^2_{t_{i1}\cdots t_{ih}}\right]$$

$$=\sum_{t_{i1},\cdots,t_{ih+1}}\left[\bar{Z}^2_{t_{i1}\cdots t_{ih}(t_{ih+1})}-2\bar{Z}_{t_{i1}\cdots t_{ih}}\bar{Z}_{t_{i1}\cdots t_{ih}(t_{ih+1})}+\bar{Z}^2_{t_{i1}\cdots t_{ih}}\right]$$

$$=\sum_{t_{i1},\cdots,t_{ih+1}}\left[\bar{Z}_{t_{i1}\cdots t_{ih}(t_{ih+1})}-\bar{Z}_{t_{i1}\cdots t_{ih}}\right]^2$$

$$=\sum_{t_{i1},\cdots,t_{ih+1}}\left[\sum_{\beta=0}^{h}\sum_{L_\beta\subset I_h}(-1)^{h-\beta}\bar{U}_{t_{l1}\cdots t_{l\beta}t_{ih+1}}-\sum_{\beta=0}^{h}\sum_{L_\beta\subset I_h}(-1)^{h-\beta}\bar{U}_{t_{l1}\cdots t_{l\beta}}\right]^2$$

$$=\sum_{t_{i1},\cdots,t_{ih+1}}\left[\sum_{\beta=0}^{h}\sum_{L_\beta\subset I_h}(-1)^{h-\beta}\bar{U}_{t_{l1}\cdots t_{l\beta}t_{ih+1}}+\sum_{\beta=0}^{h}\sum_{L_\beta\subset I_h}(-1)^{h+1-\beta}\bar{U}_{t_{l1}\cdots t_{l\beta}}\right]^2$$

$$=\sum_{t_{i1},\cdots,t_{ih+1}}\left[\sum_{\beta=0}^{h+1}\sum_{\substack{L_\beta\subset I_{h+1}\\ L_\beta\ni i_{h+1}}}(-1)^{h-\beta+1}\bar{U}_{t_{l1}\cdots t_{l\beta}}+\sum_{\beta=0}^{h}\sum_{\substack{L_\beta\subset I_{h+1}\\ L_\beta\not\ni i_{h+1}}}(-1)^{h+1-\beta}\bar{U}_{t_{l1}\cdots t_{l\beta}}\right]^2$$

$$=\sum_{t_{i1},\cdots,t_{ih+1}}\left[\sum_{\beta=0}^{h+1}\sum_{L_\beta\subset I_{h+1}}(-1)^{h+1-\beta}\bar{U}_{t_{l1}\cdots t_{l\beta}}\right]^2$$

which completes the proof of the lemma.

By making use of this Lemma the first term in (4.72) comes to be equal to (4.79), and the second term is equal to (4.80), and finally the third is equal to (4.81);

$$(4.79)\quad \sum_{t_{i_1},\cdots,t_{i_k}}\left[\sum_{\beta=0}^{k}\sum_{L_\beta\subset I_k}(-1)^{k-\beta}\overline{U}_{t_{l_1}\cdots t_{l_\beta}}\right]^2=\sum_{t_{i_1},\cdots,t_{i_k}}\left[\sum_{\beta=0}^{k}\sum_{L_\beta\subset I_k}(-1)^{k-\beta}(\overline{X}_{t_{l_1}\cdots t_{l_\beta}}-\mu)\right]^2$$

$$=\sum_{t_{i_1},\cdots,t_{i_k}}\left[\sum_{\beta=0}^{k}\sum_{L_\beta\subset I_k}(-1)^{k-\beta}\overline{X}_{t_{l_1}\cdots t_{l_\beta}}\right]^2.$$

$$(4.80)\quad \prod_{j=0}^{r} n_j(\overline{X}-\mu)^2\frac{1}{B}.$$

$$(4.81)\quad \left[\sum_{t_0,t_1,\cdots,t_r}(u_{t_0t_1\cdots t_r}-\overline{U}_{t_1\cdots t_r})^2\right]\frac{1}{\sigma_0}$$

$$=\sum_{t_0,t_1,\cdots,t_r}(x_{t_0t_1\cdots t_r}-\overline{X}_{t_1\cdots t_r})^2\frac{1}{\sigma_0}.$$

The combination of (4.79), (4.80) and (4.81) leads us to the completion of the proof.

4.4 *Estimation.* Finally we shall treat the problem of estimation of the variance components. By the usual estimates of the variance components we mean the usual ones, which is calculated as a linear form of a suitable number of mean squares in the table of the analysis of variance and is widely used as the estimates in the ordinary practice of statistical analysis. As we have already stated in the case of the 2-way layout, the completeness of the family of distribution of the sufficient statistics in our concern is yet in question, and we shall here make use of the notion of the minimum variance estimate due to Bhattacharyya to justify the usual estimates. Thus we have,

THEOREM 4.4. *In the r-way layout of random effect model, the minimum variance estimates of the variance components* $\sigma_{i_1\cdots i_k}$ $(I_k\subset R,\ k=1,\cdots\cdots,r)$ *are given by such linear forms of* $S_{(i_1\cdots i_k)}$ *and* S_0 *in (4.63) and (4.64) that these are unbiased, namely the usual estimates of the variance components, and that of the general mean is given by the sample total mean.*

PROOF. After taking the logarithm of the density function (4.60)

$$(4.82)\quad \ln f=\mathrm{K}-\frac{1}{2}\ln B-\frac{1}{2}\sum_{k=1}^{r}\sum_{I_k\subset R}(n_{i_1}-1)(n_{i_2}-1)\cdots\cdots(n_{i_k}-1)\ln B_{(i_1\cdots i_k)}$$

$$-\frac{1}{2}(n_0-1)\,n_1n_2\cdots\cdots n_r\ln\sigma_0$$

$$-\frac{1}{2}\left[\prod_{j=0}^{r}n_j(\overline{X}-\mu)^2\frac{1}{B}+\sum_{k=1}^{r}\sum_{I_k\subset R}\frac{S_{(i_1\cdots i_k)}}{B_{(i_1\cdots i_k)}}+\frac{S_0}{\sigma_0}\right],$$

we have easily

$$(4.83)\quad B\prod_{j=0}^{r}n_j\frac{\partial f}{\partial\mu}=(\overline{X}-\mu)f,$$

$$(4.84)\quad \frac{2\sigma_0^2}{(n_0-1)n_1\cdots n_r}\frac{\partial f}{\partial\sigma_0}=\left\{\frac{S_0}{(n_0-1)n_1\cdots n_r}-\sigma_0\right\}f,$$

$$(4.85)\quad \frac{2\{B_{(i_1\cdots i_k)}\}^2}{\prod_{c=1}^{k}(n_{i_c}-1)}\frac{\partial f}{\partial B_{(i_1\cdots i_k)}}+\frac{(-1)^k\{B_{(i_1\cdots i_k)}\}^2}{\prod_{j=0}^{r} n_j\cdot\prod_{c=1}^{k}(n_{i_c}-1)}\frac{\partial^2 f}{\partial \mu^2}=\left[\frac{S_{(i_1\cdots i_k)}}{\prod_{c=1}^{k}(n_{i_c}-1)}-B_{(i_1\cdots i_k)}\right]f,$$

$$(I_k\subset R,\ k=1,\cdots\cdots,\ r).$$

In view of the result given in Chapter II of Bhattacharyya [1] we observe that each one of these three relations shows the minimum variance estimates of the general mean μ, the variance of the error term σ_0 and $B_{(i_1\cdots i_k)}$ are given by the total mean $\bar{X}$, the mean square due to error $\frac{S_0}{(n_0-1)n_1n_2\cdots n_r}$, and $\frac{S_{(i_1i_2\cdots i_k)}}{(n_{i_1}-1)(n_{i_2}-1)\cdots(n_{i_k}-1)}$ respectively. The proof that the usual estimates of the variance components is of minimum variance can be obtained by taking a linear combination of a suitable number of equations in (4.85). For instance that the minimum variance estimate of $\sigma_{23\cdots r}$ is given by

$$(4.86)\quad \frac{1}{n_0n_1}\left[\frac{S_{(23\cdots r)}}{(n_2-1)(n_3-1)\cdots(n_r-1)}-\frac{S_{(12\cdots r)}}{(n_1-1)(n_2-1)\cdots(n_r-1)}\right]$$

can be proved by noting

$$(4.87)\quad \sigma_{23\cdots r}=\frac{B_{(23\cdots r)}-B_{(12\cdots r)}}{n_0n_1}$$

and by taking the linear form of (4.85) involving $B_{(23\cdots r)}$ and $B_{(12\cdots r)}$ and by using the relation

$$(4.88)\quad \frac{2\{B_{(23\cdots r)}\}^2}{n_0n_1\prod_{i=2}^{r}(n_i-1)}\frac{\partial f}{\partial B_{(23\cdots r)}}-\frac{2\{B_{(12\cdots r)}\}^2}{n_0n_1\prod_{i=0}^{r}(n_i-1)}\frac{\partial f}{\partial B_{(12\cdots r)}}$$

$$+\frac{(-1)^{r-1}\{B_{(23\cdots r)}\}^2}{n_0n_1\prod_{j=0}^{r}n_j\cdot\prod_{i=2}^{r}(n_i-1)}\frac{\partial^2 f}{\partial\mu^2}-\frac{(-1)^{r}\{B_{(12\cdots r)}\}^2}{n_0n_1\prod_{j=0}^{r}n_j\cdot\prod_{i=1}^{r}(n_i-1)}\frac{\partial^2 f}{\partial\mu^2}$$

$$=\left[\frac{1}{n_0n_1}\left\{\frac{S_{(23\cdots r)}}{\prod_{i=2}^{r}(n_i-1)}-\frac{S_{(12\cdots r)}}{\prod_{i=1}^{r}(n_i-1)}\right\}-\sigma_{23\cdots r}\right]f.$$

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