ON OSIMA'S BLOCKS OF CHARACTERS OF GROUPS OF FINITE ORDER

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Let $\mathfrak{G}$ be a group of finite order g and p be a fixed rational prime. M. OSIMA, in his paper [5], introduced a concept of blocks of group characters with regard to a subgroup $\mathfrak{H}$ of $\mathfrak{G}$ ("$\mathfrak{H}$-blocks"). Let $\mathfrak{H}_0$ be the maximal normal subgroup of $\mathfrak{G}$ contained in $\mathfrak{H}$. It is well known that the irreducible characters[1] $\phi_1, \phi_2, \cdots\cdots, \phi_k$ of $\mathfrak{H}_0$ are distributed into the classes $\mathfrak{B}_1, \mathfrak{B}_2, \cdots\cdots, \mathfrak{B}_s$ of associated characters in $\mathfrak{G}$. If $\mathfrak{B}'_1, \mathfrak{B}'_2, \cdots\cdots, \mathfrak{B}'_{s'}$ are the classes of associated irreducible characters of $\mathfrak{H}_0$ in $\mathfrak{H}$, then each class $\mathfrak{B}_\sigma$ is a collection of classes $\mathfrak{B}'_\rho$. Let $\chi_1, \chi_2, \cdots\cdots, \chi_n$ be the irreducible characters of $\mathfrak{G}$ and $\theta_1, \theta_2, \cdots\cdots, \theta_h$ be those of $\mathfrak{H}$. As is well known, there corresponds to each character χ_i exactly one class $\mathfrak{B}_\sigma$ such that

$$\chi_i(H_0) = s_{i\sigma} \sum_{\phi_\mu \in \mathfrak{B}_\sigma} \phi_\mu(H_0) \qquad (H_0 \in \mathfrak{H}_0),$$

where $s_{i\sigma}$ is a positive rational integer. If a class $\mathfrak{B}_\sigma$ corresponds to a character χ_i in this sense, we say that χ_i belongs to $\mathfrak{B}_\sigma$ by counting χ_i in $\mathfrak{B}_\sigma$. We also say that θ_λ belongs to $\mathfrak{B}_\sigma$ if θ_λ belongs to $\mathfrak{B}'_\rho$ contained in $\mathfrak{B}_\sigma$. We set

$$\chi_i(H) = \sum_{\lambda=1}^{h} r_{i\lambda}\theta_\lambda(H) \qquad (H \in \mathfrak{H}),$$

where the $r_{i\lambda}$ are non-negative rational integers. As is easily seen, if $r_{i\lambda} \neq 0$, then χ_i and θ_λ belong to the same class $\mathfrak{B}_\sigma$. Hence, χ_i and χ_j belong to the same class $\mathfrak{B}_\sigma$ if and only if χ_i and χ_j are connected by a chain $\chi_i, \chi_r, \cdots\cdots, \chi_t, \chi_j$ such that any two consecutive $\chi_l(H)$ and $\chi_m(H)$ of the chain have an irreducible constituent θ_λ in common, i. e. $r_{l\lambda} \neq 0$ and $r_{m\lambda} \neq 0$. Thus the classes $\mathfrak{B}_\sigma$ are the $\mathfrak{H}$-blocks of $\mathfrak{G}$ in OSIMA's sense.[2] From the definition of the classes $\mathfrak{B}_\sigma$, we have the following:

LEMMA 1. *Two characters χ_i and χ_j belong to the same $\mathfrak{H}$-block $\mathfrak{B}_\sigma$ of $\mathfrak{G}$ if and only if*

$$\frac{\chi_i(H_0)}{\chi_i(1)} = \frac{\chi_j(H_0)}{\chi_j(1)}$$

for all elements H_0 of $\mathfrak{H}_0$, where 1 *denotes the identity of the group* $\mathfrak{G}$. ([5])

Henceforth the term "block of a group" will always mean block with regard to a p-Sylow subgroup of the group. While BRAUER's blocks with regard to a rational prime q will be referred as q-blocks.

The purpose of this paper is to consider a connection between the blocks of $\mathfrak{G}$ and

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1) The term "irreducible character" will always mean absolutely irreducible ordinary character.
2) Cf. [5].

those of the normalizer $\mathfrak{N}(R)$ of a p-regular element R in $\mathfrak{G}$.[3)]

NOTATION: $\mathfrak{G}$ denotes a group of finite order g and p is a fixed rational prime. Ω is the field of g-th roots of unity. $K_1, K_2, \cdots\cdots, K_n$ are the classes of conjugate elements in $\mathfrak{G}$; there are n distinct irreducible characters $\chi_1, \chi_2, \cdots\cdots, \chi_n$ of $\mathfrak{G}$. $\mathfrak{N}(G)$ denotes the normalizer of an element G in $\mathfrak{G}$; the order of $\mathfrak{N}(G)$ is denoted by $n(G)$. For a rational prime q, any element G of $\mathfrak{G}$ is written uniquely as $G=SQ=QS$, where S is a q-regular element and Q is an element whose order is a power of q; S is called the q-regular factor of G and Q is called the q-factor of G.

1. Let $\mathfrak{P}$ be a p-Sylow subgroup of $\mathfrak{G}$ and $\mathfrak{P}_0$ be the maximal normal p-subgroup of $\mathfrak{G}$. We denote by $\mathfrak{B}_1, \mathfrak{B}_2, \cdots\cdots, \mathfrak{B}_s$ the blocks of $\mathfrak{G}$ with regard to $\mathfrak{P}$. For each block $\mathfrak{B}_\sigma$, we set

(1.1) $$\Delta_\sigma = \sum_{\chi_i \in \mathfrak{B}_\sigma} e_i,$$

where e_i denotes the primitive idempotent of the center Z of the group ring of $\mathfrak{G}$ over Ω which belongs to χ_i, $i=1,2,\cdots\cdots,n$. Let $G_1, G_2, \cdots\cdots, G_n$ be a complete system of representatives for the classes $K_1, K_2, \cdots\cdots, K_n$. If we interprete each class K_ν as the sum of all its elements, then we may write

(1.2) $$\Delta_\sigma = \sum_\nu a_\nu^\sigma K_\nu,$$

where

(1.3) $$a_\nu^\sigma = \frac{1}{g} \sum_{\chi_i \in \mathfrak{B}_\sigma} \chi_i(1)\bar{\chi}_i(G_\nu).$$ [4)]

We denote by Φ_σ the sum of all irreducible characters ϕ_μ of $\mathfrak{P}_0$ which belong to $\mathfrak{B}_\sigma$ and denote by ϕ^* the character of $\mathfrak{G}$ induced by a character ϕ of $\mathfrak{P}_0$. If we set

(1.4) $$\chi_i(P_0) = s_{i\sigma}\Phi_\sigma(P_0) \qquad (P_0 \in \mathfrak{P}_0)$$

where $s_{i\sigma}$ is a positive rational integer, then, by Frobenius' theorem on induced characters, we have

(1.5) $$\phi_\mu^*(G) = \sum_{\chi_i \in \mathfrak{B}_\sigma} s_{i\sigma}\chi_i(G) \qquad (G \in \mathfrak{G})$$

for each irreducible character ϕ_μ of $\mathfrak{P}_0$ belonging to $\mathfrak{B}_\sigma$.

LEMMA 2. 1) *$a_\nu^\sigma = 0$ for all classes K_ν which are not contained in $\mathfrak{P}_0$.* 2) *All $(\mathfrak{P}_0:1)a_\nu^\sigma$ are algebraic integers.*

PROOF. 1) By the above formulae (1.3)—(1.5), we have

$$a_\nu^\sigma = \frac{1}{g}\Phi_\sigma(1)\bar{\phi}_\mu^*(G_\nu),$$

where $\phi_\mu \in \mathfrak{B}_\sigma$. Since each class K_ν containing an element of $\mathfrak{P}_0$ is contained in $\mathfrak{P}_0$, $\phi_\mu^*(G_\lambda)=0$ for all $K_\lambda \not\subseteq \mathfrak{P}_0$. Hence we have $a_\nu^\sigma=0$ for these classes K_λ.

3) A summary of the results obtained herein will appear in [4].

4) If α is a complex number, the conjugate complex number of α is denoted by $\bar{\alpha}$.

2) For $K_\nu \subseteqq \mathfrak{P}_0$, we have

$$a_\nu^\sigma = \frac{1}{g}\bar{\Phi}_\sigma(G_\nu)\phi_\mu^*(1) = \frac{1}{(\mathfrak{P}_0:1)}\bar{\Phi}_\sigma(G_\nu)\phi_\mu(1),$$

where $\phi_\mu \in \mathfrak{B}_\sigma$. Since $\Phi_\sigma(G_\nu)$ and $\phi_\mu(1)$ are algebraic integers, it follows from this formula and 1) in this lemma that all $(\mathfrak{P}_0:1)a_\nu^\sigma$ are algebraic integers, if $K_\nu \subseteqq \mathfrak{P}_0$ or not.

THE CONVERSE OF LEMMA 2. *If, for a set $\mathfrak{B}$ of characters χ_i, the idempotent $\Delta = \sum_{\chi_i \in \mathfrak{B}} e_i$ of Z is expressed as a linear combination of classes K_ν contained in $\mathfrak{P}_0$, then $\mathfrak{B}$ is a collection of blocks $\mathfrak{B}_\sigma$ of $\mathfrak{G}$.*

PROOF. Suppose $\mathfrak{B} \cap \mathfrak{B}_\sigma$ is not vacuous and $\mathfrak{B} \not\supseteqq \mathfrak{B}_\sigma$. Then we may select two characters χ_i and χ_j of $\mathfrak{B}_\sigma$ such that $\chi_i \in \mathfrak{B}$ and $\chi_j \notin \mathfrak{B}$. For these characters, we have $\omega_i(\Delta)=1$ and $\omega_j(\Delta)=0$, where ω_i and ω_j are the linear characters of Z which belong to e_i and e_j, respectively. On the other hand, we have $\omega_i(\Delta)=\omega_j(\Delta)$, because we have $\omega_i(K_\nu)=\omega_j(K_\nu)$ for all $K_\nu \subseteqq \mathfrak{P}_0$ by Lemma 1. Therefore $\mathfrak{B}$ must be a collection of blocks $\mathfrak{B}_\sigma$ of $\mathfrak{G}$.

2. Let q be an arbitrarily fixed rational prime, different from p, and $\mathfrak{q}$ be a prime ideal in Ω dividing q. For each q-block B_τ of $\mathfrak{G}$, we consider the primitive idempotent η_τ of the center Z_0 of the group ring of $\mathfrak{G}$ over the ring $\mathfrak{o}_\mathfrak{q}$ of $\mathfrak{q}$-integers: $\eta_\tau = \sum_{\chi_i \in B_\tau} e_i$. If we set

(2.1) $$\eta_\tau = \sum_\nu b_\nu^\tau K_\nu$$

then, as is well known, b_ν^τ vanishes for all q-singular classes K_ν of $\mathfrak{G}$ and all the coefficients b_ν^τ are $\mathfrak{q}$-integers. The converse also holds in the following form: If, for a set B of characters χ_i, the idempotent $\eta = \sum_{\chi_i \in B} e_i$ is expressed as a linear combination of the classes K_ν of $\mathfrak{G}$ with $\mathfrak{q}$-integral coefficients, then B is a collection of q-blocks B_τ of $\mathfrak{G}$. Therefore, it follows from Lemma 2 that *each block $\mathfrak{B}_\sigma$ of $\mathfrak{G}$ is a collection of q-blocks B_τ of $\mathfrak{G}$.*

Let Q be an arbitrarily given element of $\mathfrak{G}$ whose order is a power of q. Let $B^{(\tau)}(Q)$ be the collection of q-blocks $\hat{B}_\rho$ of $\mathfrak{N}(Q)$, which determine a q-block B_τ of $\mathfrak{G}$ in BRAUER's sense, and $\hat{\eta}_\rho$ be the primitive idempotent of the center $\hat{Z}_0$ of the group ring of $\mathfrak{N}(Q)$ over the ring $\mathfrak{o}_\mathfrak{q}$ of $\mathfrak{q}$-integers which is associated with a q-block $\hat{B}_\sigma$ of $\mathfrak{N}(Q)$. We set $\hat{\eta}^{(\tau)} = \sum_{\hat{B}_\rho \subseteqq B^{(\tau)}(Q)} \hat{\eta}_\rho$ and $\eta_\tau^0 = \sum_\nu b_\nu^\tau K_\nu^0$, where K_ν^0 is the sum of all elements in $K_\nu \cap \mathfrak{N}(Q)$, $\nu = 1, 2, \cdots\cdots, n$. It is well known that

(2.2) $$\eta_\tau^0 \equiv \hat{\eta}^{(\tau)} \pmod{\mathfrak{q}Z_0}.$$ [5]

If we set $\mathfrak{B}^{(\sigma)}(Q) = \bigcup_{B_\tau \subseteqq \mathfrak{B}_\sigma} B^{(\tau)}(Q)$, then we have the following:

LEMMA 3. *Each $\mathfrak{B}^{(\sigma)}(Q)$ is a collection of blocks $\hat{\mathfrak{B}}_\gamma$ of $\mathfrak{N}(Q)$.*

5) Cf. (4.16) in [2] and [8, p. 181].

PROOF. Suppose $\hat{\mathfrak{B}}_\gamma \cap \mathfrak{B}^{(\sigma)}(Q)$ is not vacuous and $\hat{\mathfrak{B}}_\gamma \not\subseteq \mathfrak{B}^{(\sigma)}(Q)$. Then we may choose two irreducible characters $\hat{\chi}_i$ and $\hat{\chi}_j$ of $\mathfrak{N}(Q)$ in $\hat{\mathfrak{B}}_\gamma$ such that $\hat{\chi}_i \in \mathfrak{B}^{(\sigma)}(Q)$ and $\hat{\chi}_j \notin \mathfrak{B}^{(\sigma)}(Q)$. If we set $\varDelta_\sigma^0 = \sum_{B_\tau \subseteq \mathfrak{B}_\sigma} \eta_\tau^0$, then we have $\varDelta_\sigma^0 = \sum_\nu a_\nu^\sigma K_\nu^0$, where the coefficients a_ν^σ are given by (1.2) for the block $\mathfrak{B}_\sigma$. By Lemma 2, $\varDelta_\sigma^0$ is a linear combination of classes $\hat{K}_\mu$ of $\mathfrak{N}(Q)$ with $\mathfrak{q}$-integral coefficients which are contained in $\mathfrak{P}_0 \cap \mathfrak{N}(Q)$. Since $\mathfrak{P}_0 \cap \mathfrak{N}(Q)$ is contained in the maximal normal p-subgroup $\hat{\mathfrak{P}}_0$ of $\mathfrak{N}(Q)$, we have $\hat{\omega}_i(\varDelta_\sigma^0) = \hat{\omega}_j(\varDelta_\sigma^0)$ by Lemma 1, where $\hat{\omega}_i$ and $\hat{\omega}_j$ are the linear characters of the center $\hat{Z}$ of the group ring of $\mathfrak{N}(Q)$ over Ω which belong to $\hat{\chi}_i$ and $\hat{\chi}_j$, respectively. Setting $\hat{\varDelta}^{(\sigma)} = \sum_{B_\tau \subseteq \mathfrak{B}_\sigma} \hat{\eta}^{(\tau)}$, by (2.2) we have

$$\hat{\omega}_i(\hat{\varDelta}^{(\sigma)}) \equiv \hat{\omega}_j(\hat{\varDelta}^{(\sigma)}) \pmod{\mathfrak{q}}. \tag{2.3}$$

On the other hand, we have $\hat{\omega}_i(\hat{\varDelta}^{(\sigma)}) = 1$ and $\hat{\omega}_j(\hat{\varDelta}^{(\sigma)}) = 0$, hence

$$\hat{\omega}_i(\hat{\varDelta}^{(\sigma)}) \not\equiv \hat{\omega}_j(\hat{\varDelta}^{(\sigma)}) \pmod{\mathfrak{q}}. \tag{2.4}$$

(2.4) contradicts with (2.3), therefore $\mathfrak{B}^{(\sigma)}(Q)$ is a collection of blocks of $\mathfrak{N}(Q)$. This completes the proof.

3. Let R be a p-regular element of $\mathfrak{G}$. If the order of R is a product of powers of r distinct rational primes $q_1, q_2, \cdots\cdots, q_r$, then R is uniquely decomposed into

$$R = Q_1 Q_2 \cdots\cdots Q_r \qquad (Q_i Q_j = Q_j Q_i), \tag{3.1}$$

where Q_i is the q_i-factor of $R, i = 1, 2, \cdots\cdots, r$. Let a block $\mathfrak{B}_\sigma$ of $\mathfrak{G}$ be given arbitrarily. First, applying Lemma 3 for $Q_1, \mathfrak{B}_\sigma$ and $\mathfrak{G}$, we have a collection $\mathfrak{B}^{(\sigma)}(Q_1)$ of blocks of $\mathfrak{G}^{(1)} = \mathfrak{N}(Q_1)$. Secondly, working similarly for $Q_2, \mathfrak{B}^{(\sigma)}(Q_1)$, and $\mathfrak{G}^{(1)}$, we have a collection $\mathfrak{B}^{(\sigma)}(Q_1, Q_2)$ of blocks of $\mathfrak{G}^{(2)} = \mathfrak{N}(Q_1 Q_2)$. Continuing this process, we have finally a collection $\tilde{\mathfrak{B}}^{(\sigma)} = \mathfrak{B}^{(\sigma)}(Q_1, Q_2, \cdots\cdots\cdots, Q_r)$ of blocks $\tilde{\mathfrak{B}}_\rho$ of $\tilde{\mathfrak{G}} = \mathfrak{N}(R)$. If a block $\tilde{\mathfrak{B}}_\rho$ of $\tilde{\mathfrak{G}}$ belongs to the collection $\tilde{\mathfrak{B}}^{(\sigma)}$, we say that *the block $\mathfrak{B}_\sigma$ of $\mathfrak{G}$ is determined by hte block $\tilde{\mathfrak{B}}_\rho$ of $\tilde{\mathfrak{G}}$*. It follows immediately from Theorem 1 that $\tilde{\mathfrak{B}}^{(\sigma)}$ is independent of the order of $Q_1, Q_2, \cdots\cdots, Q_r$.

Let $S(R)$ be the p-regular section of R in $\mathfrak{G}$ ("Oberklasse")[6], i. e. the set of all elements of $\mathfrak{G}$ whose p-regular factors are conjugate to R in $\mathfrak{G}$. Let $\tilde{K}_1, \tilde{K}_2, \cdots\cdots, \tilde{K}_{\tilde{n}}$ be the classes of conjugate elements in $\tilde{\mathfrak{G}}$; we may assume that the p-regular section $\tilde{S}(1)$ of 1 in $\tilde{\mathfrak{G}}$ is the union of the first v classes $\tilde{K}_\alpha$. We may choose a complete system of representatives $P_1, P_2, \cdots\cdots, P_v$ for the classes $\tilde{K}_1, \tilde{K}_2, \cdots\cdots, \tilde{K}_v$ in a given p-Sylow subgroup $\tilde{\mathfrak{P}}$ of $\tilde{\mathfrak{G}}$. As is easily seen, any two elements RP_α and RP_β with $\alpha \neq \beta$ can not belong to the same class K_ν of $\mathfrak{G}$. Hence, arranging the classes K_ν of $\mathfrak{G}$ in a suitable order, we may assume that RP_α belongs to K_α, $\alpha = 1, 2, \cdots\cdots, v$; $S(R)$ is the union of $K_1, K_2, \cdots\cdots, K_v$. Further we may assume that the maximal normal p-subgroup $\tilde{\mathfrak{P}}_0$ of $\tilde{\mathfrak{G}}$ is the union of $\tilde{K}_1, \tilde{K}_2, \cdots\cdots, \tilde{K}_u$, $1 \leqq u \leqq v$. We denote by $S_0(R)$ the union of

6) Cf. [11].

$K_1, K_2, \cdots\cdots, K_u$ and denote by $S_1(R)$ the union of $K_{u+1}, K_{u+2}, \cdots\cdots, K_v$; $S(R)=S_0(R)\cup S_1(R)$.

There are u distinct blocks $\tilde{\mathfrak{B}}_1, \tilde{\mathfrak{B}}_2, \cdots\cdots, \tilde{\mathfrak{B}}_u$ of $\tilde{\mathfrak{G}}$ with regard to $\tilde{\mathfrak{P}}$ ([5]), as is immediately seen from Lemma 1. For each block $\mathfrak{B}_\sigma$ of $\mathfrak{G}$, let Δ_σ be given by (1.1). Similarly, for each block $\tilde{\mathfrak{B}}_\rho$ of $\tilde{\mathfrak{G}}$, we may define an idempotent $\tilde{\Delta}_\rho$ of the center $\tilde{Z}$ of the group ring of $\tilde{\mathfrak{G}}$ over Ω. We set $\tilde{\Delta}^{(\sigma)}=\sum_{\tilde{\mathfrak{B}}_\rho \subseteq \mathfrak{B}^{(\sigma)}} \tilde{\Delta}_\rho$ and set

$$K_\mu \Delta_\sigma = \sum_{\nu=1}^{n} a_{\mu\nu}^{\sigma} K_\nu \qquad (\mu=1,2,\cdots\cdots,n). \tag{3.2}$$

We then have the following:

THEOREM 1. *For $\alpha=1,2,\cdots\cdots,u$, we have*

$$K_\alpha \Delta_\sigma = \sum_{\beta=1}^{u} a_{\alpha\beta}^{\sigma} K_\beta \tag{3.3}$$

and

$$\tilde{K}_\alpha \tilde{\Delta}^{(\sigma)} = \sum_{\beta=1}^{u} a_{\alpha\beta}^{\sigma} \tilde{K}_\beta. \tag{3.4}$$

For $\alpha=u+1, u+2, \cdots\cdots, v$, we have

$$K_\alpha \Delta_\sigma = \sum_{\beta=u+1}^{v} a_{\alpha\beta}^{\sigma} K_\beta \tag{3.3'}$$

and

$$\tilde{K}_\alpha \tilde{\Delta}^{(\sigma)} = \sum_{\beta=u+1}^{v} a_{\alpha\beta}^{\sigma} \tilde{K}_\beta. \tag{3.4'}$$

PROOF. According to Lemma 2, we may set

$$\tilde{K}_\alpha \tilde{\Delta}^{(\sigma)} = \sum_{\beta} \tilde{a}_{\alpha\beta}^{(\sigma)} \tilde{K}_\beta \qquad (\alpha=1,2,\cdots\cdots,v), \tag{3.5}$$

where β ranges over $1,2,\cdots\cdots,u$ for $\alpha=1,2,\cdots\cdots,u$ and ranges over $u+1, u+2, \cdots\cdots, v$ for $\alpha=u+1, u+2, \cdots\cdots, v$. Denote by $\Delta^{(\sigma)}(Q_1, Q_2, \cdots\cdots, Q_f)$ the idempotent of the center of the group ring of $\mathfrak{G}^{(f)}$ over Ω which is associated with $\mathfrak{B}^{(\sigma)}(Q_1, Q_2, \cdots\cdots, Q_f)$, $f=1,2,\cdots\cdots,r$; $\tilde{\Delta}^{(\sigma)}=\Delta^{(\sigma)}(Q_1, Q_2, \cdots\cdots, Q_r)$. Since any two elements Q_rP_α and Q_rP_β with $\alpha \neq \beta$ can not belong to the same class of conjugate elements in $\mathfrak{G}^{(r-1)}$, if we denote by $K_\alpha^{(r-1)}$ the class of conjugate elements in $\mathfrak{G}^{(r-1)}$ which contains Q_rP_α, $\alpha=1,2,\cdots\cdots,v$, then $K_1^{(r-1)}, K_2^{(r-1)}, \cdots\cdots, K_v^{(r-1)}$ are v distinct classes of $\mathfrak{G}^{(r-1)}$. First, considering $\tilde{\mathfrak{B}}^{(\sigma)}$ and $\mathfrak{B}^{(\sigma)}(Q_1, Q_2, \cdots\cdots, Q_{r-1})$ as collections of q_r-blocks of $\tilde{\mathfrak{G}}$ and $\mathfrak{G}^{(r-1)}$, respectively, by Theorem 2 in [3] we have

$$K_\alpha^{(r-1)} \Delta^{(\sigma)}(Q_1, Q_2, \cdots\cdots, Q_{r-1}) = \sum_{\beta} \tilde{a}_{\alpha\beta}^{(\sigma)} K_\beta^{(r-1)}$$

with the same proviso as (3.5). Secondly, denoting by $K_\alpha^{(r-2)}$ the classes of conjugate elements in $\mathfrak{G}^{(r-2)}$ which contains $Q_{r-1}Q_rP_\alpha$, $\alpha=1,2,\cdots\cdots,v$, we have

$$K_{\alpha}^{(r-2)}\Delta^{(\sigma)}(Q_1, Q_2, \cdots\cdots, Q_{r-2})=\sum_{\beta}\tilde{a}_{\alpha\beta}^{(\sigma)}K_{\beta}^{(r-2)}$$

with the same proviso as above. Continuing this process, we have finally (3.3) and (3.3′) with $a_{\alpha\beta}^{\sigma}=\tilde{a}_{\alpha\beta}^{(\sigma)}$ $(\alpha, \beta=1, 2, \cdots\cdots, v)$. This completes the proof.

4. In this section, we shall use the notations in the preceding sections. The elements $P_1, P_2, \cdots\cdots, P_v$ need not form a complete system of representatives for the classes of conjugate elements in $\tilde{\mathfrak{P}}$. However, we may construct such a system by adding further elements P to the set $P_1, P_2, \cdots\cdots, P_v$. Each P is conjugate to a certain P_α in $\tilde{\mathfrak{G}}$, where α is uniquely determined, $\alpha=1, 2, \cdots\cdots, v$. We denote by the elements P belonging to P_α by $P_\alpha=P_\alpha^{(0)}, P_\alpha^{(1)}, P_\alpha^{(2)}, \cdots\cdots, P_\alpha^{(l_\alpha)}, l_\alpha\geqq 0$. Let $\tilde{\chi}_1, \tilde{\chi}_2, \cdots\cdots, \tilde{\chi}_{\tilde{n}}$ be the irreducible characters of $\tilde{\mathfrak{G}}$ and $\tilde{\theta}_1, \tilde{\theta}_2, \cdots\cdots, \tilde{\theta}_{\tilde{h}}$ be those of $\tilde{\mathfrak{P}}$. We set

$$\tilde{\chi}_j(P)=\sum_{\lambda=1}^{\tilde{h}}\tilde{r}_{j\lambda}\tilde{\theta}_\lambda(P) \qquad (P\in\tilde{\mathfrak{P}}) \tag{4.1}$$

and, after M. OSIMA [6], [7], we define the constants $r_{i\lambda}^{R}$ by

$$\chi_i(RP)=\sum_{\lambda=1}^{\tilde{h}}r_{i\lambda}^{R}\tilde{\theta}_\lambda(P) \qquad (P\in\tilde{\mathfrak{P}}). \tag{4.2}$$

For $\alpha, \beta=1, 2, \cdots\cdots, v$, since we see from (3.1) that

$$a_{\alpha\beta}^{\sigma}=\frac{1}{n(RP_\alpha)}\sum_{\chi_i\in\mathfrak{B}_\sigma}\chi_i(RP_\alpha)\bar{\chi}_i(RP_\beta)=\frac{1}{n(RP_\alpha)}\sum_{\chi_i\in\mathfrak{B}_\sigma}\chi_i(RP_\alpha^{(\gamma)})\bar{\chi}_i(RP_\beta^{(\delta)}),$$

$a_{\alpha\beta}^{\sigma}$ is expressed as

$$a_{\alpha\beta}^{\sigma}=\frac{1}{n(RP_\alpha)}\sum_{\lambda,\mu=1}^{\tilde{h}}\tilde{\theta}_\lambda(P_\alpha^{(\gamma)})\bar{\tilde{\theta}}_\mu(P_\beta^{(\delta)})\sum_{\chi_i\in\mathfrak{B}_\sigma}r_{i\lambda}^{R}\bar{r}_{i\mu}^{R}. \tag{4.3}$$

On the other hand, from Theorem 1 we see, for $\alpha, \beta=1, 2, \cdots\cdots, v$,

$$\begin{aligned} a_{\alpha\beta}^{\sigma}&=\frac{1}{n(RP_\alpha)}\sum_{\tilde{\chi}_j\in\tilde{\mathfrak{B}}^{(\sigma)}}\tilde{\chi}_j(P_\alpha)\bar{\tilde{\chi}}_j(P_\beta) \\ &=\frac{1}{n(RP_\alpha)}\sum_{\tilde{\theta}_\lambda, \tilde{\theta}_\mu\in\tilde{\mathfrak{B}}^{(\sigma)}}\tilde{\theta}_\lambda(P_\alpha^{(\gamma)})\bar{\tilde{\theta}}_\mu(P_\beta^{(\delta)})\sum_{\tilde{\chi}_j\in\tilde{\mathfrak{B}}^{(\sigma)}}\tilde{r}_{j\lambda}\tilde{r}_{j\mu} \\ &=\frac{1}{n(RP_\alpha)}\sum_{\tilde{\theta}_\lambda, \tilde{\theta}_\mu\in\tilde{\mathfrak{B}}^{(\sigma)}}\tilde{\theta}_\lambda(P_\alpha^{(\gamma)})\bar{\tilde{\theta}}_\mu(P_\beta^{(\delta)})\sum_{j=1}^{\tilde{n}}\tilde{r}_{j\lambda}\tilde{r}_{j\mu}. \end{aligned}$$

Setting

$$\tilde{w}_{\lambda\mu}=\sum_{j=1}^{\tilde{n}}\tilde{r}_{j\lambda}\tilde{r}_{j\mu} \qquad (\lambda, \mu=1, 2, \cdots\cdots, \tilde{h}),$$

we have

$$a_{\alpha\beta}^{\sigma}=\frac{1}{n(RP_\alpha)}\sum_{\tilde{\theta}_\lambda, \tilde{\theta}_\mu\in\tilde{\mathfrak{B}}^{(\sigma)}}\tilde{\theta}_\lambda(P_\alpha^{(\gamma)})\bar{\tilde{\theta}}_\mu(P_\beta^{(\delta)})\tilde{w}_{\lambda\mu}. \tag{4.4}$$

Since (4.3) and (4.4) hold for any pair of γ and δ ($\gamma=0,1,2,\cdots\cdots, l_\alpha$; $\delta=0,1,2,\cdots, l_\beta$), we have

$$\sum_{\chi_i\in\mathfrak{B}_\sigma} r_{i\lambda}^R r_{i\mu}^R=\begin{cases}\tilde{w}_{\lambda\mu} & (\tilde{\theta}_\lambda, \tilde{\theta}_\mu\in\tilde{\mathfrak{B}}_\rho\subseteqq\tilde{\mathfrak{B}}^{(\sigma)}),\\ 0 & (\text{elsewhere}).\end{cases} \tag{4.5}$$

In particular, we have

$$\sum_{\chi_i\in\mathfrak{B}_\sigma} r_{i\lambda}^R r_{i\lambda}^R=0 \qquad (\tilde{\theta}_\lambda\notin\tilde{\mathfrak{B}}^{(\sigma)}),$$

hence

$$r_{i\lambda}^R=0 \qquad (\chi_i\in\mathfrak{B}_\sigma,\ \tilde{\theta}_\lambda\notin\tilde{\mathfrak{B}}^{(\sigma)}).$$

Thus we obtain the following:

THEOREM 2. *If an irreducible character $\tilde{\theta}_\lambda$ of $\tilde{\mathfrak{P}}$ belongs to a block $\tilde{\mathfrak{B}}_\rho$ of $\tilde{\mathfrak{G}}$, then $r_{i\lambda}^R$ can be different from zoro only for irreducible characters χ_i of $\mathfrak{G}$ which belong to the block $\mathfrak{B}_\sigma$ of $\mathfrak{G}$ determined by the block $\tilde{\mathfrak{B}}_\rho$ of $\tilde{\mathfrak{G}}$.*

It is easy to see that $a_{\mu\nu}^\sigma=0$ implies

$$\sum_{\chi_i\in\mathfrak{B}_\sigma} \chi_i(G_\mu)\bar{\chi}_i(G_\nu)=0$$

Hence, by Theorem 1, we have a refinement of some of the orthogonality relations for group characters.[7]

THEOREM 3. 1) *If two elements L and M of $\mathfrak{G}$ belong to different p-regular sections of $\mathfrak{G}$, then*

$$\sum_{\chi_i\in\mathfrak{B}_\sigma} \chi_i(L)\bar{\chi}_i(M)=0 \tag{4.6}$$

for each block $\mathfrak{B}_\sigma$ of $\mathfrak{G}$.[8] ([10])

2) *If L and M belong to the same p-regular section $S(R)$ of $\mathfrak{G}$ and if exactly one of the p-factors of them belongs to the maximal normal p-subgroup of $\mathfrak{N}(R)$, then* (4.6) *also holds for each block $\mathfrak{B}_\sigma$ of $\mathfrak{G}$.*

From (3.3) and (3.3′), we see the following:

LEMMA 4. *If*

$$\sum_{\nu=1}^{n} a_\nu K_\nu \Delta_\sigma=0, \tag{4.7}$$

where $a_\nu\in\Omega$, then

$$\sum_{K_\nu\subseteqq S_0(R)} a_\nu K_\nu \Delta_\sigma=\sum_{K_\nu\subseteqq S_1(R)} a_\nu K_\nu \Delta_\sigma=0$$

for each p-regular section $S(R)$ of $\mathfrak{G}$.

7) Cf. [1]—[4], [9] and [10].

8) We have a refinement of this, which is a dual result of Theorem 2 in [1].

We shall describe this lemma in terms of the irreducible characters χ_i of $\mathfrak{G}$. (4.7) implies

$$\sum_{\nu=1}^{n} a_\nu \omega_i(K_\nu) = \omega_i\left(\sum_{\nu=1}^{n} a_\nu K_\nu \Delta_\sigma\right) = 0 \qquad (\chi_i \in \mathfrak{B}_\sigma),$$

where $\omega_i(K_\nu) = g\chi_i(G_\nu)/n(G_\nu)\chi_i(1)$. Conversely, if

$$\sum_{\nu=1}^{n} a_\nu \omega_i(K_\nu) = 0$$

for all $\chi_i \in \mathfrak{B}_\sigma$, where the a_ν are numbers of Ω depending only on the classes K_ν of $\mathfrak{G}$, then

$$\omega_j\left(\sum_{\nu=1}^{n} a_\nu K_\nu \Delta_\sigma\right) = \sum_{\nu=1}^{n} a_\nu \omega_j(K_\nu)\omega_j(\Delta_\sigma) = 0$$

for all irreducible characters χ_j of $\mathfrak{G}$. Hence (4.7) holds for these a_ν. Therefore Lemma 4 is equivalent to the following:

LEMMA 4′. *If, for all χ_i of $\mathfrak{B}_\sigma$,*

$$\sum_{\nu=1}^{n} a_\nu \omega_i(K_\nu) = 0$$

where the a_ν are numbers of Ω depending only on the classes K_ν of $\mathfrak{G}$, then

$$\sum_{K_\nu \subseteqq S_0(R)} a_\nu \omega_i(K_\nu) = \sum_{K_\nu \subseteqq S_1(R)} a_\nu \omega_i(K_\nu) = 0$$

for these χ_i, where R is an arbitrary p-regular element of $\mathfrak{G}$.

Evidently this lemma is equivalent to the following:

LEMMA 4″. *If, for all χ_i of $\mathfrak{B}_\sigma$,*

$$\sum_{\nu=1}^{n} a_\nu \chi_i(G_\nu) = 0$$

where the a_ν are numbers of Ω depending only on the classes K_ν of $\mathfrak{G}$, then

$$\sum_{K_\nu \subseteqq S_0(R)} a_\nu \chi_i(G_\nu) = \sum_{K_\nu \subseteqq S_1(R)} a_\nu \chi_i(G_\nu) = 0$$

for these χ_i, where R is an arbitrary p-regular element of $\mathfrak{G}$.

In consideration of Lemma 4″, the orthogonality relations for group characters imply the following refinement of some of them.

THEOREM 4. *If χ_i and χ_j are two irreducible characters of $\mathfrak{G}$ which belong to different blocks $\mathfrak{B}_\sigma$ of $\mathfrak{G}$, then*

$$\sum_{G \in S_0(R)} \chi_i(G)\bar{\chi}_j(G) = \sum_{G \in S_1(R)} \chi_i(G)\bar{\chi}_j(G) = 0$$

for each p-regular section $S(R)$ of $\mathfrak{G}$.[9]

9) This is an improvement of a result in [10]. Cf. the papers quoted in foot-note 7).

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