THE POINT ESTIMATION OF THE PARAMETERS IN THE MIXED MODEL

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1. Introduction.

In this paper we shall be concerned with the estimation of the parameters in the mixed model, which is a sort of continuation of the previous work of the author [2][1] concerning the estimation of the variance components of the r-way layout of random effect model.

The mixed model is understood to be the model of the r-way layout where some of the treatment effects are the normal variables and the effect of the remaining treatments are constants. In this paper we assume the means of the random effects are all zero, which should not be restrictive at all for the problem of estimation, and we shall be concerned with the estimation of the variances of the random effects, the constant effects and the general mean.

For the development of our arguments, we need to define two types of mixed models. Stating the model in more details, we shall take all the interactions up till the highest order in our consideration, and it seems to be reasonable to assume that all the interaction effects between the fixed main effects are constant and all the other type of interactions are random. Namely if an interaction involves at least a factor whose main effect is random, then this interaction effect is also assumed to be random and on the other hand if an interaction involves no factor whose main effect is random, then this interaction effect is assumed to be constant.

Under snch assumptions as stated above, the two types of mixed models are defined from merely a technical reason. Type I is the mixed model involving only one random main effect, whereas Type II is the one involving more than one random main effect.

The results obtained in this paper are the derivation of the joint density function of all observations in the r-way layout of the mixed model, and the proof of the completeness of the family of distributions of the sufficient statistics in our concern, which implies by making use of the result due to Lehmann and Scheffé that the estimates of the parameters which are usually adopted in the practice of statistical inferences as unbiased estimates are the unique minimum variance unbiased estimates. In Section 2, we shall define some notations similar to those in the previous paper of the author [2] and give the model equation in our concern. Section 3 is devoted to Type I model and Section 4 is to Type II model. In Section 5, we shall remark that the result in the previous paper of the author can be improved by making use of a result of this paper.

1) Numbers in brackets refer to the references of the end of the paper.

2. Preliminaries.

We shall be concerned with the r-way layout of the mixed model whose model equation is given by the following

(2.1)
$$x_{t_0t_1\cdots t_r}=\mu+\sum_{h=1}^{r-s}\sum_{J_h\subset R-S}\alpha(j_1,\cdots,j_h;t_{j_1},\cdots,t_{j_h})$$
$$+\sum_{h=0}^{r-s}\sum_{J_h\subset R-S}\sum_{k=1}^{s}\sum_{I_k\subset S}a(i_1,\cdots,i_k,j_1,\cdots,j_h;t_{i_1},\cdots,t_{i_k},t_{j_1},\cdots,t_{j_h}),$$
$$(t_{i_c}=1,2,\cdots,n_{i_c};t_{j_d}=1,2,\cdots,n_{j_d};t_0=1,2,\cdots,n_0;c=1,\cdots,s;d=1,\cdots,r-s),$$

where μ denotes the general mean, $\alpha(j_1,\cdots,j_h;t_{j_1},\cdots,t_{j_h})$ denotes the interaction between j_1-th, j_2-th, $\cdots$, j_h-th factors with the level t_{j_1}, t_{j_2}, $\cdots$, t_{j_h}, $a(i_1,\cdots,i_k,j_1,\cdots,j_h;t_{i_1},\cdots,t_{i_k},t_{j_1},\cdots,t_{j_h})$ denotes the interaction between i_1-th, i_2-th, $\cdots$, i_k-th, j_1-th, j_2-th, $\cdots$, j_h-th factors with the level $t_{i_1},\cdots t_{i_k},t_{j_1},\cdots t_{j_h}$, and $e_{t_0t_1\cdots t_r}$ denotes the error term. When $h=0$, $a(i_1,\cdots,i_k,j_1,\cdots,j_h;t_{i_1},\cdots,t_{i_k},t_{j_1},\cdots,t_{j_h})$ is understood to be $a(i_1,\cdots,i_k;t_{i_1},\cdots t_{i_k})$. In the above equation S and $R-S$ denote the sets of integers $(1,2,\cdots,s)$ and $(s+1,s+2,\cdots,r)$ respectively. I_k and J_h denote the subsets $(i_1,i_2,\cdots,i_k)$ of $S=(1,2,\cdots,s)$ and $(j_1,j_2,\cdots,j_h)$ of $R-S=(s+1,s+2,\cdots,r)$, with the relations $i_1<i_2<\cdots<i_k$ and $j_1<j_2<\cdots<j_h$ respectively. $\sum_{I_k\subset S}$ and $\sum_{J_h\subset R-S}$ denote the summations for all subsets I_k of size k in S and for all subsets J_h of size h in $R-S$ respectively.

We assume that μ and all $\alpha(j_1,\cdots,j_h;t_{j_1},\cdots,t_{j_h})$ are constants, all $a(i_1,\cdots,i_k,j_1,\cdots,j_h;t_{i_1},\cdots,t_{i_k},t_{j_1},\cdots,t_{j_h})$ and $e_{t_0t_1\cdots t_r}$ are distributed independently to each other in normal distributions with mean all equal to zero and the variance of each $a(i_1,\cdots,i_k,j_1,\cdots,j_h;t_{i_1},\cdots,t_{i_k},t_{j_1},\cdots,t_{j_h})$ equals to $\sigma^2_{I_kJ_h}$, the variance of each $a(i_1,\cdots,i_k;t_{i_1},\cdots,t_{i_k})$ equals to $\sigma^2_{I_k}$ and the variances of $e_{t_0t_1\cdots t_r}$ all equal to σ^2_0.

Further we assume that there hold relations:

(2.2)
$$\sum_{t_{j_c}=1}^{n_{j_c}}\alpha(j_1,\cdots,j_h;t_{j_1},\cdots,t_{j_h})=0,$$
$$(J_h\subset R-S;c=1,2,\cdots,h).$$

The above stated is the general formulation of the mixed model under general assumptions, and we shall define two types of them as follows. We call the mixed model, whose model equation is represented by (2.1) for $s=1$, satisfying all assumptions stated above, Type I model and the one for $s\geqq 2$ Type II model.

Throughout this paper the notations such as U_d, V_e, M_α and N_β etc. mean the sets of integers $(u_1,u_2,\cdots,u_d)$, $(v_1,v_2,\cdots,v_e)$, $(m_1,m_2,\cdots,m_\alpha)$ and $(n_1,n_2,\cdots,n_\beta)$ etc., while these mean the empty set if d,e and α etc. equal to zero, and R,S,T etc. mean the set of integers $(1,2,\cdots,r)$, $(1,2,\cdots,s)$, $(1,2,\cdots,t)$ etc. and on these sets of integers we shall make use of the set theoretical notations such as $A\cup B, A\cap B, A\subset B, A-B$ etc. Further, we assume that the summations such as $\sum_{A\subset B}a_A$, $\sum_{\substack{A\subset B\\ A\supset C}}a_A$, where A,B,C are such sets of integers as defined above, mean the sum of all numbers a_A's having A as the suffixes which are included in B, and included in B and including C, respectively.

The Kronecker product of two or any number of matrixes are defined in this paper in the way reverse to the usual ones for the convenience in handling the cumbersome notation systems.

Let $A=(a_{ij})$, $B=(b_{ij})$, the Kronecker product denoted by $A\otimes B$ is defined as the matrix (Ab_{ij}), The Kronecker product of any number of matrixes is defined as the natural generalization of two matrixes, we shall write the Kronecker product of n matrixes $A_1, A_2, \cdots, A_n$, as $\prod_{i=1}^{n}\otimes A_i$.

In this paper, we shall make use of the well-known relations concerning the Kronecker products of two matrixes such as $(A\otimes B)(C\otimes D)=AC\otimes BD$, $(A\otimes B)^{-1}=A^{-1}\otimes B^{-1}$, $(A\otimes B)'=A'\otimes B'$, and their generalizations to the products of any number of matrixes without mentioning explicitly. Throughout this paper we shall write $n\times n$ unit matrix as I_n, E_n denotes the $n\times n$ matrix with the elements all equal to 1. Let H_n be the $n\times n$ matrix with the elements all equal to zero except for the element of the first row in the first column equal to 1, and let $K_n=I_n-H_n$.

Further let T_n be defined as the orthogonal matrix with the elements of the first column all equal to $\frac{1}{\sqrt{n}}$.

Then we have easily

$$T'_n E_n T_n = nH_n. \tag{2.3}$$

3. The case of Type I.

3. 1. Determinant of the variance matrix.

In this type the model equation is given by

$$x_{t_0t_1\cdots t_r}=\mu+\sum_{h=1}^{r-1}\sum_{J_h\subset R-1}\alpha(j_1,\cdots,j_h;t_{j_1},\cdots,t_{j_h})+\sum_{h=0}^{r-1}\sum_{J_h\subset R-1}a(1,j_1,\cdots,j_h;t_1,t_{j_1},\cdots,t_{j_h})$$
$$+e_{t_0t_1\cdots t_r}, \qquad \begin{pmatrix} t_{j_c}=1,\cdots,n_{j_c}; & c=1,\cdots,s; \\ t_0=1,\cdots,n_0; & t_1=1,\cdots,n_1. \end{pmatrix}. \tag{3.1}$$

Thus we have the expression of the variance matrix in terms of the Kronecker products as follows,

$$V=\sigma_1^2E_{n_0}\otimes\prod_{\zeta=1}^{r}\otimes\,(E_{n_\zeta}^{1-\delta_1^\zeta}\times I_{n_\zeta}^{\delta_1^\zeta})+\sum_{h=1}^{r-1}\sum_{J_h\subset R-1}\sigma_{1J_h}^2E_{n_0}\otimes\prod_{\zeta=1}^{r}\otimes(E_{n_\zeta}^{1-\delta_{1J_h}^\zeta}\times I_{n_\zeta}^{\delta_{1J_h}^\zeta})$$
$$+\sigma_0^2I_{n_0}\otimes I_{n_1}\otimes\cdots\otimes I_{n_r}, \tag{3.2}$$

where $\delta_{M_\alpha}^\zeta$ is a sort of generalization of the Kronecker's delta which is

$$\delta_{M_\alpha}^\zeta=\begin{cases}1 & \text{if } \zeta \text{ is equal to either of the elements in } M_\alpha, \\ 0 & \text{otherwise,}\end{cases} \tag{3.3}$$

and E^0 of a matrix E is defined to be unit matrix I.

For the development of the arguments in this section we have to define a number of notations as follows.

DEFINITION 3.1.

$$A_{(1)}B_{(J_h)}\equiv\sum_{\beta=h}^{r-s}\sum_{\substack{N\beta\supset J_h\\ N\beta\subset R-1}}\sigma_{1N\beta}^2\prod_{\zeta=0}^{r}n_\zeta^{1-\delta_{1.N\beta}^\zeta},\tag{3.4}$$

$$A_{(1)}B_{(J_h)}^{(V_e)}\equiv\sum_{\beta=h}^{r-s-e}\sum_{\substack{N\beta\supset J_h\\ N\beta\subset R-1-V_e}}\sigma_{1N\beta}^2\prod_{\zeta=0}^{r}n_\zeta^{1-\delta_{1N\beta V_e}^\zeta},\tag{3.5}$$

$$A_{(1)}B\equiv\sum_{\beta=0}^{r-s}\sum_{N\beta\subset R-1}\sigma_{1.N\beta}^2\prod_{\zeta=0}^{r}n_\zeta^{1-\delta_{1N\beta}^\zeta},\tag{3.6}$$

$$C_{(1)}D_{(J_h)}\equiv A_{(1)}B_{(J_h)}+\sigma_0^2,\tag{3.7}$$

$$C_{(1)}D\equiv A_{(1)}B+\sigma_0^2,\tag{3.8}$$

where we assume $\sigma_{1N\beta}^2=\sigma_1^2$ when $\beta=0$.

Now with the aid of the above notations we may evaluate the determinant of the variance matrix (3.2).

THEOREM 3.1. *The determinant* $|V|$ *of the variance matrix* V *of* (3.2) *is given in the notations of* (3.7) *and* (3.8) *as follows,*

$$|V|=\prod_{h=0}^{r-1}\prod_{J_h\subset R-1}\{C_{(1)}D_{(J_h)}\}^{n_1(n_{j_1}-1)\cdots n(j_h-1)}\{\sigma_0^2\}^{(n_0-1)n_1n_2\cdots n_r}.\tag{3.9}$$

PROOF. Let us at first transform this matrix by the orthogonal matrix which is the Kronecker product of the matrixes T_{n_i} defined in Section 2, and we have

$$(T_{n_0}\otimes T_{n_1}\otimes\cdots\otimes T_{n_r})'V(T_{n_0}\otimes T_{n_1}\otimes\cdots\otimes T_{n_r})=\sigma_1^2\prod_{j=0}^{r}n_j^{1-\delta_1^j}H_{n_0}\otimes\prod_{\zeta=1}^{r}\otimes(H_{n_\zeta}^{1-\delta_1^\zeta}\times I_{n_\zeta}^{\delta_1^\zeta})\tag{3.10}$$

$$+\sum_{h=1}^{r-1}\sum_{J_h\subset R-1}\sigma_{1J_h}^2\prod_{j=0}^{r}n_j^{1-\delta_{1J_h}^j}H_{n_0}\otimes\prod_{\zeta=1}^{r}\otimes(H_{n_\zeta}^{1-\delta_{1J_h}^\zeta}\times I_{n_\zeta}^{\delta_{1J_h}^\zeta})+\sigma_0^2I_{n_0}\otimes I_{n_1}\otimes\cdots\otimes I_{n_r}$$

$$=\sigma_1^2\prod_{j=0}^{r}n_j^{1-\delta_1^j}H_{n_0}\otimes\prod_{\zeta=1}^{r}\otimes\left(H_{n_\zeta}^{1-\delta_1^\zeta}\times(H_{n_\zeta}+K_{n_\zeta})^{\delta_1^\zeta}\right)$$

$$+\sum_{h=1}^{r-1}\sum_{J_h\subset R-1}\sigma_{1J_h}^2\prod_{j=0}^{r}n_j^{1-\delta_{1J_h}^j}H_{n_0}\otimes\prod_{\zeta=1}^{r}\otimes\left(H_{n_\zeta}^{1-\delta_{1J_h}^\zeta}\times(H_{n_\zeta}+K_{n_\zeta})^{\delta_{1J_h}^\zeta}\right)$$

$$+\sigma_0^2\prod_{\zeta=0}^{r}\otimes(H_{n_\zeta}+K_{n_\zeta})$$

$$=\sigma_1^2\prod_{j=0}^{r}n_j^{1-\delta_1^j}H_{n_0}\otimes\prod_{\zeta=1}^{r}\otimes(H_{n_\zeta}+\delta_1^\zeta K_{n_\zeta})$$

$$+\sum_{k=1}^{r-1}\sum_{J_h\subset R-1}\sigma_{1J_h}^2\prod_{j=0}^{r}n_j^{1-\delta_{1J_h}^j}H_{n_0}\otimes\prod_{\zeta=1}^{r}\otimes(H_{n_\zeta}+\delta_{1J_h}^\zeta K_{n_\zeta})+\sigma_0^2\prod_{\zeta=0}^{r}\otimes(H_{n_\zeta}+K_{n_\zeta}).$$

Then we have

$$(3.11)\qquad |V|=\prod_{h=0}^{r-1}\prod_{J_h\subset R-1}\left\{\sum_{\beta=h}^{r}\sum_{N\beta\supset J_h}\sigma^2_{1N\beta}\prod_{\zeta=0}^{r}n_\zeta^{1-\delta^\zeta_{1N\beta}}+\sigma_0^2\right\}^{n_1(n_{j_1}-1)\cdots(n_{j_h}-1)}\cdot\{\sigma_0^2\}^{(n_0-1)n_1n_2\cdots n_r},$$

which is equal to (3.9).

3.2. The inverse of the variance matrix.

At first we observe the recurrence relation of $A_{(1)}B^{(V_e)}_{(J_h)}$.

LEMMA 3.1.

$$(3.12)\qquad A_{(1)}B^{(V_e)}_{(J_h)}=\frac{1}{n_{v_e}}\left[A_{(1)}B^{(V_{e-1})}_{(J_h)}-A_{(1)}B^{(V_{e-1})}_{(J_hv_e)}\right].$$

This lemma enables us to express $A_{(1)}B^{(V_e)}_{(J_h)}$ in terms of $A_{(1)}B_{(J_hT_q)}$, which is given by

LEMMA 3.2.

$$(3.13)\qquad A_{(1)}B^{(V_e)}_{(J_h)}=\frac{1}{n_{v_1}n_{v_2}\cdots n_{v_e}}\sum_{q=0}^{e}\sum_{T_q\subset V_e}(-1)^qA_{(1)}B_{(J_h,T_q)}.$$

The above two lemata are easily proved in the similar way to Section 4 of [2]. Now let us derive the inversion of the variance matrix.

THEOREM 3.2. *The inverse of the variance matrix* (3.2) *is given by*

$$(3.14)\qquad X_1E_{n_0}\otimes\prod_{\zeta=1}^{r}\otimes\left(E_{n_\zeta}^{1-\delta_1^\zeta}\times I_{n_\zeta}^{\delta_1^\zeta}\right)+\sum_{h=1}^{r-1}\sum_{J_h\subset R-1}X_{1J_h}E_{n_0}\otimes\prod_{\zeta=1}^{r}\otimes\left(E_{n_\zeta}^{1-\delta^\zeta_{1J_h}}\times I_{n_\zeta}^{\delta^\zeta_{1J_h}}\right)+X_0I_{n_0}\otimes I_{n_1}\otimes\cdots\otimes I_{n_r},$$

where

$$(3.15)\qquad X_0=\frac{1}{\sigma_0^2},$$

$$(3.16)\qquad X_R=\frac{1}{n_0}\left[\frac{1}{C_{(1)}D_{(R-1)}}-\frac{1}{\sigma_0^2}\right],$$

$$(3.17)\qquad X_1=\frac{1}{\prod_{\zeta=0}^{r}n_\zeta^{1-\delta_1^\zeta}}\left[\sum_{\beta=0}^{r-1}\sum_{N\beta\subset R-1}\frac{(-1)^\beta}{C_{(1)}D_{(N\beta)}}\right],$$

$$(3.18)\qquad X_{1J_h}=\frac{1}{\prod_{\zeta=0}^{r}n_\zeta^{1-\delta^\zeta_{1J_h}}}\left[\sum_{\beta=0}^{r-1-h}\sum_{N\beta\subset R-1-J_h}\frac{(-1)^\beta}{C_{(1)}D_{(J_hN\beta)}}\right],\qquad (J_h\subset R-1;\ h=1,2,\cdots,r-1).$$

PROOF. Anticipating the inverse to be the matrix of the form

$$(3.19)\qquad X_GE_{n_0}\otimes E_{n_1}\otimes\cdots\otimes E_{n_r}+X_1E_{n_0}\otimes\prod_{\zeta=1}^{r}\otimes\left(E_{n_\zeta}^{1-\delta_1^\zeta}\times I_{n_\zeta}^{\delta_1^\zeta}\right)$$

$$+\sum_{h=1}^{r-1}\sum_{J_h\subset R-1} X_{1J_h}E_{n_0}\otimes\prod_{\zeta=1}^{r}\otimes\left(E_{n_\zeta}^{1-\delta_{1J_h}^{\zeta}}\times I_{n_\zeta}^{\delta_{1J_h}^{\zeta}}\right)$$

$$+\sum_{h=1}^{r-1}\sum_{J_h\subset R-1} X_{J_h}E_{n_0}\otimes\prod_{\zeta=1}^{r}\otimes\left(E_{n_\zeta}^{1-\delta_{J_h}^{\zeta}}\times I_{n_\zeta}^{\delta_{J_h}^{\zeta}}\right)+X_0 I_{n_0}\otimes I_{n_1}\otimes\cdots\otimes I_{n_r},$$

let us seek for the condition that (3.19) is actually the inverse of (3.2). The product of the variance matrix (3.2) and the matrix (3.19) is

$$(3.20)\quad E_{n_0}\otimes E_{n_1}\otimes\cdots\otimes E_{n_r}\Bigg[X_G\Bigg\{\sum_{\beta=0}^{r-1}\sum_{N\beta\subset R-1}\sigma_{1N\beta}^2\prod_{\zeta=0}^{r} n_\zeta^{1-\delta_{1N\beta}^{\zeta}}+\sigma_0^2\Bigg\}$$

$$+\sum_{h=1}^{r-1}\sum_{J_h\subset R-1} X_{J_h}\Bigg\{\sum_{\beta=0}^{r-1-h}\sum_{N\beta\subset R-1-J_h}\sigma_{1N\beta}^2\prod_{\zeta=0}^{r} n_\zeta^{1-\delta_{1N\beta J_h}^{\zeta}}\Bigg\}\Bigg]$$

$$+E_{n_0}\otimes\prod_{\zeta=1}^{r}\otimes\left(E_{n_\zeta}^{1-\delta_1^\zeta}\times I_{n_\zeta}^{\delta_1^\zeta}\right)\Bigg[X_1\Bigg\{\sum_{\beta=0}^{r-1}\sum_{N\beta\supset R-1}\sigma_{1N\beta}^2\prod_{\zeta=0}^{r} n_\zeta^{1-\delta_{1N\beta}^{\zeta}}+\sigma_0^2\Bigg\}$$

$$+\sum_{e=1}^{r-1}\sum_{V_e\subset R-1} X_{1V_e}\Bigg\{\sum_{\beta=0}^{r-1-e}\sum_{N\beta\subset R-1-V_e}\sigma_{1N\beta}^2\prod_{\zeta=0}^{r} n_\zeta^{1-\delta_{1N\beta V_e}^{\zeta}}\Bigg\}+X_0\sigma_1^2\Bigg]$$

$$+E_{n_0}\otimes\prod_{\zeta=1}^{r}\otimes\left(E_{n_\zeta}^{1-\delta_R^\zeta}\times I_{n_\zeta}^{\delta_R^\zeta}\right)\left[X_R(n_0\sigma_R^2+\sigma_0^2)+X_0\sigma_R^2\right]$$

$$+\sum_{h=1}^{r-2}\sum_{J_h\subset R-1} E_{n_0}\otimes I_{n_1}\otimes\prod_{\zeta=2}^{r}\otimes\left(E_{n_\zeta}^{1-\delta_{J_h}^\zeta}\times I_{n_\zeta}^{\delta_{J_h}^\zeta}\right)\Bigg[X_{1J_h}\Bigg\{\sum_{\beta=h}^{r-1}\sum_{\substack{N\beta\supset J_h\\ N\beta\subset R-1}}\sigma_{1N\beta}^2\prod_{\zeta=0}^{r} n_\zeta^{1-\delta_{1N\beta}^{\zeta}}+\sigma_0^2\Bigg\}$$

$$+\sum_{e=1}^{r-1-h}\sum_{V_e\subset R-1-J_h} X_{1J_hV_e}\Bigg\{\sum_{\beta=h}^{r-1-e}\sum_{\substack{N\beta\supset J_h\\ N\beta\subset R-1-V_e}}\sigma_{1N\beta}^2\prod_{\zeta=0}^{r} n_\zeta^{1-\delta_{1N\beta V_e}^{\zeta}}\Bigg\}+X_0\sigma_{1J_h}^2\Bigg]$$

$$+E_{n_0}\otimes\prod_{\zeta=1}^{r}\otimes\left(E_{n_\zeta}^{1-\delta_{R-1}^\zeta}\times I_{n_\zeta}^{\delta_{R-1}^\zeta}\right)\left[X_{R-1}\{n_0\sigma_R^2+\sigma_0^2\}\right]$$

$$+\sum_{h=1}^{r-2}\sum_{J_h\subset R-1} E_{n_0}\otimes\prod_{\zeta=1}^{r}\otimes\left(E_{n_\zeta}^{1-\delta_{J_h}^\zeta}\times I_{n_\zeta}^{\delta_{J_h}^\zeta}\right)\Bigg[X_{J_h}\Bigg\{\sum_{\beta=h}^{r-1}\sum_{\substack{N\beta\supset J_h\\ N\beta\subset R-1}}\sigma_{1N\beta}^2\prod_{\zeta=0}^{r} n_\zeta^{1-\delta_{1N\beta}^{\zeta}}+\sigma_0^2\Bigg\}$$

$$+\sum_{e=1}^{r-1-h}\sum_{V_e\subset R-1-J_h} X_{J_hV_e}\Bigg\{\sum_{\beta=h}^{r-1-e}\sum_{\substack{N\beta\supset J_h\\ N\beta\subset R-1-V_e}}\sigma_{1N\beta}^2\prod_{\zeta=0}^{r} n_\zeta^{1-\delta_{1N\beta V_e}^{\zeta}}\Bigg\}\Bigg]$$

$$+I_{n_0}\otimes I_{n_1}\otimes\cdots\otimes I_{n_r}X_0\sigma_0^2$$

$$=E_{n_0}\otimes E_{n_1}\otimes\cdots\otimes E_{n_r}\Bigg[X_G\{C_{(1)}D\}+\sum_{h=1}^{r-1}\sum_{J_h\subset R-1}X_{J_h}\{A_{(1)}B^{(J_h)}\}\Bigg]$$

$$+E_{n_0}\otimes\prod_{\zeta=1}^{r}\otimes\left(E_{n_\zeta}^{1-\delta_1^\zeta}\times I_{n_\zeta}^{\delta_1^\zeta}\right)\Bigg[X_1\{C_{(1)}D\}+\sum_{e=1}^{r-1}\sum_{V_e\subset R-1}X_{1V_e}\{A_{(1)}B^{(V_e)}\}+X_0\sigma_1^2\Bigg]$$

$$+E_{n_0}\otimes\prod_{\zeta=1}^{r}\otimes\left(E_{n_\zeta}^{1-\delta_R^\zeta}\times I_{n_\zeta}^{\delta_R^\zeta}\right)\left[X_R\{C_{(1)}D_{(R-1)}\}+X_0\sigma_R^2\right]$$

$$+\sum_{h=1}^{r-2}\sum_{J_h\subset R-1}E_{n_0}\otimes I_{n_1}\otimes\prod_{\zeta=2}^{r}\otimes\left(E_{n_\zeta}^{1-\delta_{J_h}^\zeta}\times I_{n_\zeta}^{\delta_{J_h}^\zeta}\right)\Big[X_{1J_h}\{C_{(1)}D_{(J_h)}\}$$

$$+\sum_{e=1}^{r-1-h}\sum_{V_e\subset R-1-J_h}X_{1J_hV_e}\{A_{(1)}B_{(J_h)}^{(V_e)}\}+X_0\sigma_{1J_h}^2\Big]$$

$$+E_{n_0}\otimes\prod_{\zeta=1}^{r}\otimes\left(E_{n_\zeta}^{1-\delta_{R-1}^\zeta}\times I_{n_\zeta}^{\delta_{R-1}^\zeta}\right)\Big[X_{R-1}\{C_{(1)}D_{(R-1)}\}\Big]$$

$$+\sum_{h=1}^{r-2}\sum_{J_h\subset R-1}E_{n_0}\otimes\prod_{\zeta=1}^{r}\otimes\left(E_{n_\zeta}^{1-\delta_{J_h}^\zeta}\times I_{n_\zeta}^{\delta_{J_h}^\zeta}\right)\Big[X_{J_h}\{C_{(1)}D_{(J_h)}\}+\sum_{e=1}^{r-1-h}\sum_{V_e\subset R-1-J_h}X_{J_hV_e}\{A_{(1)}B_{(J_h)}^{(V_e)}\}\Big]$$

$$+I_{n_0}\otimes I_{n_1}\otimes\cdots\otimes I_{n_r}X_0\sigma_0^2.$$

Thus the condition we are seeking for is expressed by the following equations,

(3.21) $$X_0\sigma_0^2=1,$$

(3.22) $$X_R\{C_{(1)}D_{(R-1)}\}=-X_0\sigma_R^2,$$

(3.23) $$X_{1J_h}\{C_{(1)}D_{(J_h)}\}=-\sum_{e=1}^{r-1-h}\sum_{V_e\subset R-1-J_h}X_{1J_hV_e}\{A_{(1)}B_{(J_h)}^{(V_e)}\}-X_0\sigma_{1J_h}^2,$$

$(J_h\subset R-1;\ h=1,2,\cdots,r-2)$,

(3.24) $$X_1\{C_{(1)}D\}=-\sum_{e=1}^{r-1}\sum_{V_e\subset R-1}X_{1V_e}\{A_{(1)}B^{(V_e)}\}-X_0\sigma_1^2,$$

(3.25) $$X_{R-1}\{C_{(1)}D_{(R-1)}\}=0,$$

(3.26) $$X_{J_h}\{C_{(1)}D_{(J_h)}\}=-\sum_{e=1}^{r-1-h}\sum_{V_e\subset R-1-J_h}X_{J_hV_e}\{A_{(1)}B_{(J_h)}^{(V_e)}\},$$

$(J_h\subset R-1;\ h=1,2,\cdots,r-2)$,

(3.27) $$X_G\{C_{(1)}D\}=-\sum_{h=1}^{r-1}\sum_{J_h\subset R-1}X_{J_h}\{A_{(1)}B^{(J_h)}\}.$$

The solutions of the last three systems of the equations (3.25), (3.26) and (3.27) are given as follows, it is obvious, by solving successively,

(3.28) $$X_{J_h}=0, \qquad (J_h\subset R-1;\ h=1,2,\cdots,r-1),$$

$$X_G=0.$$

Since, as is easily seen, to solve the remaining four systems of the equations, (3.21), ⋯, (3.24) corresponds to solving (4.27), (4.28), (4.29) in [2], the solutions of (3.21), ⋯, (3.24) are given by (3.15), (3.16), (3.17) and (3.18).

Thus the proof of this theorem is completed.

3.3. The joint density function.

The joint density function in Type I is given in the following, THEOREM 3.3. *The joint density function of all observations in our case is given by*

(3.29) $$f(X)=(2\pi)^{-n_0n_1\cdots n_r/2}\prod_{h=0}^{r-1}\prod_{J_h\subset R-1}\{C_{(1)}D_{(J_h)}\}^{-n_1(n_{j_1}-1)\cdots(n_{j_h}-1)/2}\{\sigma_0^2\}^{-(n_0-1)n_1\cdots n_r/2}$$

$$\cdot\, exp\left[-\frac{1}{2}\left\{\frac{S'_{(1)}}{C_{(1)}D}+\sum_{h=1}^{r-1}\sum_{J_h\subset R-1}\frac{S'_{(1J_h)}}{C_{(1)}D_{(J_h)}}+\frac{S_0}{\sigma_0^2}\right\}\right],$$

where

(3.30) $$\bar{X}_{t_1t_{l_1}\cdots t_{l\beta}}=\frac{1}{\prod_{\zeta=0}^{r}n_\zeta^{1-\delta_{1L\beta}^{\zeta}}}\sum_{\substack{t_{i_1},\cdots,t_{i_{r-1-\beta}}\\ I_{r-1-\beta}\subset R-1-L_\beta}}\sum_{t_0}x_{t_0t_1\cdots t_r},\qquad (L_\beta\subset R-1;\ \beta=0,1,\cdots,r-1),$$

(3.31) $$S'_{(1)}=\prod_{\zeta=0}^{r}n_\zeta^{1-\delta_1^\zeta}\sum_{t_1=1}^{n_1}(\bar{X}_{t_1}-\mu)^2,$$

(3.32) $$S'_{(1J_h)}=\prod_{\zeta=0}^{r}n_\zeta^{1-\delta_{1J_h}^{\zeta}}\sum_{t_1,t_{j_1},\cdots,t_{j_h}}\left\{\sum_{\beta=0}^{h}\sum_{L_\beta\subset J_h}(-1)^{h-\beta}\bar{X}_{t_1t_{l_1}\cdots t_{l\beta}}-\alpha(j_1,\cdots,j_h;\ t_{j_1},\cdots,t_{j_h})\right\}^2,$$

$$(J_h\subset R-1;\ h=1,\cdots,r-1),$$

(3.33) $$S_0=\sum_{t_0t_1\cdots t_r}(x_{t_0t_1\cdots t_r}-\bar{X}_{t_0t_1\cdots t_r})^2,$$

where in (3.30) $\bar{X}_{t_1t_{l_1}\cdots t_{l\beta}}=\bar{X}_{t_1}$ *when* $\beta=0$.

PROOF. As it is obvious that the type of the density function is the normal distribution, the constant factor in (3.29) is easily derived from Theorem 3.2. Before evaluating the quadratic form of $x_{t_0t_1\cdots t_r}$, let us introduce new variables defined by

(3.34) $$u_{t_0t_1\cdots t_r}=x_{t_0t_1\cdots t_r}-\mu-\sum_{h=1}^{r-1}\sum_{J_h\subset R-1}\alpha(j_1,\cdots,j_h;\ t_{j_1},\cdots,t_{j_h}),$$

(3.35) $$U_{t_1t_{l_1}\cdots t_{l\beta}}=\sum_{\substack{t_{d_1},\cdots,t_{d_{r-1-\beta}}\\ D_{r-1-\beta}\subset R-1-L_\beta}}\sum_{t_0}u_{t_0t_1\cdots t_r},\qquad (L_\beta\subset R-1;\ \beta=0,\cdots,r-1),$$

(3.36) $$\bar{U}_{t_1t_{l_1}\cdots t_{l\beta}}=\frac{1}{\prod_{\zeta=0}^{r}n_\zeta^{1-\delta_{1L\beta}^{\zeta}}}U_{t_1t_{l_1}\cdots t_{l\beta}},\qquad (L_\beta\subset R-1;\ \beta=0,\cdots,r-1),$$

where when $\beta=0$ we shall use the convention for $U_{t_1t_{l_1}\cdots t_{l\beta}}$ and $\bar{U}_{t_1t_{l_1}\cdots t_{l\beta}}$ same to that stated in this theorem. Furthermore, let us remark that (3.36) may be expressed in terms of $\bar{X}_{t_1t_{l_1}\cdots t_{l\beta}}$, μ and $\alpha(v_1,v_2,\cdots,v_p;\ t_{v_1},t_{v_2},\cdots,t_{v_p})$, by the assumptions (2.2), as follows,

(3.37) $$\bar{U}_{t_1t_{l_1}\cdots t_{l\beta}}=\bar{X}_{t_1t_{l_1}\cdots t_{l\beta}}-\mu-\sum_{p=1}^{\beta}\sum_{V_p\subset L_\beta}\alpha(v_1,\cdots,v_p;\ t_{v_1},\cdots,t_{v_p}),$$

(3.38) $$\bar{U}_{t_1}=\bar{X}_{t_1}-\mu.$$

Using the inverse matrix derived in Theorem 3.2 we have the term of the quadratic form in the joint density function that

$$(3.39)\quad S=X_1\Big\{\sum_{t_1}\Big(\sum_{t_0,t_2,\cdots,t_r} u_{t_0t_1\cdots t_r}\Big)^2\Big\}+\sum_{h=1}^{r-2}\sum_{J_h\subset R-1} X_{1J_h}\Big\{\sum_{t_1,t_{j_1},\cdots,t_{j_h}}\Big(\sum_{\substack{t_0,t_{d_1},\cdots,t_{d_{r-h-1}}\\ D_{r-h-1}\subset R- -}} u_{t_0t_1\cdots t_r}\Big)^2\Big\}$$

$$+X_R\sum_{t_1,\cdots,t_r}\Big(\sum_{t_0} u_{t_0t_1\cdots t_r}\Big)^2+X_0\sum_{t_0,t_1,\cdots,t_r} u^2_{t_0t_1\cdots t_r}$$

$$=X_1\sum_{t_1} U^2_{t_1}+\sum_{h=1}^{r-2}\sum_{J_h\subset R-1} X_{1J_h}\Big\{\sum_{t_1,t_{j_1},\cdots,t_{j_h}} U^2_{t_1t_{j_1}\cdots t_{j_h}}\Big\}$$

$$+X_R\sum_{t_1,\cdots,t_r} U^2_{t_1\cdots t_r}+X_0\sum_{t_0,t_1,\cdots,t_r} u^2_{t_0t_1\cdots t_r}$$

$$=\frac{1}{\prod_{\zeta=0}^{r} n_\zeta^{1-\delta_1^\zeta}}\Big[\sum_{\beta=0}^{r-1}\sum_{N\beta\subset R-1}\frac{(-1)^\beta}{C_{(1)}D_{(N\beta)}}\Big]\Big(\sum_{t_1} U^2_{t_1}\Big)$$

$$+\sum_{h=1}^{r-2}\sum_{J_h\subset R-1}\frac{1}{\prod_{\zeta=0}^{r} n_\zeta^{1-\delta_{1J_h}^\zeta}}\Big[\sum_{\beta=0}^{r-1-h}\sum_{N\beta\subset R-1-J_h}\frac{(-1)^\beta}{C_{(1)}D_{(J_hN\beta)}}\Big]\Big(\sum_{t_1,t_{j_1},\cdots,t_{j_h}} U^2_{t_1t_{j_1}\cdots t_{j_h}}\Big)$$

$$+\frac{1}{n_0}\Big[\frac{1}{C_{(1)}D_{(R-1)}}-\frac{1}{\sigma_0^2}\Big]\Big(\sum_{t_1,\cdots,t_r} U^2_{t_1\cdots t_r}\Big)+\frac{1}{\sigma_0^2}\Big(\sum_{t_0,\cdots,t_r} u^2_{t_0t_1\cdots t_r}\Big).$$

The coefficients to $\frac{1}{C_{(1)}D}$, $\frac{1}{C_{(1)}D_{(L\beta)}}$ and $\frac{1}{C_{(1)}D_{(R-1)}}$ in (3.39) are given by

$$(3.40)\quad \frac{1}{\prod_{\zeta=0}^{r} n_\zeta^{1-\delta_1^\zeta}}\Big(\sum_{t_1} U^2_{t_1}\Big),$$

$$(3.41)\quad \sum_{h=0}^{\beta}\sum_{J_h\subset L\beta}\frac{(-1)^{\beta-h}}{\prod_{\zeta=0}^{r} n_\zeta^{1-\delta_{1J_h}^\zeta}}\Big(\sum_{t_1,t_{j_1},\cdots,t_{j_h}} U^2_{t_1t_{j_1}\cdots t_{j_h}}\Big),$$

and

$$(3.42)\quad \sum_{h=0}^{r-1}\sum_{J_h\subset R-1}\frac{(-)^{r-1-h}}{\prod_{\zeta=0}^{r} n_\zeta^{1-\delta_{1J_h}^\zeta}}\Big(\sum_{t_1,t_{j_1},\cdots,t_{j_h}} U^2_{t_1t_{j_1}\cdots t_{j_h}}\Big),$$

respectively.

Therefore, by Lemma 4.4 in [2], (3.39) is equal to

$$(3.43)\quad \sum_{\beta=1}^{r-1}\sum_{L\beta\subset R-1}\sum_{h=0}^{\beta}\sum_{J_h\subset L\beta}\frac{(-1)^{\beta-h}}{\prod_{\zeta=0}^{r} n_\zeta^{1-\delta_{1J_h}^\zeta}}\Big(\sum_{t_1,t_{j_1},\cdots,t_{j_h}} U^2_{t_1t_{j_1}\cdots t_{j_h}}\Big)\frac{1}{C_{(1)}D_{(L\beta)}}$$

$$+\Big[\sum_{t_0,t_1,\cdots,t_r} u^2_{t_0t_1\cdots t_r}-\frac{1}{n_0}\sum_{t_1\cdots t_r} U^2_{t_1\cdots t_r}\Big]\frac{1}{\sigma_0^2}+\frac{1}{\prod_{\zeta=0}^{r} n_\zeta^{1-\delta_1^\zeta}}\Big(\sum_{t_1} U^2_{t_1}\Big)\frac{1}{C_{(1)}D}$$

$$=\sum_{\beta=1}^{r-1}\sum_{L\beta\subset R-1}\sum_{h=0}^{\beta}\sum_{J_h\subset L\beta}(-1)^{\beta-h}\prod_{\zeta=0}^{r} n_\zeta^{1-\delta_{1J_h}^\zeta}\Big(\sum_{t_1,t_{j_1},\cdots,t_{j_h}} \bar{U}^2_{t_1t_{j_1}\cdots t_{j_h}}\Big)\frac{1}{C_{(1)}D_{(L\beta)}}$$

$$+\prod_{\zeta=0}^{r} n_\zeta^{1-\delta_1^\zeta}\Big(\sum_{t_1}\bar{U}_{t_1}^2\Big)\frac{1}{C_{(1)}D}+\Big[\sum_{t_0,t_1,\cdots,t_r} u_{t_0t_1\cdots t_r}^2-n_0\sum_{t_1}\bar{U}_{t_1}^2\Big]\frac{1}{\sigma_0^2}$$

$$=\sum_{\beta=1}^{r-1}\sum_{L_\beta\subset R-1}\prod_{\zeta=0}^{r} n_\zeta^{1-\delta_{1L_\beta}^\zeta}\Big[\sum_{t_1,t_{l_1},\cdots,t_{l_\beta}}\sum_{h=0}^{\beta}\sum_{J_h\subset L_\beta}(-1)^{\beta-h}\bar{U}_{t_1t_{j_1}\cdots t_{j_h}}^2\Big]\frac{1}{C_{(1)}D_{(L_\beta)}}$$

$$+\prod_{\zeta=0}^{r} n_\zeta^{1-\delta_1^\zeta}\Big(\sum_{t_1}\bar{U}_{t_1}^2\Big)\frac{1}{C_{(1)}D}+\Big[\sum_{t_0,t_1,\cdots,t_r} u_{t_0t_1\cdots t_r}^2-n_0\sum_{t_1\cdots t_r}\bar{U}_{t_1\cdots t_r}^2\Big]\frac{1}{\sigma_0^2}$$

$$=\sum_{\beta=1}^{r-1}\sum_{L_\beta\subset R-1}\prod_{\zeta=0}^{r} n_\zeta^{1-\delta_{1L_\beta}^\zeta}\Big[\sum_{t_1,t_{l_1},\cdots,t_{l_\beta}}\Big\{\sum_{h=0}^{\beta}\sum_{J_h\subset L_\beta}(-1)^{\beta-h}\bar{U}_{t_1t_{j_1}\cdots t_{j_h}}\Big\}^2\Big]\frac{1}{C_{(1)}D_{(L_\beta)}}$$

$$+\prod_{\zeta=0}^{r} n_\zeta^{1-\delta_1^\zeta}\Big(\sum_{t_1}\bar{U}_{t_1}^2\Big)\frac{1}{C_{(1)}D}+\sum_{t_0,t_1,\cdots,t_r}(u_{t_0t_1\cdots t_r}-\bar{U}_{t_1\cdots t_r})^2\frac{1}{\sigma_0^2}$$

$$=\sum_{h=1}^{r-1}\sum_{J_h\subset R-1}\prod_{\zeta=0}^{r} n_\zeta^{1-\delta_{1J_h}^\zeta}\sum_{t_1,t_{j_1},\cdots,t_{j_h}}\Big\{\sum_{\beta=0}^{h}\sum_{L_\beta\subset J_h}(-1)^{h-\beta}\bar{U}_{t_1t_{l_1}\cdots t_{l_\beta}}\Big\}^2\frac{1}{C_{(1)}D_{(J_h)}}$$

$$+\prod_{\zeta=0}^{r} n_\zeta^{1-\delta_1^\zeta}\Big(\sum_{t_1}\bar{U}_{t_1}^2\Big)\frac{1}{C_{(1)}D}+\sum_{t_0,t_1,\cdots,t_r}(u_{t_0t_1\cdots t_r}-\bar{U}_{t_1\cdots t_r})^2\frac{1}{\sigma_0^2}.$$

After inserting (3.37) and (3.38) in the above formula, (3.43) is given by

(3.44)

$$\sum_{h=1}^{r-1}\sum_{J_h\subset R-1}\prod_{\zeta=0}^{r} n_\zeta^{1-\delta_{1J_h}^\zeta}\sum_{t_1,t_{j_1},\cdots,t_{j_h}}\Big[\sum_{\beta=0}^{h}\sum_{L_\beta\subset J_h}(-1)^{h-\beta}\Big\{\bar{X}_{t_1t_{l_1}\cdots t_{l_\beta}}-\mu-\sum_{p=1}^{\beta}\sum_{V_p\subset L_\beta}\alpha(v_1,\cdots,v_p;t_{v_1},\cdots,t_{v_p})\Big\}\Big]^2$$

$$\cdot\frac{1}{C_{(1)}D_{(J_h)}}+\prod_{\zeta=0}^{r} n_\zeta^{1-\delta_1^\zeta}\Big\{\sum_{t_1}(\bar{X}_{t_1}-\mu)^2\Big\}\frac{1}{C_{(1)}D}$$

$$+\sum_{t_0,\cdots,t_r}(x_{t_0t_1\cdots t_r}-\bar{X}_{t_1\cdots t_r})^2\frac{1}{\sigma_0^2}.$$

As the formula which is squared in the first term is simplified as follows,

(3.45) $$\sum_{\beta=0}^{h}\sum_{L_\beta\subset J_h}(-1)^{h-\beta}\bar{X}_{t_1t_{l_1}\cdots t_{l_\beta}}-\sum_{\beta=0}^{h}\sum_{L_\beta\subset J_h}(-1)^{h-\beta}\sum_{p=1}^{\beta}\sum_{V_p\subset L_\beta}\alpha(v_1,v_2,\cdots,v_p;\ t_{v_1},t_{v_2},\cdots,t_{v_p})$$

$$=\sum_{\beta=0}^{h}\sum_{L_\beta\subset J_h}(-1)^{h-\beta}\bar{X}_{t_1t_{l_1}\cdots t_{l_\beta}}-\sum_{k=1}^{h-1}\sum_{V_k\subset J_h}\{{}_{h-k}C_0(-1)^{h-k}+{}_{h-k}C_1(-1)^{h-k-1}+\cdots+{}_{h-k}C_{h-k}(-1)^0\}$$

$$\cdot\,\alpha(v_1,\cdots,v_k;\ t_{v_1},\cdots,t_{v_k})-\alpha(j_1,\cdots,j_h;\ t_{j_1},\cdots,t_{j_h})$$

$$=\sum_{\beta=0}^{h}\sum_{L_\beta\subset J_h}(-1)^{h-\beta}\bar{X}_{t_1t_{l_1}\cdots t_{l_\beta}}-\alpha(j_1,\cdots,j_h;\ t_{j_1},\cdots,t_{j_h}),$$

finally (3.43) is equal to

(3.46)

$$\sum_{h=1}^{r-1}\sum_{J_h\subset R-1}\prod_{\zeta=0}^{r} n_\zeta^{1-\delta_{1J_h}^\zeta}\sum_{t_1,t_{j_1},\cdots,t_{j_h}}\Big\{\sum_{\beta=0}^{h}\sum_{L_\beta\subset J_h}(-1)^{h-\beta}\bar{X}_{t_1t_{l_1}\cdots t_{l_\beta}}-\alpha(j_1,\cdots,j_h;\ t_{j_1},\cdots,t_{j_h})\Big\}^2\frac{1}{C_{(1)}D_{(J_h)}}$$

$$+\prod_{\zeta=0}^{r} n_\zeta^{1-\delta_1^\zeta}\sum_{t_1}(\bar{X}_{t_1}-\mu)^2\frac{1}{C_{(1)}D}+\sum_{t_0,t_1,\cdots,t_r}(x_{t_0t_1\cdots t_r}-\bar{X}_{t_1\cdots t_r})^2\frac{1}{\sigma_0^2},$$

which completes the theorem.

3.4. Estimation.

In order to show the sufficient statistic for the distribution whose joint density function is given by Theorem 3.3 and to derive the density function of the sufficient statistic, we need to modificate (3.29) further.

Now we have

LEMMA 3.3. *The joint density function of all observations,* (3.29), *is equal to*

$$f(X)=K\varphi(\sigma^2,\alpha,\mu)$$

(3.47) $$\cdot\, exp\Big[Z^{(1)}\frac{\mu}{C_{(1)}D}+\sum_{h=1}^{r-1}\sum_{J_h\subset R-1}\sum_{t_{j_1}=1}^{n_{j_1}-1}\sum_{t_{j_2}=1}^{n_{j_2}-1}\cdots\sum_{t_{j_h}=1}^{n_{j_h}-1}Z^{(2)}_{(J_h;t_{j_1},\cdots,t_{j_h})}\frac{\alpha(j_1,\cdots,j_h;t_{j_1},\cdots,t_{j_h})}{C_{(1)}D_{(J_h)}}$$
$$-\frac{1}{2}Z^{(3)}\frac{1}{C_{(1)}D}-\frac{1}{2}\sum_{h=1}^{r-1}\sum_{J_h\subset R-1}Z^{(4)}_{(J_h)}\frac{1}{C_{(1)}D_{(J_h)}}-\frac{S_0}{2\sigma_0^2}\Big],$$

where

(3.48) $$Z^{(1)}=\prod_{i=0}^{r}n_i\bar{X},$$

(3.49) $$Z^{(2)}_{(J_h;t_{j_1},\cdots,t_{j_h})}=\prod_{\zeta=0}^{r}n_\zeta^{1-\delta^\zeta_{J_h}}\Big(\bar{X}_{t_{j_1}\cdots t_{j_h}}+\sum_{\gamma=1}^{h}\sum_{D_{h-\gamma}\subset J_h}(-1)^\gamma\bar{X}_{t_{d_1}\cdots t_{d_{h-\gamma}}n_{d_{h-\gamma+1}}\cdots n_{d_h}}\Big),$$

(3.50) $$Z^{(3)}=\prod_{\zeta=0}^{r}n_\zeta^{1-\delta_1^\zeta}\Big(\sum_{t_1}\bar{X}_{t_1}^2\Big),$$

(3.51) $$Z^{(4)}_{(J_h)}=\prod_{\zeta=0}^{r}n_\zeta^{1-\delta^\zeta_{1J_h}}\sum_{t_1,t_{j_1},\cdots,t_{j_h}}\Big(\sum_{\beta=0}^{h}\sum_{L_\beta\subset J_h}(-1)^{h-\beta}\bar{X}_{t_1t_{l_1}\cdots t_{l_\beta}}\Big)^2.$$

PROOF. It should be enough to work out on (3.32), which is equal to except for the constant term, in virtue of (2.2),

(3.52) $$\sum_{t_1,t_{j_1},\cdots,t_{j_h}}\Big\{\sum_{\beta=0}^{h}\sum_{L_\beta\subset J_h}(-1)^{h-\beta}\bar{X}_{t_1t_{l_1}\cdots t_{l_\beta}}\Big\}^2$$
$$-2\sum_{t_1,t_{j_1},\cdots,t_{j_h}}\Big\{\sum_{\beta=0}^{h}\sum_{L_\beta\subset J_h}(-1)^{h-\beta}\bar{X}_{t_1t_{l_1}\cdots t_{l_\beta}}\Big\}\alpha(j_1,\cdots,j_h;\ t_{j_1},\cdots,t_{j_h})+g(\alpha)$$
$$=\sum_{t_1,t_{j_1},\cdots,t_{j_h}}\Big\{\sum_{\beta=0}^{h}\sum_{L_\beta\subset J_h}(-1)^{h-\beta}\bar{X}_{t_1t_{l_1}\cdots t_{l_\beta}}\Big\}^2$$
$$-2n_1\sum_{t_{j_1},\cdots,t_{j_h}}\bar{X}_{t_{j_1}\cdots t_{j_h}}\alpha(j_1,\cdots,j_h;\ t_{j_1},\cdots,t_{j_h})+g(\alpha).$$

On the other hand we have that

(3.53) $$\sum_{t_{j_1},\cdots,t_{j_k}}\bar{X}_{t_{j_1}\cdots t_{j_k}}\alpha(j_1,\cdots,j_k;\ t_{j_1},\cdots,t_{j_k})$$

$$=\sum_{t_{j_1}=1}^{n_{j_1}-1}\sum_{t_{j_2}=1}^{n_{j_2}-1}\cdots\sum_{t_{j_k}=1}^{n_{j_k}-1}\left\{X_{t_{j_1}\cdots t_{j_k}}+\sum_{\gamma=1}^{k}\sum_{D_{k-\gamma}\subset J_k}(-1)^{\gamma}\overline{X}_{t_{d_1}\cdots t_{d_{k-\gamma}}n_{d_{k-\gamma+1}}\cdots n_{d_k}}\right\}$$

$$\cdot\,\alpha(j_1,\cdots,j_k;\ t_{j_1},\cdots,t_{j_k}),\qquad (J_k\subset R-1;\ k=1,2,\cdots,r-1),$$

for, when $k=1$

(3.54) $$\sum_{t_{j_1}}\overline{X}_{t_{j_1}}\alpha(j_1;t_{j_1})=\sum_{t_{j_1}=1}^{n_{j_1}-1}X_{t_{j_1}}\alpha(j_1;t_{j_1})+\overline{X}_{n_{j_1}}\alpha(j_1;n_{j_1})$$

$$=\sum_{t_{j_1}=1}^{n_{j_1}-1}\overline{X}_{t_{j_1}}\alpha(j_1;t_{j_1})+\overline{X}_{n_{j_1}}\left(-\sum_{t_{j_1}=1}^{n_{j_1}-1}\alpha(j_1;t_{j_1})\right)$$

$$=\sum_{t_{j_1}=1}^{n_{j_1}-1}(\overline{X}_{t_{j_1}}-\overline{X}_{n_{j_1}})\,\alpha(j_1;\ t_{j_1}),$$

and when we assume that (3.51) holds true for $k=c$

(3.55) $$\sum_{t_{j_1},\cdots,t_{j_{c+1}}}\overline{X}_{t_{j_1}\cdots t_{j_{c+1}}}\alpha(j_1\cdots j_{c+1};\ t_{j_1}\cdots t_{j_{c+1}})$$

$$=\sum_{t_{j_1},\cdots,t_{j_c}}\left\{\sum_{t_{j_{c+1}}=1}^{n_{j_{c+1}}-1}\overline{X}_{t_{j_1}\cdots t_{j_{c+1}}}\alpha(j_1,\cdots,j_{c+1};\ t_{j_1},\cdots,t_{j_{c+1}})\right.$$
$$\left.+\overline{X}_{t_{j_1}\cdots t_{j_c}n_{j_{c+1}}}\alpha(j_1,\cdots,j_{c+1};\ t_{j_1},\cdots,t_{j_c},n_{j_{c+1}})\right\}$$

$$=\sum_{t_{j_1},\cdots,t_{j_c}}\left\{\sum_{t_{i_{c+1}}=1}^{n_{j_{c+1}}-1}(\overline{X}_{t_{j_1}\cdots t_{j_{c+1}}}-\overline{X}_{t_{j_1}\cdots t_{j_c}n_{j_{c+1}}})\alpha(j_1,\cdots,j_{c+1};\ t_{j_1},\cdots,t_{j_{c+1}})\right\}$$

$$=\sum_{t_{j_1}=1}^{n_{j_1}-1}\cdots\sum_{t_{j_{c+1}}=1}^{n_{j_{c+1}}-1}\left\{\overline{X}_{t_{j_1}\cdots t_{j_{c+1}}}+\sum_{\gamma=1}^{c}\sum_{D_{c-\gamma}\subset J_c}(-1)^{\gamma}\overline{X}_{t_{d_1}\cdots t_{d_{c-\gamma}}n_{d_{c-\gamma+1}}\cdots n_{d_c}t_{j_{c+1}}}\right\}$$
$$\cdot\,\alpha(j_1,\cdots,j_{c+1};\ t_{j_1},\cdots,t_{j_{c+1}})$$
$$-\sum_{t_{j_1}=1}^{n_{j_1}-1}\cdots\sum_{t_{j_c}=1}^{n_{j_c}-1}\left\{\overline{X}_{t_{j_1}\cdots t_{j_c}n_{j_{c+1}}}+\sum_{\gamma=1}^{c}\sum_{D_{c-\gamma}\subset J_c}(-1)^{\gamma}\overline{X}_{t_{d_1}\cdots t_{d_{c-\gamma}}n_{d_{c-\gamma+1}}\cdots n_{d_c}n_{j_{c+1}}}\right\}$$
$$\cdot\,\alpha(j_1,\cdots,j_{c+1};\ t_{j_1},\cdots,t_{j_{c+1}})$$

$$=\sum_{t_{j_1}=1}^{n_{j_1}-1}\cdots\sum_{t_{j_{c+1}}=1}^{n_{j_{c+1}}-1}\left\{\overline{X}_{t_{j_1}\cdots t_{j_{c+1}}}+\sum_{\gamma=1}^{c+1}\sum_{D_{c+1-\gamma}\subset J_{c+1}}(-1)^{\gamma}\overline{X}_{t_{d_1}\cdots t_{d_{c+1-\gamma}}n_{d_{c+2-\gamma}}\cdots n_{j_{c+1}}}\right\}$$
$$\cdot\,\alpha(j_1,\cdots,j_{c+1};\ t_{j_1},\cdots,t_{j_{c+1}}),$$

therefore (3.52) should be given by

$$\sum_{t_1,t_{j_1},\cdots,t_{j_h}}\left\{\sum_{\beta=0}^{h}\sum_{L_\beta\subset J_h}(-)^{h-\beta}\overline{X}_{t_1t_{l_1}\cdots t_{l_\beta}}\right\}^2$$

$$-2n_1\sum_{t_{j_1}=1}^{n_{j_1}-1}\cdots\sum_{t_{j_h}=1}^{n_{j_h}-1}\left\{\overline{X}_{t_{j_1}\cdots t_{j_h}}+\sum_{\gamma=1}^{h}\sum_{D_{h-\gamma}\subset J_h}(-)^{\gamma}\overline{X}_{t_{d_1}\cdots t_{d_{h-\gamma}}n_{d_{h-\gamma+1}}\cdots n_{d_h}}\right\}$$
$$\cdot\,\alpha(j_1,\cdots,j_h;\ t_{j_1},\cdots,t_{j_h})+g(\alpha).$$

This completes the proof of Lemma 3.3.

For the development of the arguments we need to define a number of notations and consider about the completeness of the distributions of the sufficient statistics.

Let X be a random variable, R^X be a sample space of X, P_θ^X be the probability distribution of X defined over R^X, which is indexed by a subscript θ taking its value in an abstract space Ω, and $\mathfrak{P}^X=\{P_\theta^X|\theta\in\Omega\}$ be a family of the probability distributions of X. And let $U=u(X)$ be the sufficient statistic for $\mathfrak{P}^X$, R^U be a space of the sufficient statistic U and $\mathfrak{P}^U=\{P_\theta^U|\theta\in\Omega\}$ be a family of the probability distributions of U defined over R_U.

We remark the well-known result, (see [6], [7]), that if there exists a sufficient statistic U for $\mathfrak{P}^X$ such that $\mathfrak{P}^U$ is complete then a statistic is a minimum variance estimate of its expected value if and only if it is the function of U (a. e. $\mathfrak{P}^X$), and yet it is unique for U.

As for completeness, we have the following theorem:
If $X=(X_1, X_2, \cdots, X_n)$, $\theta=(\theta_1, \theta_2, \cdots, \theta_k)$ if Ω contains a non-degenerate k-dimensional interval, and if P_θ^X has a density of the form

(3.56) $$dP_\theta^X/d\mu^X=C(\theta)exp\left[\sum_{i=1}^{k}\theta_i u_i(X)\right]$$

with respect to a measure μ^X on the class $\mathfrak{F}^X$ of Borel sets in the n-dimensional Euclidean space W^X, then $U=(u_1(X), u_2(X), \cdots, u_k(X))$ is a sufficient statistic for $\mathfrak{P}^X$ with a probability density

(3.57) $$dP_\theta^U/d\nu^U=C'(\theta)h(u)exp\left[\sum_{i=1}^{k}\theta_i u_i\right]$$

with respect to a measure ν^U on R^U, and the family $\mathfrak{P}_U$ is strongly complete;(see [7]).

In our mixed model of Type I, the random variable is $X=(X_{11\cdots1}, \cdots, X_{n_011\cdots1}; X_{121\cdots1}, \cdots, X_{n_021\cdots1}; \cdots\cdots; X_{1n_1\cdots n_r}, \cdots, X_{n_0n_1\cdots n_r})$, the sample space R^X is a $n_0n_1\cdots n_r$-dimensional Euclidean space, and the family $\mathfrak{P}^X$ is specified by the parameter $\theta=(\mu, \alpha(j_1, \cdots, j_h; t_{j_1}, \cdots, t_{j_h}), \sigma_1^2, \sigma_{1J_h}^2, \sigma_0^2; J_h\subset R-1, t_{j_c}=1, \cdots, n_{j_c}-1, c=1, \cdots, h, h=1, \cdots, r-1)$ whose space is of $(2^{r-1}+1+n_1n_2\cdots n_r)$-dimension, where $-\infty<\mu<\infty$, $-\infty<\alpha(j_1, \cdots, j_h; t_{j_1}, \cdots, t_{j_h})<\infty$, $0<\sigma_0^2<\infty$, $0<\sigma_1^2<\infty$, $0<\sigma_{1J_h}^2<\infty$.

As we have seen in Lemma 3.3, the probability density of X is given by (3.47). In order to derive the sufficient statistic for $\mathfrak{P}^X$, and to show the completeness of $\mathfrak{P}^U$, we consider the transformations of the original parameters such that

(3.58) $$\tau^{(1)}=\frac{\mu}{C_{(1)}D},$$

(3.59) $$\tau^{(2)}_{(J_h;t_{j_1},\cdots,t_{j_h})}=\frac{\alpha(j_1, \cdots, j_h; t_{j_1}, \cdots, t_{j_h})}{C_{(1)}D_{(J_h)}}, \qquad \begin{pmatrix}J_h\subset R-1;\ t_{j_c}=1, \cdots, n_{j_c}-1;\\ c=1, \cdots, h;\ h=1, \cdots, r-1\end{pmatrix}$$

(3.60) $$\tau^{(3)}=-\frac{1}{2C_{(1)}D},$$

(3.61) $\tau^{(4)}_{(J_h)}=-\frac{1}{2C_{(1)}D_{(J_h)}}$, $(J_h\subset R-1;\ h=1,\cdots,r-1)$,

(3.62) $\tau^{(5)}=-\frac{1}{2\sigma_0^2}$.

After observing the independency of the class of parsametric functions $\{C_{(1)}D_{(J_h)};\ J_h\subset R-1,\ h=1,\cdots,r-1\}$, it is noted that the transformation (3.58), $\cdots$, (3.62) from θ to $\tau=(\tau^{(1)},\ \tau^{(2)}_{(J_h;t_{j_1},\cdots,t_{j_h})},\ \tau^{(3)},\ \tau^{(4)}_{(J_h)},\ \tau^{(5)};\ J_h\subset R-1,\ t_{j_c}=1,\ \cdots,\ n_{j_c}-1,\ c=1,\cdots,\ h,\ h=1,\cdots,\ r-1)$ is one-to-one. Therefore, considering as if the given parameter were τ instead of θ, we can say that $\mathfrak{B}^X$ is specified by τ, where $-\infty<\tau^{(1)}<\infty,\ -\infty<\tau^{(2)}_{(J_h;t_{j_1},\cdots,t_{j_h})}<\infty,\ -\infty<\tau^{(5)}<\tau^{(4)}_{(M)}<\tau^{(4)}_{(N)}<\tau^{(3)}<0$ for any pair (M,N) such that $M\supset N$, $M\subset R-1$, $N\subset R-1$.

Then the probability density function of X is given in the form

(3.63)
$$K\varphi_{(\tau)}exp\Big[\tau^{(1)}Z^{(1)}+\sum_{h=1}^{r-1}\sum_{J_h\subset R-1}\sum_{t_{j_1}=1}^{n_{j_1}-1}\cdots\sum_{t_{j_h}=1}^{n_{j_h}-1}\tau^{(2)}_{(J_h;t_{j_1},\cdots,t_{j_h})}Z^{(2)}_{(J_h;t_{j_1},\cdots,t_{j_h})}$$
$$+\tau^{(3)}Z^{(3)}+\sum_{h=1}^{r-1}\sum_{J_h\subset R-1}\tau^{(4)}_{(J_h)}Z^{(4)}_{(J_h)}+\tau^{(5)}S_0\Big].$$

Therefore, from the above-reviewed results on sufficiency and completeness, the sufficient statistic for $\mathfrak{B}^X$ is $U=(Z^{(1)},\ Z^{(2)}_{(J_h;t_{j_1},\cdots,t_{j_h})},\ Z^{(3)},\ Z^{(4)}_{(J_h)},\ S_0;\ J_h\subset R-1,\ t_{j_c}=1,\ \cdots,\ n_{j_c}-1,\ c=1,\cdots,\ h,\ h=1,2,\cdots,\ r-1)$ and the family $\mathfrak{B}^U$ is strongly complete.

In the estimation problem of the parameters, the estimates usually adopted in the practice of statistical inferences are unbiased and based on the sufficient statistic U defined above. As we have observed its completeness, we can obtain the following:

THEOREM 3.4. *In the mixed model of Type I, the usual estimates of the parameters, such as the fixed treatment effects, the variance components of random treatment effects, etc., are the best unbiased estimates among all unbiased estimates.*

4. The case of Type II.

4.1. The determinant of the variance matrix.

At first under the model equation (2.1) we have the expression of the variance matrix in terms of the Kronecker products as follows,

(4.1)
$$V=\sum_{k=1}^{s}\sum_{I_k\subset S}\sigma^2_{I_k}E_{n_0}\otimes\prod_{\zeta=1}^{r}\otimes\Big(E_{n_\zeta}^{1-\delta^\zeta_{I_k}}\times I_{n_\zeta}^{\delta^\zeta_{I_k}}\Big)$$
$$+\sum_{k=1}^{s}\sum_{I_k\subset S}\sum_{h=1}^{r-s}\sum_{J_h\subset R-S}\sigma^2_{I_kJ_h}E_{n_0}\otimes\prod_{\zeta=1}^{r}\otimes\Big(E_{n_\zeta}^{1-\delta^\zeta_{I_kJ_h}}\times I_{n_\zeta}^{\delta^\zeta_{I_kJ_h}}\Big)+\sigma_0^2I_{n_0}\otimes I_{n_1}\otimes\cdots\otimes I_{n_r},$$

where $\delta^\zeta_{M_\alpha}$ is defined in (3.3).

Our next step is to introduce a number of notations given by

DEFINITION 4.1.

(4.2)
$$A_{(I_k)}B_{(J_h)}=\sum_{\alpha=k}^{s}\sum_{\substack{M_\alpha\supset I_k\\ M_\alpha\subset S}}\sum_{\beta=h}^{r-s}\sum_{\substack{N_\beta\supset J_h\\ N_\beta\subset R-S}}\sigma^2_{M_\alpha N_\beta}\prod_{\zeta=0}^{r}n_\zeta^{1-\delta^\zeta_{M_\alpha N_\beta}},$$

(4.3) $$A_{(I_k)}B \equiv \sum_{\alpha=k}^{s} \sum_{\substack{M_\alpha \supset I_k \\ M_\alpha \subset S}} \sum_{\beta=0}^{r-s} \sum_{N_\beta \subset R-S} \sigma^2_{M_\alpha N_\beta} \prod_{\zeta=0}^{r} n_\zeta^{1-\delta^\zeta_{M_\alpha N_\beta}},$$

(4.4) $$AB_{(J_h)} \equiv \sum_{\alpha=1}^{s} \sum_{M_\alpha \subset S} \sum_{\beta=h}^{r-s} \sum_{\substack{N_\beta \supset J_h \\ N_\beta \subset R-S}} \sigma^2_{M_\alpha N_\beta} \prod_{\zeta=0}^{r} n_\zeta^{1-\delta^\zeta_{M_\alpha N_\beta}},$$

(4.5) $$AB \equiv \sum_{\alpha=1}^{s} \sum_{M_\alpha \subset S} \sum_{\beta=0}^{r-s} \sum_{N_\beta \subset R-S} \sigma^2_{M_\alpha N_\beta} \prod_{\zeta=0}^{r} n_\zeta^{1-\delta^\zeta_{M_\alpha N_\beta}},$$

(4.6) $$A^{(U_d)}_{(I_k)} B^{(V_e)}_{(J_h)} \equiv \sum_{\alpha=k}^{s-d} \sum_{\substack{M_\alpha \supset I_k \\ M_\alpha \subset S-U_d}} \sum_{\beta=h}^{r-s-e} \sum_{\substack{N_\beta \supset J_h \\ N_\beta \subset R-S-V_e}} \sigma^2_{M_\alpha N_\beta} \prod_{\zeta=0}^{r} n_\zeta^{1-\delta^\zeta_{M_\alpha N_\beta U_d V_e}},$$

(4.7) $$C_{(I_k)}D_{(J_h)} \equiv A_{(I_k)}B_{(J_h)} + \sigma_0^2,$$

(4.8) $$C_{(I_k)}D \equiv A_{(I_k)}B + \sigma_0^2,$$

(4.9) $$CD_{(J_h)} \equiv AB_{(J_h)} + \sigma_0^2,$$

(4.10) $$CD \equiv AB + \sigma_0^2$$

(4.11) $$G_{(J_h)} \equiv \sum_{p=1}^{s} \sum_{L_p \subset S} (-1)^{p-1} C_{(L_p)} D_{(J_h)},$$

(4.12) $$G \equiv \sum_{p=1}^{s} \sum_{L_p \subset S} (-1)^{p} C_{(L_p)} D,$$

where $I_k \cap U_d = \phi$ and $J_h \cap V_e = \phi$.

Then we have the following relations:

(4.13) $$CD_{(J_h)} = \sum_{p=1}^{s} \sum_{L_p \subset S} (-1)^{p-1} C_{(L_p)} D_{(J_h)},$$

$$CD = \sum_{p=1}^{s} \sum_{L_p \subset S} (-1)^{p-1} C_{(L_p)} D.$$

Now we can evaluate the determinant of the variance matrix (4.1).

THEOREM 4.1. *The determinant* $|V|$ *of the variance matrix* (4.1) *is given in the form*

(4.14) $$G \prod_{d=1}^{r-s} \prod_{U_d \subset R-S} \{G_{(U_d)}\}^{(n_{u_1}-1)(n_{u_2}-1)\cdots(n_{u_d}-1)}$$

$$\cdot \prod_{k=1}^{s} \prod_{I_k \subset S} \prod_{h=0}^{r-s} \prod_{J_h \subset R-S} \{C_{(I_k)} D_{(J_h)}\}^{(n_{i_1}-1)\cdots(n_{i_k}-1)(n_{j_1}-1)\cdots(n_{j_h}-1)}$$

$$\cdot \{\sigma_0^2\}^{(n_0-1)n_1 n_2 \cdots n_r}.$$

PROOF. We shall transform the matrix (4.1) by the orthogonal matrix which is the Kronecker product of the matrixes T_{n_i} defined in Section 2, then we have

$$(4.15)\quad \sum_{k=1}^{s}\sum_{I_k\subset S}\sigma_{I_k}^2\prod_{t=0}^{r} n_t^{1-\delta_{I_k}^t}H_{n_0}\otimes\prod_{\zeta=1}^{r}\otimes\left(H_{n_\zeta}^{1-\delta_{I_k}^\zeta}\times I_{n_\zeta}^{\delta_{I_k}^\zeta}\right)$$

$$+\sum_{k=1}^{s}\sum_{I_k\subset S}\sum_{h=1}^{r-s}\sum_{J_h\subset R-S}\sigma_{I_kJ_h}^2\prod_{t=0}^{r} n_t^{1-\delta_{I_kJ_h}^t}H_{n_0}\otimes\prod_{\zeta=1}^{r}\otimes\left(H_{n_\zeta}^{1-\delta_{I_kJ_h}^\zeta}\times I_{n_\zeta}^{\delta_{I_kJ_h}^\zeta}\right)$$

$$+\sigma_0^2 I_{n_0}\otimes I_{n_1}\otimes\cdots\otimes I_{n_r}$$

$$=\sum_{k=1}^{s}\sum_{I_k\subset S}\sigma_{I_k}^2\prod_{t=0}^{r} n_t^{1-\delta_{I_k}^t}H_{n_0}\otimes\prod_{\zeta=1}^{r}\otimes\left(H_{n_\zeta}^{1-\delta_{I_k}^\zeta}\times (H_{n_\zeta}+K_{n_\zeta})^{\delta_{I_k}^\zeta}\right)$$

$$+\sum_{k=1}^{s}\sum_{J_k\subset S}\sum_{h=1}^{r-s}\sum_{J_h\subset R-S}\sigma_{I_kJ_h}^2\prod_{t=0}^{r} n_t^{1-\delta_{I_kJ_h}^t}H_{n_0}\otimes\prod_{\zeta=1}^{r}\otimes\left(H_{n_\zeta}^{1-\delta_{I_kJ_h}^\zeta}\times (H_{n_\zeta}+K_{n_\zeta})^{\delta_{I_kJ_h}^\zeta}\right)$$

$$+\sigma_0^2\prod_{\zeta=0}^{r}\otimes(H_{n_\zeta}+K_{n_\zeta})$$

$$=\sum_{k=1}^{s}\sum_{I_k\subset S}\sigma_{I_k}^2\prod_{t=0}^{r} n_t^{1-\delta_{I_k}^t}H_{n_0}\otimes\prod_{\zeta=1}^{r}\otimes(H_{n_\zeta}+\delta_{I_k}^\zeta K_{n_\zeta})$$

$$+\sum_{k=1}^{s}\sum_{I_k\subset S}\sum_{h=1}^{r-s}\sum_{J_h\subset R-S}\sigma_{I_kJ_h}^2\prod_{t=0}^{r} n_t^{1-\delta_{I_kJ_h}^t}H_{n_0}\otimes\prod_{\zeta=1}^{r}\otimes(H_{n_\zeta}+\delta_{I_kJ_h}^\zeta K_{n_\zeta})$$

$$+\sigma_0^2\prod_{\zeta=0}^{r}\otimes(H_{n_\zeta}+K_{n_\zeta}),$$

whose determinant is given by

$$(4.16)\quad \left\{\sum_{\alpha=1}^{s}\sum_{M_\alpha\subset S}\sum_{\beta=0}^{r-s}\sum_{N_\beta\subset R-S}\sigma_{M_\alpha N_\beta}^2\prod_{\zeta=0}^{r} n_\zeta^{1-\delta_{M_\alpha N_\beta}^\zeta}+\sigma_0^2\right\}$$

$$\cdot\prod_{d=1}^{r-s}\prod_{U_d\subset R-S}\cdot\left\{\sum_{\alpha=1}^{s}\sum_{M_\alpha\subset S}\sum_{\beta=d}^{r-s}\sum_{\substack{N_\beta\supset U_d\\ N_\beta\subset R-S}}\sigma_{M_\alpha N_\beta}^2\prod_{\zeta=0}^{r} n_\zeta^{1-\delta_{M_\alpha N_\beta}^\zeta}+\sigma_0^2\right\}^{(n_{u_1}-1)\cdots(n_{u_d}-1)}$$

$$\cdot\prod_{k=1}^{s}\prod_{I_k\subset S}\prod_{h=0}^{r-s}\prod_{J_h\subset R-S}$$

$$\cdot\left\{\sum_{\alpha=k}^{s}\sum_{\substack{M_\alpha\supset I_k\\ M_\alpha\subset S}}\sum_{\beta=h}^{r-s}\sum_{\substack{N_\beta\supset J_h\\ N_\beta\subset R-S}}\sigma_{M_\alpha N_\beta}^2\prod_{\zeta=0}^{r} n_\zeta^{1-\delta_{M_\alpha N_\beta}^\zeta}+\sigma_0^2\right\}^{(n_{i_1}-1)\cdots(n_{i_k}-1)(n_{j_1}-1)\cdots(n_{j_h}-1)}$$

$$\cdot\{\sigma_0^2\}^{(n_0-1)n_1n_2\cdots n_r}.$$

This formula is equivalent to (4.14).

4.2. The inverse of the variance matrix.

Before finding out the inverse of the variance matrix, we have to be prepared with some relations between the notations, defined in Definition 4.1, and some other results in order to simplify the complicated and tedious algebraic calculations.

LEMMA 4.1.

(4.17) $$A_{(I_k)}^{(U_d)}B_{(J_h)}^{(V_e)}=\frac{1}{n_{u_d}}[A_{(I_k)}^{(U_{d-1})}B_{(J_h)}^{(V_e)}-A_{(I_k u_d)}^{(U_{d-1})}B_{(J_h)}^{(V_e)}],$$

$$\begin{pmatrix} I_k\subset S;\ U_d\subset S;\ J_h\subset R-S;\ V_e\subset R-S; \\ k=0,\cdots,s;\ d=1,\cdots,s;\ e,h=0,\cdots,r-s. \end{pmatrix}.$$

PROOF.

(4.18) $$A_{(I_k)}^{(U_d)}B_{(J_h)}^{(V_e)}$$

$$=\sum_{\beta=h}^{r-s-e}\sum_{\substack{N_\beta\supset J_h\\ N_\beta\subset R-S-V_e}}\Bigg[\sum_{\alpha=0}^{s-d+1}\sum_{\substack{M_\alpha\subset S-U_{d-1}\\ M_\alpha\supset I_k}}\sigma^2_{M_\alpha N_\beta}\prod_{\zeta=0}^{r}n_\zeta^{1-\delta^\zeta_{M_\alpha N_\beta U_d V_e}}$$

$$-\sum_{\alpha=0}^{s-d+1}\sum_{\substack{M_\alpha\supset(I_k u_d)\\ M_\alpha\subset S-U_{d-1}}}\sigma^2_{M_\alpha N_\beta}\prod_{\zeta=0}^{r}n_\zeta^{1-\delta^\zeta_{M_\alpha N_\beta U_d V_e}}\Bigg]$$

$$=\frac{1}{n_{u_d}}\Bigg[\sum_{\beta=h}^{r-s-e}\sum_{\substack{N_\beta\supset J_h\\ N_\beta\subset R-S-V_e}}\Bigg\{\sum_{\alpha=0}^{s-d+1}\sum_{\substack{M_\alpha\subset S-U_{d-1}\\ M_\alpha\supset I_k}}\sigma^2_{M_\alpha N_\beta}\prod_{\zeta=0}^{r}n_\zeta^{1-\delta^\zeta_{M_\alpha N_\beta U_{d-1} V_e}}$$

$$-\sum_{\alpha=0}^{s-d+1}\sum_{\substack{M_\alpha\supset(I_k u_d)\\ M_\alpha\subset S-U_{d-1}}}\sigma^2_{M_\alpha N_\beta}\prod_{\zeta=0}^{r}n_\zeta^{1-\delta^\zeta_{M_\alpha N_\beta U_{d-1} V_e}}\Bigg\}\Bigg]$$

$$=\frac{1}{n_{u_d}}[A_{(I_k)}^{(U_{d-1})}B_{(J_h)}^{(V_e)}-A_{(I_k u_d)}^{(U_{d-1})}B_{(J_h)}^{(V_e)}].$$

The next result is stated without explicit proof.

LEMMA 4.2.

(4.19) $$A_{(I_k)}^{(U_d)}B_{(J_h)}^{(V_e)}=\frac{1}{n_{v_e}}[A_{(I_k)}^{(U_d)}B_{(J_h)}^{(V_{e-1})}-A_{(I_k)}^{(U_d)}B_{(J_h v_e)}^{(V_{e-1})}],$$

$$\begin{pmatrix} I_k\subset S;\ U_d\subset S;\ J_h\subset R-S;\ V_e\subset R-S; \\ k,d=0,\cdots,s;\ e=1,\cdots,r-s;\ h=0,\cdots,r-s. \end{pmatrix}.$$

From Lemma 4.1 and 4.2 we have

LEMMA 4.3.

(4.20) $$A_{(I_k)}^{(U_d)}B_{(J_h)}^{(V_e)}=\frac{1}{\prod_{\zeta=1}^{r}n_\zeta^{\delta^\zeta_{U_d}}}\Bigg[\sum_{p=0}^{d}\sum_{L_p\subset U_d}(-1)^pA_{(I_kL_p)}B_{(J_h)}^{(V_e)}\Bigg],$$

$$\begin{pmatrix} I_k\subset S;\ U_d\subset S;\ J_h\subset R-S;\ V_e\subset R-S; \\ k=0,\cdots,s;\ d=1,\cdots,s;\ e,h=0,\cdots,r-s. \end{pmatrix}$$

and

LEMMA 4.4.

(4.21) $$A_{(I_k)}^{(U_d)}B_{(J_h)}^{(V_e)}=\frac{1}{\prod_{\zeta=1}^{r}n_\zeta^{\delta^\zeta_{V_e}}}\Bigg[\sum_{q=0}^{e}\sum_{T_q\subset V_e}(-1)^qA_{(I_k)}^{(U_d)}B_{(J_hT_q)}\Bigg],$$

$$\begin{pmatrix} I_k\subset S;\ U_d\subset S;\ J_h\subset R-S;\ V_e\subset R-S; \\ k,d=0,\cdots,s;\ e=1,\cdots,r-s;\ h=0,\cdots,r-s. \end{pmatrix}.$$

These lemata can be proved easily by mathematical induction in the similar way to Lemma 4.2 in [2].

In the sequel we can express $A_{(I_k)}^{(U_d)}B_{(J_h)}^{(V_e)}$ in terms of $A_{(I_kL_p)}B_{(J_hT_q)}$ using Lemma 4.3 and 4.4 in turn, which is given by

LEMMA 4.5.

$$A_{(I_k)}^{(U_d)}B_{(J_h)}^{(V_e)}=\frac{1}{\prod_{\zeta=1}^{r} n_\zeta^{\delta_{U_dV_e}^{\zeta}}}\left[\sum_{p=0}^{d}\sum_{L_p\subset U_d}\sum_{q=0}^{e}\sum_{T_p\subset V_e}(-1)^{p+q}A_{(I_kL_p)}B_{(J_hT_q)}\right], \tag{4.22}$$

$$\begin{pmatrix} I_k\subset S;\ U_d\subset S;\ J_h\subset R-S;\ V_e\subset R-S; \\ k=0,\cdots,s;\ d=1,\cdots,s;\ h=0,\cdots,r-s;\ e=1,\cdots,r-s. \end{pmatrix}.$$

On the other hand we have

LEMMA 4.6. *Let G and E be the functions, which are finite and not vanished, defined on the subsets of the set of integers $(1, 2, \cdots, r)$, then we have*

$$\sum_{d=1}^{k}\sum_{U_d\subset A_k}\left\{\sum_{\alpha=0}^{k-d}\sum_{M_\alpha\subset A_k-U_d}\frac{(-1)^\alpha}{G_{(U_dM_\alpha)}}\right\}\left\{\sum_{p=0}^{d}\sum_{L_p\subset U_d}(-1)^pE_{(L_p)}\right\} \tag{4.23}$$

$$=\sum_{d=1}^{k}\sum_{U_d\subset A_k}(-1)^d\left\{\frac{E_{(U_d)}-E}{G_{(U_d)}}\right\},$$

where E is the value of $E_{(U_d)}$ corresponding to $U_d=\phi$.

PROOF. Among the left side (4.23) the partial sum for $d+\alpha=c(c=1, 2, \cdots, k)$ is given by

$$\sum_{d=1}^{c}\sum_{U_d\subset A_k}\sum_{M_{c-d}\subset A_k-U_d}\sum_{p=0}^{d}\sum_{L_p\subset U_d}(-1)^{c-d+p}\frac{E_{(L_p)}}{G_{(U_dM_{c-d})}}. \tag{4.24}$$

This is divided into three parts, the sum for $p=0$, the sum for $p=h(1\leqq h\leqq c-1)$ and the sum for $p=c$. And these are evaluated in (4.25), (4.26) and (4.27) respectively:

$$\sum_{d=1}^{c}\sum_{U_d\subset A_k}\sum_{M_{c-d}\subset A_k-U_d}(-1)^{c-d}\frac{E}{G_{(U_dM_{c-d})}} \tag{4.25}$$

$$=\sum_{U_c\subset A_k}(-1)^0\frac{E}{G_{(U_c)}}+\sum_{U_{c-1}\subset A_k}\sum_{M_1\subset A_k-U_{c-1}}(-1)^1\frac{E}{G_{(U_{c-1}M_1)}}$$

$$+\sum_{U_{c-2}\subset A_k}\sum_{M_2\subset A_k-U_{c-2}}(-1)^2\frac{E}{G_{(U_{c-2}M_2)}}+\cdots\cdots+\sum_{U_1\subset A_k}\sum_{M_{c-1}\subset A_k-U_1}(-1)^{c-1}\frac{E}{G_{(U_1M_{c-1})}}$$

$$=[{}_cC_0(-1)^0+{}_cC_1(-1)^1+\cdots+{}_cC_{c-1}(-1)^{c-1}]\sum_{U_c\subset A_k}\frac{E}{G_{(U_c)}}$$

$$=-(-1)^c\sum_{U_c\subset A_k}\frac{E}{G_{(U_c)}}.$$

$$\sum_{U_h\subset A_k}\sum_{M_{c-h}\subset A_k-U_h}\sum_{L_h\subset U_h}(-1)^c\frac{E_{(L_h)}}{G_{(U_hM_{c-h})}}+\sum_{U_{h+1}\subset A_k}\sum_{M_{c-h-1}\subset A_k-U_{h+1}}\sum_{L_h\subset U_{h+1}}\frac{(-1)^1E_{(L_h)}}{G_{(U_{h+1}M_{c-h-1})}} \tag{4.26}$$

$$+\cdots\cdots\cdots\cdots+\sum_{U_c\subset A_k}\sum_{L_h\subset U_c}\frac{(-1)^h E_{(L_h)}}{G_{(U_c)}}$$

$$=\left[\sum_{j=0}^{h}(-1)^{c-j}{}_{c-h}C_j\right]\sum_{U_h\subset A_k}\sum_{M_{c-h}\subset A_k-U_h}\sum_{L_h\subset U_h}\frac{E_{(L_h)}}{G_{(U_hM_{c-h})}}=0.$$

(4.27) $$\sum_{U_c\subset A_k}(-1)^c\frac{E_{(U_c)}}{G_{(U_c)}}.$$

Finally the sum of (4.24) with respect to c is equal to the right side of (4.23), which completes the proof.

Now we have made our necessary preparation, and we can enter into the calculation of the inversion of the variance matrix.

THEOLEM 4.2. *The inverse of the variance matrix* (4.1) *is given by*

(4.28) $$X_G E_{n_0}\otimes E_{n_1}\otimes\cdots\otimes E_{n_r}+\sum_{k=1}^{s}\sum_{I_k\subset S}X_{I_k}E_{n_0}\otimes\prod_{\zeta=1}^{r}\otimes\left(E_{n_\zeta}^{1-\delta_{I_k}^{\zeta}}\times I_{n_\zeta}^{\delta_{I_k}^{\zeta}}\right)$$

$$+\sum_{k=1}^{s}\sum_{I_k\subset S}\sum_{h=1}^{r-s}\sum_{J_h\subset R-S}X_{I_kJ_h}E_{n_0}\otimes\prod_{\zeta=1}^{r}\otimes\left(E_{n_\zeta}^{1-\delta_{I_kJ_h}^{\zeta}}\times I_{n_\zeta}^{\delta_{I_kJ_h}^{\zeta}}\right)$$

$$+\sum_{h=1}^{r-s}\sum_{J_h\subset R-S}X_{J_h}E_{n_0}\otimes\prod_{\zeta=1}^{r}\otimes\left(E_{n_\zeta}^{1-\delta_{J_h}^{\zeta}}\times I_{n_\zeta}^{\delta_{J_h}^{\zeta}}\right)+X_0 I_{n_0}\otimes I_{n_1}\otimes\cdots\otimes I_{n_r},$$

where

(4.29) $$X_0=\frac{1}{\sigma_0^2},$$

(4.30) $$X_R=\frac{1}{n_0}\left(\frac{1}{C_{(S)}D_{(R-S)}}-\frac{1}{\sigma_0^2}\right),$$

(4.31) $$X_{I_k,R-S}=\frac{1}{\prod_{\zeta=0}^{r}n_\zeta^{1-\delta_{I_k,R-S}^{\zeta}}}\left[\sum_{\alpha=0}^{s-k}\sum_{M_\alpha\subset S-I_k}\frac{(-1)^\alpha}{C_{(I_kM_\alpha)}D_{(R-S)}}\right],\qquad (I_k\subset S;\ k=1,\cdots,s),$$

(4.32) $$X_{S,J_h}=\frac{1}{\prod_{\zeta=0}^{r}n_\zeta^{1-\delta_{S,J_h}^{\zeta}}}\left[\sum_{\beta=0}^{r-s-h}\sum_{N_\beta\subset R-S-J_h}\frac{(-1)^\beta}{C_{(S)}D_{(J_hN_\beta)}}\right],\qquad (J_h\subset R-S;\ h=1,\cdots,r-s),$$

(4.33) $$X_{I_kJ_h}=\frac{1}{\prod_{\zeta=0}^{r}n_\zeta^{1-\delta_{I_kJ_h}^{\zeta}}}\left[\sum_{\alpha=0}^{s-k}\sum_{M_\alpha\subset S-I_k}\sum_{\beta=0}^{r-s-h}\sum_{N_\beta\subset R-S-J_h}\frac{(-1)^{\alpha+\beta}}{C_{(I_kM_\alpha)}D_{(J_hN_\beta)}}\right],$$

$$(I_k\subset S;\ J_h\subset R-S;\ k=1,\cdots,s;\ h=1,\cdots,r-s),$$

(4.34) $$X_{I_k}=\frac{1}{\prod_{\zeta=0}^{r}n_\zeta^{1-\delta_{I_k}^{\zeta}}}\left[\sum_{\alpha=0}^{s-k}\sum_{M_\alpha\subset S-I_k}\sum_{\beta=0}^{r-s}\sum_{N_\beta\subset R-S}\frac{(-1)^{\alpha+\beta}}{C_{(I_kM_\alpha)}D_{(N_\beta)}}\right],\qquad (I_k\subset S;\ k=1,\cdots,s),$$

(4.35)
$$X_{J_h}=\frac{1}{\prod_{\zeta=0}^{r} n_\zeta^{1-\delta_{J_h}^{\zeta}}}\left[\sum_{\alpha=1}^{s}\sum_{M_\alpha\subset S}\sum_{\beta=0}^{r-s-h}\sum_{N_\beta\subset R-S-J_h}\frac{(-1)^{\alpha+\beta}}{C_{(M_\alpha)}D_{(J_hN_\beta)}}+\sum_{\beta=0}^{r-s-h}\sum_{N_\beta\subset R-S-J_h}\frac{(-1)^{\beta}}{E_{(J_hN_\beta)}}\right],\qquad (J_h\subset R-S;\ h=1,\cdots,r-s),$$

(4.36)
$$X_G=\frac{1}{\prod_{\zeta=0}^{r} n_\zeta}\left[\frac{1}{G}+\sum_{\beta=1}^{r-s}\sum_{N_\beta\subset R-S}\frac{(-1)^{\beta}}{G_{(N_\beta)}}+\sum_{\alpha=1}^{s}\sum_{M_\alpha\subset S}\sum_{\beta=0}^{r-s}\sum_{N_\beta\subset R-S}\frac{(-1)^{\alpha+\beta}}{C_{(M_\alpha)}D_{(N_\beta)}}\right].$$

PROOF. Anticipating the inverse to be the form of (4.28), we search for the condition that (4.28) is actually the inverse.

The product of the variance matrix (4.1) and the matrix (4.28) is

(4.37)
$$\begin{aligned}
&E_{n_0}\otimes E_{n_1}\otimes\cdots\otimes E_{n_r}\\
&\quad\cdot\Big[X_G\{CD\}+\sum_{k=1}^{s}\sum_{I_k\subset S}X_{I_k}\{A^{(I_k)}B\}\\
&\qquad+\sum_{k=1}^{s}\sum_{I_k\subset S}\sum_{h=1}^{r-s}\sum_{J_h\subset R-S}X_{I_kJ_h}\{A^{(I_k)}B^{(J_h)}\}+\sum_{h=1}^{r-s}\sum_{J_h\subset R-S}X_{J_h}\{AB^{(J_h)}\}\Big]\\
&+E_{n_0}\otimes\prod_{\zeta=1}^{r}\otimes\Big(E_{n_\zeta}^{1-\delta_S^{\zeta}}\times I_{n_\zeta}^{\delta_S^{\zeta}}\Big)\\
&\quad\cdot\Big[X_S\{C_{(S)}D\}+\sum_{e=1}^{r-s}\sum_{V_e\subset R-S}X_{SV_e}\{A_{(S)}B^{(V_e)}\}+X_0\sigma_S^2\Big]\\
&+\sum_{k=1}^{s-1}\sum_{I_k\subset S}E_{n_0}\otimes\prod_{\zeta=1}^{r}\otimes\Big(E_{n_\zeta}^{1-\delta_{I_k}^{\zeta}}\times I_{n_\zeta}^{\delta_{I_k}^{\zeta}}\Big)\\
&\quad\cdot\Big[X_{I_k}\{C_{(I_k)}D\}+\sum_{d=1}^{s-k}\sum_{U_d\subset S-I_k}X_{I_kU_d}\{A_{(I_k)}^{(U_d)}B\}\\
&\qquad+\sum_{d=1}^{s-k}\sum_{U_d\subset S-I_k}\sum_{e=1}^{r-s}\sum_{V_e\subset R-S}X_{I_kU_dV_e}\{A_{(I_k)}^{(U_d)}B^{(V_e)}\}\\
&\qquad+\sum_{e=1}^{r-s}\sum_{V_e\subset R-S}X_{I_kV_e}\{A_{(I_k)}B^{(V_e)}\}+X_0\sigma_{I_k}^2\Big]\\
&+E_{n_0}\otimes I_{n_1}\otimes I_{n_2}\otimes\cdots\otimes I_{n_r}\\
&\quad\cdot\Big[X_R\{C_{(S)}D_{(R-S)}\}+X_0\sigma_R^2\Big]\\
&+\sum_{k=1}^{s-1}\sum_{I_k\subset S}E_{n_0}\otimes\prod_{\zeta=1}^{r}\otimes\Big(E_{n_\zeta}^{1-\delta_{I_k,R-S}^{\zeta}}\times I_{n_\zeta}^{\delta_{I_k,R-S}^{\zeta}}\Big)\\
&\quad\cdot\Big[X_{I_k,R-S}\{C_{(I_k)}D_{(R-S)}\}+\sum_{d=1}^{s-k}\sum_{U_d\subset S-I_k}X_{I_kU_dR-S}\{A_{(I_k)}^{(U_d)}B_{(R-S)}\}+X_0\sigma_{I_kR-S}^2\Big]
\end{aligned}$$

$$+\sum_{h=1}^{r-s-1}\sum_{J_h\subset R-S} E_{n_0}\otimes\prod_{\zeta=1}^{r}\otimes\left(E_{n_\zeta}^{1-\delta_{SJ_h}^{\zeta}}\times I_{n_\zeta}^{\delta_{SJ_h}^{\zeta}}\right)$$

$$\cdot\left[X_{SJ_h}\{C_{(S)}D_{(J_h)}\}+\sum_{e=1}^{r-s-h}\sum_{V_e\subset R-S-J_h} X_{SJ_hV_e}\{A_{(S)}B_{(J_h)}^{(V_e)}\}+X_0\sigma_{SJ_h}^2\right]$$

$$+\sum_{k=1}^{s-1}\sum_{I_k\subset S}\sum_{h=1}^{r-s-1}\sum_{J_h\subset R-S} E_{n_0}\otimes\prod_{\zeta=1}^{r}\otimes\left(E_{n_\zeta}^{1-\delta_{I_kJ_h}^{\zeta}}\times I_{n_\zeta}^{\delta_{I_kJ_h}^{\zeta}}\right)$$

$$\cdot\left[X_{I_kJ_h}\{C_{(I_k)}D_{(J_h)}\}+\sum_{e=1}^{r-s-h}\sum_{V_e\subset R-S-J_h} X_{I_kJ_hV_e}\{A_{(I_k)}B_{(J_h)}^{(V_e)}\}\right.$$

$$+\sum_{d=1}^{s-k}\sum_{U_d\subset S-I_k} X_{I_kU_dJ_h}\{A_{(I_k)}^{(U_d)}B_{(J_h)}\}$$

$$\left.+\sum_{d=1}^{s-k}\sum_{U_d\subset S-I_k}\sum_{e=1}^{r-s-h}\sum_{V_e\subset R-S-J_h} X_{I_kU_dJ_hV_e}\{A_{(I_k)}^{(U_d)}B_{(J_h)}^{(V_e)}\}+X_0\sigma_{I_kJ_h}^2\right]$$

$$+E_{n_0}\otimes\prod_{\zeta=1}^{r}\otimes\left(E_{n_\zeta}^{1-\delta_{R-S}^{\zeta}}\times I_{n_\zeta}^{\delta_{R-S}^{\zeta}}\right)$$

$$\cdot\left[X_{R-S}\{CD_{(R-S)}\}+\sum_{d=1}^{s}\sum_{U_d\subset S} X_{U_dR-S}\{A^{(U_d)}B_{(R-S)}\}\right]$$

$$+\sum_{h=1}^{r-s-1}\sum_{J_h\subset R-S} E_{n_0}\otimes\prod_{\zeta=1}^{r}\otimes\left(E_{n_\zeta}^{1-\delta_{J_h}^{\zeta}}\times I_{n_\zeta}^{\delta_{J_h}^{\zeta}}\right)$$

$$\cdot\left[X_{J_h}\{CD_{(J_h)}\}+\sum_{e=1}^{r-s-h}\sum_{V_e\subset R-S-J_h} X_{J_hV_e}\{AB_{(J_h)}^{(V_e)}\}\right.$$

$$+\sum_{d=1}^{s}\sum_{U_d\subset S} X_{U_dJ_h}\{A^{(U_d)}B_{(J_h)}\}$$

$$\left.+\sum_{d=1}^{s}\sum_{U_d\subset S}\sum_{e=1}^{r-s-h}\sum_{V_e\subset R-S-J_h} X_{U_dJ_hV_e}\{A^{(U_d)}B_{(J_h)}^{(V_e)}\}\right]$$

$$+I_{n_0}\otimes I_{n_1}\otimes I_{n_2}\otimes\cdots\otimes I_{n_r}X_0\sigma_0^2.$$

Thus the condition is expressed by the following equations,

(4.38) $$X_0\sigma_0^2=1,$$

(4.39) $$X_R\{C_{(S)}D_{(R-S)}\}=-X_0\sigma_R^2,$$

(4.40) $$X_{I_kR-S}\{C_{(I_k)}D_{(R-S)}\}=-\sum_{d=1}^{s-k}\sum_{U_d\subset S-I_k} X_{I_kU_dR-S}\{A_{(I_k)}^{(U_d)}B_{(R-S)}\}-X_0\sigma_{I_kR-S}^2,$$

$$(I_k\subset S;\ k=1,\cdots,s),$$

(4.41) $$X_{SJ_h}\{C_{(S)}D_{(J_h)}\}=-\sum_{e=1}^{r-s-h}\sum_{V_e\subset R-S-J_h} X_{SJ_hV_e}\{A_{(S)}B_{(J_h)}^{(V_e)}\}-X_0\sigma_{SJ_h}^2,$$

$$(J_h\subset R-S;\ h=1,\cdots,r-s),$$

(4.42) $$X_{I_kJ_h}\{C_{(I_k)}D_{(J_h)}\}$$

$$=-\sum_{e=1}^{r-s-h}\sum_{V_e\subset R-S-J_h}X_{I_kJ_hV_e}\{A_{(I_k)}B_{(J_h)}^{(V_e)}\}-\sum_{d=1}^{s-k}\sum_{U_d\subset S-I_k}X_{I_kU_dJ_h}\{A_{(I_k)}^{(U_d)}B_{(J_h)}\}$$

$$-\sum_{d=1}^{s-k}\sum_{U_d\subset S-I_k}\sum_{e=1}^{r-s-h}\sum_{V_e\subset R-S-J_h}X_{I_kU_dJ_hV_e}\{A_{(I_k)}^{(U_d)}B_{(J_h)}^{(V_e)}\}-X_0\sigma_{I_kJ_h}^2,$$

$$\begin{pmatrix}I_k\subset S;\ J_h\subset R-S\\ k=1,\cdots,s-1;\ h=1,\cdots,r-s-1,\end{pmatrix},$$

(4.43) $$X_S\{C_{(S)}D\}=-\sum_{e=1}^{r-s}\sum_{V_e\subset R-S}X_{SV_e}\{A_{(S)}B^{(V_e)}\}-X_0\sigma_S^2,$$

(4.44) $$X_{R-S}\{CD_{(R-S)}\}=-\sum_{d=1}^{s}\sum_{U_d\subset S}X_{U_dR-S}\{A^{(U_d)}B_{(R-S)}\},$$

(4.45) $$X_{I_k}\{C_{(I_k)}D\}=-\sum_{d=1}^{s-k}\sum_{U_d\subset S-I_k}X_{I_kU_d}\{A_{(I_k)}^{(U_d)}B\}-\sum_{e=1}^{r-s}\sum_{V_e\subset R-S}X_{I_kV_e}\{A_{(I_k)}B^{(V_e)}\}$$

$$-\sum_{d=1}^{s-k}\sum_{U_d\subset S-I_k}\sum_{e=1}^{r-s}\sum_{V_e\subset R-S}X_{I_kU_dV_e}\{A_{(I_k)}^{(U_d)}B^{(V_e)}\}-X_0\sigma_{I_k}^2,$$

$$(I_k\subset S;\ k=1,\cdots,s-1),$$

(4.46) $$X_{J_h}\{CD_{(J_h)}\}$$

$$=-\sum_{e=1}^{r-s-h}\sum_{V_e\subset R-S-J_h}X_{J_hV_e}\{AB_{(J_h)}^{(V_e)}\}-\sum_{d=1}^{s}\sum_{U_d\subset S}X_{U_dJ_h}\{A^{(U_d)}B_{(J_h)}\}$$

$$-\sum_{d=1}^{s}\sum_{U_d\subset S}\sum_{e=1}^{r-s-h}\sum_{V_e\subset R-S-J_h}X_{U_dJ_hV_e}\{A^{(U_d)}B_{(J_h)}^{(V_e)}\},$$

$$(J_h\subset R-S;\ h=1,\cdots,r-s-1),$$

(4.47) $$X_G\{CD\}=-\sum_{k=1}^{s}\sum_{I_k\subset S}X_{I_k}\{A^{(I_k)}B\}-\sum_{h=1}^{r-s}\sum_{J_h\subset R-S}X_{J_h}\{AB^{(J_h)}\}$$

$$-\sum_{h=1}^{r-s}\sum_{J_h\subset R-S}\sum_{k=1}^{s}\sum_{I_k\subset S}X_{I_kJ_h}\{A^{(I_k)}B^{(J_h)}\}.$$

Now the proof of this theorem is completed by proving the following:

LEMMA 4.6. *The solutions of the equations* (4.38), ⋯, (4.47) *are given by* (4.29), ⋯, (4.36).

PROOF. (4.29) comes from (4.38) directly and (4.30) comes from (4.29) and (4.39).

(i) Solutions for X_{I_kR-S}, $(I_k\subset S;\ k=1,2,\cdots,s-1)$.

(4.31) is obtained by mathematical induction in k in (4.40) and from (4.29) and (4.30), which is as follows.

At the first stage, we shall prove (4.31) holds true for all $I_{s-1}\subset S$. The equation to be solved is

(4.48) $$X_{I_{s-1}R-S}\{C_{(I_{s-1})}D_{(R-S)}\}=-\Big[X_R\{A_{(I_{s-1})}^{(S-I_{s-1})}B_{(R-S)}\}+X_0\sigma_{I_{s-1}R-S}^2\Big]$$

$$=-\frac{1}{n_0}\Big[\frac{1}{C_{(S)}D_{(R-S)}}\ \{A_{(I_{s-1})}^{(S-I_{s-1})}B_{(R-S)}\}\Big]$$

$$=\frac{1}{n_0 n_{i_s}}\left[\frac{1}{C_{(S)}D_{(R-S)}}\{A_{(S)}B_{(R-S)}-A_{(I_s-1)}B_{(R-S)}\}\right]$$

$$=\frac{1}{n_0 n_{i_s}}\left[\frac{1}{C_{(S)}D_{(R-S)}}\{C_{(S)}D_{(R-S)}-C_{(I_s-1)}D_{(R-S)}\}\right].$$

Hence we have

$$X_{I_s-1R-S}=\frac{1}{n_0 n_{i_s}}\left[\frac{1}{C_{(I_s-1)}D_{(R-S)}}-\frac{1}{C_{(S)}D_{(R-S)}}\right], \tag{4.49}$$

which completes the first stage.

At the second stage, we shall prove, assuming that this holds true for all $I_k \subset S$ when $k=q,\ q+1, \cdots, s-1$, this also holds true for all $I_{q-1}\subset S$. Under this assumptions we obtain by Lemma 4.5 and 4.6

$$X_{I_{q-1}R-S}\{C_{(I_{q-1})}D_{(R-S)}\} \tag{4.50}$$

$$=-\left[\sum_{d=1}^{s-q+1}\sum_{U_d\subset S-I_{q-1}}X_{I_{q-1}U_dR-S}\{A^{(U_d)}_{(I_{q-1})}B_{(R-S)}\}+X_0\sigma^2_{I_{q-1}R-S}\right]$$

$$=-\left[\sum_{d=1}^{s-q}\sum_{U_d\subset S-I_{q-1}}X_{I_{q-1}U_dR-S}\{A^{(U_d)}_{(I_{q-1})}B_{(R-S)}\}+X_R\{A^{(S-I_{q-1})}_{(I_{q-1})}B_{(R-S)}\}\right.$$
$$\left.+\frac{X_0}{n_0}A^{(S-I_{q-1})}_{(I_{q-1})}B_{(R-S)}\right]$$

$$=-\frac{1}{\prod_{\zeta=0}^{r}n_\zeta^{1-\delta^\zeta_{I_{q-1}R-S}}}\left[\sum_{d=1}^{s-q+1}\sum_{U_d\subset S-I_{q-1}}\left\{\sum_{\alpha=0}^{s-q+1-d}\sum_{M_\alpha\subset S-(I_{q-1}\cup U_d)}\frac{(-1)^\alpha}{C_{(I_{q-1}U_dM_\alpha)}D_{(R-S)}}\right\}\right.$$
$$\left.\cdot\left\{\sum_{p=0}^{d}\sum_{L_p\subset U_d}(-1)^p A_{(I_{q-1}L_p)}B_{(R-S)}\right\}\right]$$

$$=-\frac{1}{\prod_{\zeta=0}^{r}n_\zeta^{1-\delta^\zeta_{I_{q-1}R-S}}}\left[\sum_{d=1}^{s-q+1}\sum_{U_d\subset S-I_{q-1}}(-1)^d\left\{\frac{A_{(I_{q-1}U_d)}B_{(R-S)}-A_{(I_{q-1})}B_{(R-S)}}{C_{(I_{q-1}U_d)}D_{(R-S)}}\right\}\right]$$

$$=\frac{1}{\prod_{\zeta=0}^{r}n_\zeta^{1-\delta^\zeta_{I_{q-1}R-S}}}\left[\sum_{d=1}^{s-q+1}\sum_{U_d\subset S-I_{q-1}}(-1)^d\left\{\frac{C_{(I_{q-1})}D_{(R-S)}-C_{(I_{q-1}U_d)}D_{(R-S)}}{C_{(I_{q-1}U_d)}D_{(R-S)}}\right\}\right].$$

Hence we have

$$X_{I_{q-1}R-S}=\frac{1}{\prod_{\zeta=0}^{r}n_\zeta^{1-\delta^\zeta_{I_{q-1}R-S}}}\left[\sum_{d=1}^{s-q+1}\sum_{U_d\subset S-I_{q-1}}\frac{(-1)^d}{C_{(I_{q-1}U_d)}D_{(R-S)}}-\sum_{d=1}^{s-q+1}\sum_{U_d\subset S-I_{q-1}}\frac{(-1)^d}{C_{(I_{q-1})}D_{(R-S)}}\right] \tag{4.51}$$

$$=\frac{1}{\prod_{\zeta=0}^{r}n_\zeta^{1-\delta^\zeta_{I_{q-1}R-S}}}\left[\sum_{d=0}^{s-q+1}\sum_{U_d\subset S-I_{q-1}}\frac{(-1)^d}{C_{(I_{q-1}U_d)}D_{(R-S)}}\right].$$

(ii) Solutions for X_{SJ_h}, $(J_h\subset R-S;\ h=1,2,\cdots,r-s-1)$.

(4.32) is obtained by mathematical induction in h in (4.41) and from (4.29) and (4.30), which is as follows. At the first stage, we shall prove (4.32) holds true for all $J_{r-s-1}\subset R-S$. The equation to be solved is

$$X_{SJ_{r-s-1}}\{C_{(S)}D_{(J_{r-s-1})}\}=-\left[X_R\{A_{(S)}B_{(J_{r-s-1})}^{(R-S-J_{r-s-1})}\}+X_0\sigma_{SJ_{r-s-1}}^2\right] \tag{4.52}$$

$$=-\frac{1}{n_0}\left[\frac{A_{(S)}B_{(J_{r-s-1})}^{(R-S-J_{r-s-1})}}{C_{(S)}D_{(R-S)}}\right]=\frac{1}{n_0 n_{j_{r-s-1}}}\left[\frac{A_{(S)}B_{(R-S)}-A_{(S)}B_{(J_{r-s-1})}}{C_{(S)}D_{(R-S)}}\right]$$

$$=\frac{1}{n_0 n_{j_{r-s-1}}}\left[\frac{C_{(S)}D_{(R-S)}-C_{(S)}D_{(J_{r-s-1})}}{C_{(S)}D_{(R-S)}}\right].$$

And we have

$$X_{SJ_{r-s-1}}=\frac{1}{n_0 n_{j_{r-s-1}}}\left[\frac{1}{C_{(S)}D_{(J_{r-s-1})}}-\frac{1}{C_{(S)}D_{(R-S)}}\right]. \tag{4.53}$$

At the second stage, we shall prove, assuming that this holds true for all $J_h\subset R-S$ when $h=c,\ c+1,\cdots,\ r-s-1$, this also holds true for all $J_{c-1}\subset R-S$. Under these assumptions, by Lemma 4.5 and 4.6 the equation to be solved is given as follows

$$X_{SJ_{c-1}}\{C_{(S)}D_{(J_{c-1})}\} \tag{4.54}$$

$$=-\left[\sum_{e=1}^{r-s-c+1}\sum_{V_e\subset R-S-J_{c-1}}X_{SJ_{c-1}V_e}\{A_{(S)}B_{(J_{c-1})}^{(V_e)}\}+X_0\sigma_{SJ_{c-1}}^2\right]$$

$$=-\frac{1}{N_{SJ_{c-1}}}\left[\sum_{e=1}^{r-s-c+1}\sum_{V_e\subset R-S-J_{c-1}}\left\{\sum_{\beta=0}^{r-s-c+1-e}\sum_{N_\beta\subset R-S-(J_{c-1}\cup V_e)}\frac{(-1)^\beta}{C_{(S)}D_{(J_{c-1}V_eN_\beta)}}\right\}\right.$$

$$\left.\cdot\left\{\sum_{q=0}^{e}\sum_{T_q\subset V_e}(-1)^q A_{(S)}B_{(J_{c-1},T_q)}\right\}\right]$$

$$=-\frac{1}{N_{SJ_{c-1}}}\left[\sum_{e=1}^{r-s-c+1}\sum_{V_e\subset R-S-J_{c-1}}(-1)^e\left\{\frac{A_{(S)}B_{(J_{c-1}V_e)}-A_{(S)}B_{(J_{c-1})}}{C_{(S)}D_{(J_{c-1}V_e)}}\right\}\right]$$

$$=\frac{1}{N_{SJ_{c-1}}}\left[\sum_{e=1}^{r-s-c+1}\sum_{V_e\subset R-S-J_{c-1}}(-1)^e\left\{\frac{C_{(S)}D_{(R-S)}-C_{(S)}D_{(J_{c-1}V_e)}}{C_{(S)}D_{(J_{c-1}V_e)}}\right\}\right],$$

where $N_{SJ_{c-1}}=\prod_{\zeta=0}^{r}n_\zeta^{1-\delta_{SJ_{c-1}}^{\zeta}}$.

Then we have

$$X_{SJ_{c-1}}=\frac{1}{N_{SJ_{c-1}}}\left[\sum_{e=1}^{r-s-c+1}\sum_{V_e\subset R-S-J_{c-1}}\frac{(-1)^e}{C_{(S)}D_{(J_{c-1}V_e)}}-\sum_{e=1}^{r-s-c+1}\sum_{V_e\subset R-S-J_{c-1}}\frac{(-1)^e}{C_{(S)}D_{(J_{c-1})}}\right] \tag{4.55}$$

$$=\frac{1}{\prod_{\zeta=0}^{r}n_\zeta^{1-\delta_{SJ_{c-1}}^{\zeta}}}\left[\sum_{e=0}^{r-s-c+1}\sum_{V_e\subset R-S-J_{c-1}}\frac{(-1)^e}{C_{(S)}D_{(J_{c-1}V_e)}}\right],$$

which completes the second stage.

In the following cases we shall make use of Lemma 4.5 and 4.6 without refering to them explicitly.

(iii) Solutions for $X_{I_kJ_h}$, $(I_k\subset S;\ J_h\subset R-S;\ k=1,2,\cdots,s-1;\ h=1,2,\cdots,r-s-1)$.

(4.33) is obtained by mathematical induction in $k+h$ in (4.42) and from (4.29), (4.30), (4.31) and (4.32).

At the first stage we shall prove (4.33) holds true for all $I_{s-1}\subset S$ and $J_{r-s-1}\subset R-S$. The equation to be solved is

(4.56)
$$X_{I_{s-1}J_{r-s-1}}\{C_{(I_{s-1})}D_{(J_{r-s-1})}\}$$
$$=-\Big[X_{I_{s-1}R-S}\{A_{(I_{s-1})}B_{(J_{r-s-1})}^{(R-S-J_{r-s-1})}\}+X_{SJ_{r-s-1}}\{A_{(I_{s-1})}^{(S-I_{s-1})}B_{(J_{r-s-1})}\}$$
$$+X_R\{A_{(I_{s-1})}^{(S-I_{s-1})}B_{(J_{r-s-1})}^{(R-S-J_{r-s-1})}\}+X_0\sigma^2_{I_{s-1}J_{r-s-1}}\Big]$$
$$=\frac{-1}{n_0n_{i_s}n_{j_{r-s}}}\Big[\frac{A_{(I_{s-1})}B_{(J_{r-s-1})}-A_{(I_{s-1})}B_{(R-S)}}{C_{(I_{s-1})}D_{(R-S)}}$$
$$+\frac{A_{(I_{s-1})}B_{(J_{r-s-1})}-A_{(S)}B_{(J_{r-s-1})}}{C_{(S)}D_{(J_{r-s-1})}}+\frac{A_{(S)}B_{(R-S)}-A_{(I_{s-1})}B_{(J_{r-s-1})}}{C_{(S)}D_{(R-S)}}\Big]$$
$$=\frac{1}{n_0n_{i_s}n_{j_{r-s}}}\Big[\frac{C_{(I_{s-1})}D_{(R-S)}-C_{(I_{s-1})}D_{(J_{r-s-1})}}{C_{(I_{s-1})}D_{(R-S)}}$$
$$+\frac{C_{(S)}D_{(J_{r-s-1})}-C_{(I_{s-1})}D_{(J_{r-s-1})}}{C_{(S)}D_{(J_{r-s-1})}}+\frac{C_{(I_{s-1})}D_{(J_{r-s-1})}-C_{(S)}D_{(R-S)}}{C_{(S)}D_{(R-S)}}\Big].$$

Hence we have

(4.57)
$$X_{I_{s-1}J_{r-s-1}}=\frac{1}{n_0n_{i_s}n_{j_{r-s}}}\Big[\frac{1}{C_{(I_{s-1})}D_{(J_{r-s-1})}}-\frac{1}{C_{(I_{s-1})}D_{(R-S)}}$$
$$-\frac{1}{C_{(S)}D_{(J_{r-s-1})}}+\frac{1}{C_{(S)}D_{(R-S)}}\Big],$$

which completes the first stage.

At the second stage, we shall prove. assuming that this holds true for all $I_k\subset S$ and J_{c-k} when $k=1,2,\cdots,c-1$, this also holds true for all $I_k\subset S$ nad J_{c-1-k} when $k=1,2,\cdots,c-2$. Then we observe

(4.58)
$$X_{I_kJ_{c-1-k}}\{C_{(I_k)}D_{(J_{c-1-k})}\}$$
$$=-\Big[\sum_{e=1}^{r-s-c+1+k}\sum_{V_e\subset R-S-J_{c-1-k}}X_{I_kJ_{c-1-k}V_e}\{A_{(I_k)}B_{(J_{c-1-k})}^{(V_e)}\}$$
$$+\sum_{d=1}^{s-k}\sum_{U_d\subset S-I_k}X_{I_kU_dJ_{c-1-k}}\{A_{(I_k)}^{(U_d)}B_{(J_{c-1-k})}\}$$
$$+\sum_{d=1}^{s-k}\sum_{U_d\subset S-I_k}\sum_{e=1}^{r-s-c+1+k}\sum_{V_e\subset R-S-J_{c-1-k}}X_{I_kU_dJ_{c-1-k}V_e}\{A_{(I_k)}^{(U_d)}B_{(J_{c-1-k})}^{(V_d)}\}+X_0\sigma^2_{I_kJ_{c-1-k}}\Big]$$

$$
\begin{aligned}
&= \frac{-1}{N_{I_k J_{c-1-k}}} \Bigg[\sum_{d=1}^{s-k} \sum_{U_d \subset S-I_k} \Bigg\{ \sum_{\alpha=0}^{s-k-d} \sum_{M_\alpha \subset S-(I_k \cup U_d)} \sum_{\beta=0}^{r-s-c+k+1} \sum_{N_\beta \subset R-S-J_{c-1-k}} \frac{(-1)^{\alpha+\beta}}{C_{(I_k U_d M_\alpha)} D_{(J_{c-1-k} N_\beta)}} \Bigg\} \\
&\qquad \cdot \Bigg\{ \sum_{p=0}^{d} \sum_{L_p \subset U_d} (-1)^p A_{(I_k L_p)} B_{(J_{c-1-k})} \Bigg\} \\
&\quad + \sum_{e=1}^{r-s-c+k+1} \sum_{V_e \subset R-S-J_{c-1-k}} \Bigg\{ \sum_{\alpha=0}^{s-k} \sum_{M_\alpha \subset S-I_k} \sum_{\beta=0}^{r-s-c+k+1-e} \sum_{N_\beta \subset R-S-(J_{c-1-k} \cup V_e)} \frac{(-1)^{\alpha+\beta}}{C_{(I_k M_\alpha)} D_{(J_{c-1-k} V_e N_\beta)}} \Bigg\} \\
&\qquad \cdot \Bigg\{ \sum_{q=0}^{e} \sum_{T_q \subset V_e} (-1)^q A_{(I_k)} B_{(J_{c-1-k} T_q)} \Bigg\} \\
&\quad + \sum_{d=1}^{s-k} \sum_{U_d \subset S-I_k} \sum_{e=1}^{r-s-c+k+1} \sum_{V_e \subset R-S-J_{c-1-k}} \Bigg\{ \sum_{\alpha=0}^{s-k-d} \sum_{M_\alpha \subset S-(I_k \cup U_d)} \\
&\qquad \cdot \sum_{\beta=0}^{r-s-c+k+1-e} \sum_{N_\beta \subset R-S-(J_{c-1-k} \cup V_e)} \frac{(-1)^{\alpha+\beta}}{C_{(I_k U_d M_\alpha)} D_{(J_{c-1-k} V_e N_\beta)}} \Bigg\} \\
&\qquad \cdot \Bigg\{ \sum_{p=0}^{d} \sum_{L_p \subset U_d} \sum_{q=0}^{e} \sum_{T_q \subset V_e} (-1)^{p+q} A_{(I_k L_p)} B_{(J_{c-1-k} T_q)} \Bigg\} \Bigg]
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{-1}{N_{I_k J_{c-1-k}}} \Bigg[\sum_{d=1}^{s-k} \sum_{U_d \subset S-I_k} (-1)^d \Bigg\{ \frac{A_{(I_k U_d)} B_{(J_{c-1-k})} - A_{(I_k)} B_{(J_{c-1-k})}}{C_{(I_k U_d)} D_{(J_{c-1-k})}} \Bigg\} \\
&\quad + \sum_{d=1}^{s-k} \sum_{U_d \subset S-I_k} \sum_{e=1}^{r-s-c+k+1} \sum_{V_e \subset R-S-J_{c-1-k}} (-1)^{d+e} \Bigg\{ \frac{A_{(I_k U_d)} B_{(J_{c-1-k})} - A_{(I_k)} B_{(J_{c-1-k})}}{C_{(I_k U_d)} D_{(J_{c-1-k} V_e)}} \Bigg\} \\
&\quad + \sum_{e=1}^{r-s-c+k+1} \sum_{V_e \subset R-S-J_{c-1-k}} (-1)^e \Bigg\{ \frac{A_{(I_k)} B_{(J_{c-1-k} V_e)} - A_{(I_k)} B_{(J_{c-1-k})}}{C_{(I_k)} D_{(J_{c-1-k} V_e)}} \Bigg\} \\
&\quad + \sum_{d=1}^{s-k} \sum_{U_d \subset S-I_k} \sum_{e=1}^{r-s-c+k+1} \sum_{V_e \subset R-S-J_{c-1-k}} (-1)^{d+e} \Bigg\{ \frac{A_{(I_k)} B_{(J_{c-1-k} V_e)} - A_{(I_k)} B_{(J_{c-1-k})}}{C_{(I_k U_d)} D_{(J_{c-1-k} V_e)}} \Bigg\} \\
&\quad + \sum_{d=1}^{s-k} \sum_{U_d \subset S-I_k} \sum_{e=1}^{r-s-c+k+1} \sum_{V_e \subset R-S-J_{c-1-k}} (-1)^{d+e} \\
&\qquad \cdot \Bigg\{ \frac{A_{(I_k U_d)} B_{(J_{c-1-k} V_e)} - A_{(I_k)} B_{(J_{c-1-k} V_e)} - A_{(I_k U_d)} B_{(J_{c-1-k})} + A_{(I_k)} B_{(J_{c-1-k})}}{C_{(I_k U_d)} D_{(J_{c-1-k} V_e)}} \Bigg\} \Bigg]
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{N_{I_k J_{c-1-k}}} \Bigg[\sum_{d=1}^{s-k} \sum_{U_d \subset S-I_k} (-1)^d \Bigg\{ \frac{C_{(I_k)} D_{(J_{c-1-k})} - C_{(I_k U_d)} D_{(J_{c-1-k})}}{C_{(I_k U_d)} D_{(J_{c-1-k})}} \Bigg\} \\
&\quad + \sum_{e=1}^{r-s-c+k+1} \sum_{V_e \subset R-S-J_{c-1-k}} (-1)^e \Bigg\{ \frac{C_{(I_k)} D_{(J_{c-1-k})} - C_{(I_k)} D_{(J_{c-1-k} V_e)}}{C_{(I_k)} D_{(J_{c-1-k} V_e)}} \Bigg\} \\
&\quad + \sum_{d=1}^{s-k} \sum_{U_d \subset S-I_k} \sum_{e=1}^{r-s-c+k+1} \sum_{V_e \subset R-S-J_{c-1-k}} (-1)^{d+e} \Bigg\{ \frac{C_{(I_k)} D_{(J_{c-1-k})} - C_{(I_k U_d)} D_{(J_{c-1-k} V_e)}}{C_{(I_k U_d)} D_{(J_{c-1-k} V_e)}} \Bigg\} \Bigg]
\end{aligned}
$$

where $N_{I_k J_{c-1-k}} = \prod_{\zeta=0}^{r} n_\zeta^{1-\delta^{\zeta}_{I_k J_{c-1-k}}}$.

And so we have

(4.59)
$$X_{I_kJ_{c-1-k}}=\frac{1}{N_{I_kJ_{c-1-k}}}\Bigg[\sum_{d=1}^{s-k}\sum_{U_d\subset S-I_k}(-1)^d\Big\{\frac{1}{C_{(I_kU_d)}D_{(J_{c-1-k})}}-\frac{1}{C_{(I_k)}D_{(J_{c-1-k})}}\Big\}$$
$$+\sum_{e=1}^{r-s-c+k+1}\sum_{V_e\subset R-S-J_{c-1-k}}(-1)^e\Big\{\frac{1}{C_{(I_k)}D_{(J_{c-1-k}V_e)}}-\frac{1}{C_{(I_k)}D_{(J_{c-1-k})}}\Big\}$$
$$+\sum_{d=1}^{s-k}\sum_{U_d\subset S-I_k}\sum_{e=1}^{r-s-c+k+1}\sum_{V_e\subset R-S-J_{c-1-k}}(-1)^{d+e}\Big\{\frac{1}{C_{(I_kU_d)}D_{(J_{c-1-k}V_e)}}-\frac{1}{C_{(I_k)}D_{(J_{c-1-k})}}\Big\}\Bigg]$$
$$=\frac{1}{N_{I_kJ_{c-1-k}}}\Bigg[\sum_{d=0}^{s-k}\sum_{U_d\subset S-I_k}\sum_{e=0}^{r-s-c+k+1}\sum_{V_e\subset R-S-J_{c-1-k}}\frac{(-1)^{d+e}}{C_{(I_kU_d)}D_{(J_{c-1-k}V_e)}}\Bigg].$$

(iv) Solution for X_S.

The solution for X_S is obtained by substituting (4.29), (4.30) and (4.32) in (4.43):

(4.60)
$$X_S\{C_{(S)}D\}=-\Bigg[\sum_{e=1}^{r-s}\sum_{V_e\subset R-S}X_{SV_e}\{A_{(S)}B^{(V_e)}\}+X_0\sigma_S^2\Bigg]$$
$$=\frac{-1}{\prod_{\zeta=0}^{r}n_\zeta^{1-\delta_S^\zeta}}\Bigg[\sum_{e=1}^{r-s}\sum_{V_e\subset R-S}\Big\{\sum_{\beta=0}^{r-s-e}\sum_{N_\beta\subset R-S-V_e}\frac{(-1)^\beta}{C_{(S)}D_{(V_eN_\beta)}}\Big\}\Big\{\sum_{q=0}^{e}\sum_{T_q\subset V_e}(-1)^qA_{(S)}B_{(T_q)}\Big\}\Bigg]$$
$$=\frac{1}{N_S}\Bigg[\sum_{e=1}^{r-s}\sum_{V_e\subset R-S}(-1)^e\Big\{\frac{A_{(S)}B-A_{(S)}B_{(V_e)}}{C_{(S)}D_{(V_e)}}\Big\}\Bigg]$$
$$=\frac{1}{N_S}\Bigg[\sum_{e=1}^{r-s}\sum_{V_e\subset R-S}(-1)^e\Big\{\frac{C_{(S)}D-C_{(S)}D_{(V_e)}}{C_{(S)}D_{(V_e)}}\Big\}\Bigg],$$

which gives us

(4.61)
$$X_S=\frac{1}{N_S}\Bigg[\sum_{e=1}^{r-s}\sum_{V_e\subset R-S}(-1)^e\Big\{\frac{1}{C_{(S)}D_{(V_e)}}-\frac{1}{C_{(S)}D}\Big\}\Bigg]$$
$$=\frac{1}{\prod_{\zeta=0}^{r}n_\zeta^{1-\delta_S^\zeta}}\Bigg[\sum_{e=0}^{r-s}\sum_{V_e\subset R-S}\frac{(-1)^e}{C_{(S)}D_{(V_e)}}\Bigg].$$

(v) Solution for X_{R-S}.

The solution for X_{R-S} is obtained by inserting (4.29), (4.30) and (4.31) in (4.44):

(4.62)
$$X_{R-S}\{CD_{(R-S)}\}=-\Bigg[\sum_{d=1}^{s}\sum_{U_d\subset S}X_{U_dR-S}\{A^{(U_d)}B_{(R-S)}\}\Bigg]$$
$$=\frac{-1}{\prod_{\zeta=0}^{r}n_\zeta^{1-\delta_{R-S}^\zeta}}\Bigg[\sum_{d=1}^{s}\sum_{U_d\subset S}\Big\{\sum_{\alpha=0}^{s-d}\sum_{M_\alpha\subset S-U_d}\frac{(-1)^\alpha}{C_{(U_dM_\alpha)}D_{(R-S)}}\Big\}$$
$$\cdot\Big\{\sum_{p=0}^{d}\sum_{L_p\subset U_d}(-1)^pA_{(L_p)}B_{(R-S)}\Big\}\Bigg]$$
$$=\frac{-1}{N_{R-S}}\Bigg[\sum_{d=1}^{s}\sum_{U_d\subset S}(-1)^d\Big\{\frac{A_{(U_d)}B_{(R-S)}-AB_{(R-S)}}{C_{(U_d)}D_{(R-S)}}\Big\}\Bigg]$$

$$=\frac{1}{N_{R-S}}\left[\sum_{d=1}^{s}\sum_{U_d\subset S}(-1)^d\left\{\frac{CD_{(R-S)}-C_{(U_d)}D_{(R-S)}}{C_{(U_d)}D_{(R-S)}}\right\}\right].$$

On the other hand it holds that

(4.63) $$CD_{(J_h)}=G_{(J_h)}, \qquad (J_h\subset R-S;\ h=0.1,\cdots,r-s),$$

since

(4.64) $$C^{(S)}D_{(J_h)}=\frac{1}{\prod_{\zeta=0}^{r}n_\zeta^{1-\delta_S^\zeta}}\left[CD_{(J_h)}+\sum_{p=1}^{s}\sum_{L_p\subset S}(-1)^pC_{(L_p)}D_{(J_h)}\right]=0,$$

and so

(4.65) $$CD_{(J_h)}=\sum_{p=1}^{s}\sum_{L_p\subset S}(-1)^{p-1}C_{(L_p)}D_{(J_h)}=G_{(J_h)}.$$

Therefore we have

(4.66) $$X_{R-S}=\frac{1}{N_{R-S}}\left[\sum_{d=1}^{s}\sum_{U_d\subset S}(-1)^d\left\{\frac{1}{C_{(U_d)}D_{(R-S)}}-\frac{1}{G_{(R-S)}}\right\}\right]$$

$$=\frac{1}{\prod_{\zeta=0}^{r}n_\zeta^{1-\delta_{R-S}^\zeta}}\left[\sum_{d=1}^{s}\sum_{U_d\subset S}\frac{(-1)^d}{C_{(U_d)}D_{(R-S)}}+\frac{1}{G_{(R-S)}}\right].$$

(**vi**) Solutions for X_{I_k}, $(I_k\subset S,\ k=1,2,\cdots,s-1)$.

(4.34) is obtained by mathematical induction in k in (4.45) and from (4.29), (4.30), (4.31), (4.32), (4.33) and (4.61).

At the first stage we shall prove (4.34) holds true for all $I_{s-1}\subset S$. The equation to be solved is

(4.67) $$X_{I_{s-1}}\{C_{(I_{s-1})}D\}=-\left[X_S\{A_{(I_{s-1})}^{(S-I_{s-1})}B\}+\sum_{e=1}^{r-s}\sum_{V_e\subset R-S}X_{I_{s-1}V_e}\{A_{(I_{s-1})}B^{(V_e)}\}\right.$$

$$\left.+\sum_{e=1}^{r-s}\sum_{V_e\subset R-S}X_{SV_e}\{A_{(I_{s-1})}^{(S-I_{s-1})}B^{(V_e)}\}+X_0\sigma_{I_{s-1}}^2\right]$$

$$=\frac{1}{\prod_{\zeta=0}^{r}n_\zeta^{1-\delta_{I_{s-1}}^\zeta}}\left[\left\{\sum_{\beta=0}^{r-s}\sum_{N_\beta\subset R-S}\frac{(-1)^\beta}{C_{(S)}D_{(N_\beta)}}\right\}\{A_{(I_{s-1})}B-A_{(S)}B\}\right.$$

$$+\sum_{e=1}^{r-s}\sum_{V_e\subset R-S}\left\{\sum_{\alpha=0}^{1}\sum_{M_\alpha\subset S-I_{s-1}}\sum_{\beta=0}^{r-s-e}\sum_{N_\beta\subset R-S-V_e}\frac{(-1)^{\alpha+\beta}}{C_{(I_{s-1}M_\alpha)}D_{(V_eN_\beta)}}\right\}$$

$$\cdot\left\{\sum_{q=0}^{e}\sum_{T_q\subset V_e}(-1)^qA_{(I_{s-1})}B_{(T_q)}\right\}$$

$$+\sum_{e=1}^{r-s}\sum_{V_e\subset R-S}\left\{\sum_{\beta=0}^{r-s-e}\sum_{N_\beta\subset R-S-V_e}\frac{(-1)^\beta}{C_{(S)}D_{(V_eN_\beta)}}\right\}$$

$$\cdot\left\{\sum_{p=0}^{1}\sum_{L_p\subset S-I_s-1}\sum_{q=0}^{e}\sum_{T_q\subset V_e}(-1)^{p+q}A_{(I_s-1L_p)}B_{(T_q)}\right\}\Bigg]$$

$$=\frac{-1}{N_{I_s-1}}\Bigg[\sum_{\beta=0}^{r-s}\sum_{N_\beta\subset R-S}(-1)^{\beta}\left\{\frac{A_{(I_s-1)}B-A_{(S)}B}{C_{(S)}D_{(N_\beta)}}\right\}$$

$$+\sum_{\alpha=0}^{1}\sum_{M_\alpha\subset S-I_s-1}\sum_{e=1}^{r-s}\sum_{V_e\subset R-S}(-1)^{\alpha+e}\left\{\frac{A_{(I_s-1)}B_{(V_e)}-A_{(I_s-1)}B}{C_{(I_s-1M_\alpha)}D_{(V_e)}}\right\}$$

$$+\sum_{p=0}^{1}\sum_{L_p\subset S-I_s-1}\sum_{e=1}^{r-s}\sum_{V_e\subset R-S}(-1)^{p+e}\left\{\frac{A_{(I_s-1L_p)}B_{(V_e)}-A_{(I_s-1L_p)}B}{C_{(S)}D_{(V_e)}}\right\}\Bigg]$$

$$=\frac{-1}{N_{I_s-1}}\Bigg[\frac{A_{(I_s-1)}B-A_{(S)}B}{C_{(S)}D}$$

$$+\sum_{e=1}^{r-s}\sum_{V_e\subset R-S}(-1)^{e}\left\{\frac{A_{(I_s-1)}B-A_{(S)}B_{(V_e)}}{C_{(S)}D_{(V_e)}}+\frac{A_{(I_s-1)}B_{(V_e)}-A_{(I_s-1)}B}{C_{(I_s-1)}D_{(V_e)}}\right\}\Bigg]$$

$$=\frac{1}{N_{I_s-1}}\Bigg[\frac{C_{(S)}D-C_{(I_s-1)}D}{C_{(S)}D}$$

$$+\sum_{e=1}^{r-s}\sum_{V_e\subset R-S}(-1)^{e}\left\{\frac{C_{(S)}D_{(V_e)}-C_{(I_s-1)}D}{C_{(S)}D_{(V_e)}}+\frac{C_{(I_s-1)}D-C_{(I_s-1)}D_{(V_e)}}{C_{(I_s-1)}D_{(V_e)}}\right\}\Bigg].$$

Then we have

$$X_{I_s-1}=\frac{1}{\prod_{\zeta=0}^{r}n_\zeta^{1-\delta_{I_s-1}^{\zeta}}}\Bigg[\frac{1}{C_{(I_s-1)}D}-\frac{1}{C_{(S)}D} \tag{4.68}$$

$$+\sum_{e=1}^{r-s}\sum_{V_e\subset R-S}(-)^{e}\left\{\frac{1}{C_{(I_s-1)}D_{(V_e)}}-\frac{1}{C_{(S)}D_{(V_e)}}\right\}\Bigg],$$

which completes the first stage.

At the second stage we shall prove, assuming that this holds true for all $I_k\subset S$ when $k=c,\ c+1,\cdots,s-1$, that (4.29), (4.30), (4.31), (4.32), (4.33), and (4.61) hold, this also holds true for all $I_{c-1}\subset S$. The equation is to be given by

$$X_{I_c-1}\{C_{(I_c-1)}D\} \tag{4.69}$$

$$=-\Bigg[\sum_{d=1}^{s-c+1}\sum_{U_d\subset S-I_c-1}X_{I_c-1U_d}\{A_{(I_c-1)}^{(U_d)}B\}+\sum_{e=1}^{r-s}\sum_{V_e\subset R-S}X_{I_c-1V_e}\{A_{(I_c-1)}B^{(V_e)}\}$$

$$+\sum_{d=1}^{s-c+1}\sum_{U_d\subset S-I_c-1}\sum_{e=1}^{r-s}\sum_{V_e\subset R-S}X_{I_c-1U_dV_e}\{A_{(I_c-1)}^{(U_d)}B^{(V_e)}\}+X_0\sigma_{I_c-1}^2\Bigg]$$

$$=\frac{-1}{\prod_{\zeta=0}^{r}n_\zeta^{1-\delta_{I_c-1}^{\zeta}}}\Bigg[\sum_{d=1}^{s-c+1}\sum_{U_d\subset S-I_c-1}\left\{\sum_{\alpha=0}^{s-c+1-d}\sum_{M_\alpha\subset S-(I_c-1\cup U_d)}\sum_{\beta=0}^{r-s}\sum_{N_\beta\subset R-S}\frac{(-1)^{\alpha+\beta}}{C_{(I_c-1U_dM_\alpha)}D_{(N_\beta)}}\right\}$$

$$\cdot\left\{\sum_{p=0}^{d}\sum_{L_p\subset U_d}(-1)^{p}A_{(I_c-1L_p)}B\right\}$$

$$+\sum_{e=1}^{r-s}\sum_{V_e\subset R-S}\left\{\sum_{\alpha=0}^{s-c+1}\sum_{M_\alpha\subset S-I_{c-1}}\sum_{\beta=0}^{r-s-e}\sum_{N\beta\subset R-S-V_e}\frac{(-1)^{\alpha+\beta}}{C_{(I_{c-1}M_\alpha)}D_{(V_eN\beta)}}\right\}$$

$$\cdot\left\{\sum_{q=0}^{e}\sum_{T_q\subset V_e}(-1)^q A_{(I_{c-1})}B_{(T_q)}\right\}$$

$$+\sum_{d=1}^{s-c+1}\sum_{U_d\subset S-I_{c-1}}\sum_{e=1}^{r-s}\sum_{V_e\subset R-S}$$

$$\cdot\left\{\sum_{\alpha=0}^{s-c+1-d}\sum_{M_\alpha\subset S-(I_{c-1}\cup U_d)}\sum_{\beta=0}^{r-s-e}\sum_{N\beta\subset R-S-V_e}\frac{(-1)^{\alpha+\beta}}{C_{(I_{c-1}U_dM_\alpha)}D_{(V_eN\beta)}}\right\}$$

$$\left.\cdot\left\{\sum_{p=0}^{d}\sum_{L_p\subset U_d}\sum_{q=0}^{e}\sum_{T_q\subset V_e}(-1)^{p+q}A_{(I_{c-1}L_p)}B_{(T_q)}\right\}\right]$$

$$=\frac{1}{N_{I_{c-1}}}\left[\sum_{d=1}^{s-c+1}\sum_{U_d\subset S-I_{c-1}}(-1)^d\left\{\frac{C_{(I_{c-1})}D-C_{(I_{c-1}U_d)}D}{C_{(I_{c-1}U_d)}D}\right\}\right.$$

$$+\sum_{e=1}^{r-s}\sum_{V_e\subset R-S}(-1)^e\left\{\frac{C_{(I_{c-1})}D-C_{(I_{c-1})}D_{(V_e)}}{C_{(I_{c-1})}D_{(V_e)}}\right\}$$

$$\left.+\sum_{d=1}^{s-c+1}\sum_{U_d\subset S-I_{c-1}}\sum_{e=1}^{r-s}\sum_{V_e\subset R-S}(-1)^{d+e}\left\{\frac{C_{(I_{c-1})}D-C_{(I_{c-1}U_d)}D_{(V_e)}}{C_{(I_{c-1}U_d)}D_{(V_e)}}\right\}\right].$$

Hence we have

$$X_{I_{c-1}}=\frac{1}{N_{I_{c-1}}}\left[\sum_{d=1}^{s-c+1}\sum_{U_d\subset S-I_{c-1}}\frac{(-1)^d}{C_{(I_{c-1}U_d)}D}+\sum_{e=1}^{r-s}\sum_{V_e\subset R-S}\frac{(-1)^e}{C_{(I_{c-1})}D_{(V_e)}}\right.$$
$$\left.+\sum_{d=1}^{s-c+1}\sum_{U_d\subset S-I_{c-1}}\sum_{e=1}^{r-s}\sum_{V_e\subset R-S}\frac{(-1)^{d+e}}{C_{(I_{c-1}U_d)}D_{(V_e)}}+\frac{1}{C_{(I_{c-1})}D}\right]$$
$$=\frac{1}{\prod_{\zeta=0}^{r}n_\zeta^{1-\delta^\zeta_{I_{c-1}}}}\left[\sum_{d=0}^{s-c+1}\sum_{U_d\subset S-I_{c-1}}\sum_{e=0}^{r-s}\sum_{V_e\subset R-S}\frac{(-1)^{d+e}}{C_{(I_{c-1}U_d)}D_{(V_e)}}\right], \tag{4.70}$$

which completes the second stage.

(vii) Solutions for X_h, $(J_h\subset R-S;\ h=1,2,\cdots,r-s-1)$.

(4.35) is obtained by mathematical induction in h in (4.46) and from (4.30), (4.31), (4.32), (4.33) and (4.66).

At the first stage, we shall prove (4.35) holds true for all $J_{r-s-1}\subset R-S$. The equation to be solved is

$$X_{J_{r-s-1}}\{G_{(J_{r-s-1})}\}$$
$$=-\left[X_{R-S}\{AB_{(J_{r-s-1})}-AB_{(R-S)}\}+\sum_{d=1}^{s}\sum_{U_d\subset S}X_{U_dJ_{r-s-1}}\{A^{(U_d)}B_{(J_{r-s-1})}\}\right.$$
$$\left.+\sum_{d=1}^{s}\sum_{U_d\subset S}X_{U_dR-S}\{A^{(U_d)}B_{(J_{r-s-1})}-A^{(U_d)}B_{(R-S)}\}\right] \tag{4.71}$$

$$=\frac{-1}{\prod_{\zeta=0}^{r} n_\zeta^{1-\delta^\zeta_{J_{r-s-1}}}}\left[\left\{\sum_{d=1}^{s}\sum_{U_d\subset S}\frac{(-1)^d}{C_{(U_d)}D_{(R-S)}}+\frac{1}{G_{(R-S)}}\right\}\{AB_{(J_{r-s-1})}-AB_{(R-S)}\}\right.$$

$$+\sum_{d=1}^{s}\sum_{U_d\subset S}\left\{\sum_{\alpha=0}^{s-d}\sum_{M_\alpha\subset S-U_d}\sum_{\beta=0}^{1}\sum_{N_\beta\subset R-S-J_{r-s-1}}\frac{(-1)^{\alpha+\beta}}{C_{(U_dM_\alpha)}D_{(J_{r-s-1}N_\beta)}}\right\}$$

$$\cdot\left\{\sum_{p=0}^{d}\sum_{L_p\subset U_d}(-1)^pA_{(L_p)}B_{(J_{r-s-1})}\right\}$$

$$+\sum_{d=1}^{s}\sum_{U_d\subset S}\left\{\sum_{\alpha=0}^{s-d}\sum_{M_\alpha\subset S-U_d}\frac{(-1)^\alpha}{C_{(U_dM_\alpha)}D_{(R-S)}}\right\}$$

$$\left.\cdot\left\{\sum_{p=0}^{d}\sum_{L_p\subset U_d}\sum_{q=0}^{1}\sum_{T_q\subset R-S-J_{r-s-1}}(-1)^{p+q}A_{(L_p)}B_{(J_{r-s-1}T_q)}\right\}\right]$$

$$=\frac{-1}{N_{J_{r-s-1}}}\left[\sum_{d=1}^{s}\sum_{U_d\subset S}(-1)^d\left\{\frac{AB_{(J_{r-s-1})}-AB_{(R-S)}}{C_{(U_d)}D_{(R-S)}}\right\}+\frac{AB_{(J_{r-s-1})}-AB_{(R-S)}}{G_{(R-S)}}\right.$$

$$+\sum_{\beta=0}^{1}\sum_{N_\beta\subset R-S-J_{r-s-1}}\sum_{d=1}^{s}\sum_{U_d\subset S}(-1)^{\beta+d}\left\{\frac{A_{(U_d)}B_{(J_{r-s-1})}-AB_{(J_{r-s-1})}}{C_{(U_d)}D_{(J_{r-s-1}N_\beta)}}\right\}$$

$$+\sum_{d=1}^{s}\sum_{U_d\subset S}(-1)^d\left\{\frac{A_{(U_d)}B_{(J_{r-s-1})}-AB_{(J_{r-s-1})}}{C_{(U_d)}D_{(R-S)}}\right\}$$

$$\left.-\sum_{d=1}^{s}\sum_{U_d\subset S}(-1)^d\left\{\frac{A_{(U_d)}B_{(R-S)}-AB_{(R-S)}}{C_{(U_d)}D_{(R-S)}}\right\}\right]$$

$$=\frac{1}{N_{J_{r-s-1}}}\left[\sum_{d=1}^{s}\sum_{U_d\subset S}(-1)^d\left\{\frac{C_{(U_d)}D_{(R-S)}-CD_{(J_{r-s-1})}}{C_{(U_d)}D_{(R-S)}}\right.\right.$$

$$\left.\left.+\frac{CD_{(J_{r-s-1})}-C_{(U_d)}D_{(J_{r-s-1})}}{C_{(U_d)}D_{(J_{r-s-1})}}\right\}+\frac{CD_{(R-S)}-CD_{(J_{r-s-1})}}{G_{(R-S)}}\right]$$

$$=\frac{1}{N_{J_{r-s-1}}}\left[\sum_{d=1}^{s}\sum_{U_d\subset S}(-1)^d\left\{\frac{C_{(U_d)}D_{(R-S)}-G_{(J_{r-s-1})}}{C_{(U_d)}D_{(R-S)}}\right.\right.$$

$$\left.\left.+\frac{G_{(J_{r-s-1})}-C_{(U_d)}D_{(J_{r-s-1})}}{C_{(U_d)}D_{(J_{r-s-1})}}\right\}+\frac{G_{(R-S)}-G_{(J_{r-s-1})}}{G_{(R-S)}}\right].$$

And so we have

$$X_{J_{r-s-1}}=\frac{1}{\prod_{\zeta=0}^{r} n_\zeta^{1-\delta^\zeta_{J_{r-s-1}}}}\left[\sum_{d=1}^{s}\sum_{U_d\subset S}(-1)^d\left\{\frac{1}{C_{(U_d)}D_{(J_{r-s-1})}}-\frac{1}{C_{(U_d)}D_{(R-S)}}\right\}\right.$$

$$\left.+\frac{1}{G_{(J_{r-s-1})}}-\frac{1}{G_{(R-S)}}\right]. \tag{4.72}$$

At the second stage we shall prove, assuming that this holds true for all $J_h\subset R-S$ when $h=c,\ c+1,\cdots,r-s-1$, that (4.30), (4.31), (4.32), (4.33) and (4.66) hold, this also holds true for all $J_{c-1}\subset R-S$. The equation to be solved is given by

(4.73) $$X_{J_{c-1}}\{G_{(J_{c-1})}\}$$

$$=-\Big[\sum_{e=1}^{r-s-c+1}\sum_{V_e\subset R-S-J_{c-1}}X_{J_{c-1}V_e}\{AB^{(V_e)}_{(J_{c-1})}\}+\sum_{d=1}^{s}\sum_{U_d\subset S}X_{U_dJ_{c-1}}\{A^{(U_d)}B_{(J_{c-1})}\}$$

$$+\sum_{d=1}^{s}\sum_{U_d\subset S}\sum_{e=1}^{r-s-c+1}\sum_{V_e\subset R-S-J_{c-1}}X_{U_dJ_{c-1}V_e}\{A^{(U_d)}B^{(V_e)}_{(J_{c-1})}\}\Big]$$

$$=\frac{-1}{\prod_{\zeta=0}^{r}n_\zeta^{1-\delta^\zeta_{J_{c-1}}}}\Big[\sum_{e=1}^{r-s-c+1}\sum_{V_e\subset R-S-J_{c-1}}\Big\{\sum_{\alpha=1}^{s}\sum_{M_\alpha\subset S}\sum_{\beta=0}^{r-s-c+1-e}\sum_{N_\beta\subset R-S-(J_{c-1}\cup V_e)}$$

$$\cdot\frac{(-1)^{\alpha+\beta}}{C_{(M_\alpha)}D_{(J_{c-1}V_eN_\beta)}}+\sum_{\beta=0}^{r-s-c+1-e}\sum_{N_\beta\subset R-S-(J_{c-1}\cup V_e)}\frac{(-1)^\beta}{G_{(J_{c-1}V_eN_\beta)}}\Big\}$$

$$\cdot\Big\{\sum_{q=0}^{e}\sum_{T_q\subset V_e}(-1)^qAB_{(J_{c-1}T_q)}\Big\}$$

$$+\sum_{d=1}^{s}\sum_{U_d\subset S}\Big\{\sum_{\alpha=0}^{s-d}\sum_{M_\alpha\subset S-U_d}\sum_{\beta=0}^{r-s-c+1}\sum_{N_\beta\subset R-S-J_{c-1}}\frac{(-1)^{\alpha+\beta}}{C_{(U_dM_\alpha)}D_{(J_{c-1}N_\beta)}}\Big\}$$

$$\cdot\Big\{\sum_{p=0}^{d}\sum_{L_p\subset U_d}(-1)^pA_{(L_p)}B_{(J_{c-1})}\Big\}$$

$$+\sum_{d=1}^{s}\sum_{U_d\subset S}\sum_{e=1}^{r-s-c+1}\sum_{V_e\subset R-S-J_{c-1}}\Big\{\sum_{\alpha=0}^{s-d}\sum_{M_\alpha\subset S-U_d}\sum_{\beta=0}^{r-s-c+1-e}\sum_{N_\beta\subset R-S-(J_{c-1}\cup V_e)}$$

$$\cdot\frac{(-1)^{\alpha+\beta}}{C_{(U_dM_\alpha)}D_{(J_{c-1}V_eN_\beta)}}\Big\}\Big\{\sum_{p=0}^{d}\sum_{L_p\subset U_d}\sum_{q=0}^{e}\sum_{T_q\subset V_e}(-1)^{p+q}A_{(L_p)}B_{(J_{c-1}T_q)}\Big\}\Big]$$

$$=\frac{-1}{N_{J_{c-1}}}\Big[\sum_{d=1}^{s}\sum_{U_d\subset S}(-1)^d\Big\{\frac{A_{(U_d)}B_{(J_{c-1})}-AB_{(J_{c-1})}}{C_{(U_d)}D_{(J_{c-1})}}\Big\}$$

$$+\sum_{d=1}^{s}\sum_{U_d\subset S}\sum_{e=1}^{r-s-c+1}\sum_{V_e\subset R-S-J_{c-1}}(-1)^{d+e}\Big\{\frac{A_{(U_d)}B_{(J_{c-1}V_e)}-AB_{(J_{c-1})}}{C_{(U_d)}D_{(J_{c-1}V_e)}}\Big\}$$

$$+\sum_{e=1}^{r-s-c+1}\sum_{V_e\subset R-S-J_{c-1}}(-1)^e\Big\{\frac{AB_{(J_{c-1}V_e)}-AB_{(J_{c-1})}}{G_{(J_{c-1}V_e)}}\Big\}\Big]$$

$$=\frac{1}{N_{J_{c-1}}}\Big[\sum_{d=1}^{s}\sum_{U_d\subset S}(-1)^d\Big\{\frac{G_{(J_{c-1})}-C_{(U_d)}D_{(J_{c-1})}}{C_{(U_d)}D_{(J_{c-1})}}\Big\}$$

$$+\sum_{d=1}^{s}\sum_{U_d\subset S}\sum_{e=1}^{r-s-c+1}\sum_{V_e\subset R-S-J_{c-1}}(-1)^{d+e}\Big\{\frac{G_{(J_{c-1})}-C_{(U_d)}D_{(J_{c-1}V_e)}}{C_{(U_d)}D_{(J_{c-1}V_e)}}\Big\}$$

$$+\sum_{e=1}^{r-s-c+1}\sum_{V_e\subset R-S-J_{c-1}}(-1)^e\Big\{\frac{G_{(J_{c-1})}-G_{(J_{c-1}V_e)}}{G_{(J_{c-1}V_e)}}\Big\}\Big].$$

Therefore we have

(4.74)
$$X_{J_{c-1}}=\frac{1}{\prod_{\zeta=0}^{r} n_{\zeta}^{1-\delta_{J_{c-1}}^{\zeta}}}\left[\sum_{d=1}^{s}\sum_{U_d\subset S}\sum_{e=0}^{r-s-c+1}\sum_{V_e\subset R-S-J_{c-1}}\frac{(-1)^{d+e}}{C_{(U_d)}D_{(J_{c-1}V_e)}}+\sum_{e=0}^{r-s-c+1}\sum_{V_e\subset R-S-J_{c-1}}\frac{(-1)^{e}}{G_{(J_{c-1}V_e)}}\right].$$

(**viii**) Solution for X_G.

(4.36) is obtained by inserting (4.30), ⋯, (4.35), (4.61) and (4.66) in (4.47). After inserting them in (4.47), we have

(4.75)
$$\begin{aligned}
X_G\{G\}=&-\Big[\sum_{d=1}^{s}\sum_{U_d\subset S}X_{U_d}\{A^{(U_d)}B\}\\
&+\sum_{d=1}^{s}\sum_{U_d\subset S}\sum_{e=1}^{r-s}\sum_{V_e\subset R-S}X_{U_dV_e}\{A^{(U_d)}B^{(V_e)}\}+\sum_{e=1}^{r-s}\sum_{V_e\subset R-S}X_{V_e}\{AB^{(V_e)}\}\Big]\\
=&\frac{-1}{\prod_{\zeta=0}^{r}n_{\zeta}}\Big[\sum_{d=1}^{s}\sum_{U_d\subset S}\Big\{\sum_{\alpha=0}^{s-d}\sum_{M_\alpha\subset S-U_d}\sum_{\beta=0}^{r-s}\sum_{N_\beta\subset R-S}\frac{(-1)^{\alpha+\beta}}{C_{(U_dM_\alpha)}D_{(N_\beta)}}\Big\}\\
&\cdot\Big\{\sum_{p=0}^{d}\sum_{L_p\subset U_d}(-1)^{p}A_{(L_p)}B\Big\}\\
&+\sum_{d=1}^{s}\sum_{U_d\subset S}\sum_{e=1}^{r-s}\sum_{V_e\subset R-S}\Big\{\sum_{\alpha=0}^{s-d}\sum_{M_\alpha\subset S-U_d}\sum_{\beta=0}^{r-s-e}\sum_{N_\beta\subset R-S-V_e}\frac{(-1)^{\alpha+\beta}}{C_{(U_dM_\alpha)}D_{(V_eN_\beta)}}\Big\}\\
&\cdot\Big\{\sum_{p=0}^{d}\sum_{L_p\subset U_d}\sum_{q=0}^{e}\sum_{T_q\subset V_e}(-1)^{p+q}A_{(L_p)}B_{(T_q)}\Big\}\\
&+\sum_{e=1}^{r-s}\sum_{V_e\subset R-S}\Big\{\sum_{\alpha=1}^{s}\sum_{M_\alpha\subset S}\sum_{\beta=0}^{r-s-e}\sum_{N_\beta\subset R-S-V_e}\frac{(-1)^{\alpha+\beta}}{C_{(M_\alpha)}D_{(V_eN_\beta)}}\\
&+\sum_{\beta=0}^{r-s-e}\sum_{N_\beta\subset R-S-V_e}\frac{(-1)^{\beta}}{G_{(V_eN_\beta)}}\Big\}\Big\{\sum_{q=0}^{e}\sum_{T_q\subset V_e}(-1)^{q}AB_{(T_q)}\Big\}\\
=&\frac{-1}{\prod_{\zeta=0}^{r}n_{\zeta}}\Big[\sum_{\beta=0}^{r-s}\sum_{N_\beta\subset R-S}\sum_{d=1}^{s}\sum_{U_d\subset S}(-1)^{\beta+d}\Big\{\frac{A_{(U_d)}B-AB}{C_{(U_d)}D_{(N_\beta)}}\Big\}\\
&+\sum_{d=1}^{s}\sum_{U_d\subset S}\sum_{e=1}^{r-s}\sum_{V_e\subset R-S}(-1)^{d+e}\Big\{\frac{A_{(U_d)}B_{(V_e)}-AB_{(V_e)}}{C_{(U_d)}D_{(V_e)}}\Big\}\\
&-\sum_{d=1}^{s}\sum_{U_d\subset S}\sum_{e=1}^{r-s}\sum_{V_e\subset R-S}(-1)^{d+e}\Big\{\frac{A_{(U_d)}B-AB}{C_{(U_d)}D_{(V_e)}}\Big\}\\
&+\sum_{\alpha=1}^{s}\sum_{M_\alpha\subset S}\sum_{e=1}^{r-s}\sum_{V_e\subset R-S}(-1)^{\alpha+e}\Big\{\frac{AB_{(V_e)}-AB}{C_{(M_\alpha)}D_{(V_e)}}\Big\}\\
&+\sum_{e=1}^{r-s}\sum_{V_e\subset R-S}(-1)^{e}\Big\{\frac{AB_{(V_e)}-AB}{G_{(V_e)}}\Big\}\Big]
\end{aligned}$$

$$=\frac{1}{\prod_{\zeta=0}^{r} n_\zeta}\left[\sum_{d=1}^{s}\sum_{U_d\subset S}(-1)^d\left\{\frac{G-C_{(U_d)}D}{C_{(U_d)}D}\right\}\right.$$

$$+\sum_{d=1}^{s}\sum_{U_d\subset S}\sum_{e=1}^{r-s}\sum_{V_e\subset R-S}(-1)^{d+e}\left\{\frac{G-C_{(U_d)}D_{(V_e)}}{C_{(U_d)}D_{(V_e)}}\right\}$$

$$\left.+\sum_{e=1}^{r-s}\sum_{V_e\subset R-S}(-1)^e\left\{\frac{G-G_{(V_e)}}{G_{(V_e)}}\right\}\right].$$

Hence we have

$$X_G=\frac{1}{\prod_{\zeta=0}^{r} n_\zeta}\left[\frac{1}{G}+\sum_{e=1}^{r-s}\sum_{V_e\subset R-S}\frac{(-1)^e}{G_{(V_e)}}\right. \tag{4.76}$$

$$\left.+\sum_{d=1}^{s}\sum_{U_d\subset S}\sum_{e=0}^{r-s}\sum_{V_e\subset R-S}\frac{(-1)^{d+e}}{C_{(U_d)}D_{(V_e)}}\right].$$

Finally the proof of Lemma 4.6 is completed, and also Theorem 4.2 is proved.

4.3. The joint density function.

The joint density function in Type II is given in the following,

THEOREM 4.3. *The joint density function of all observations in our case is given by*

$$f(X)=(2\pi)^{-n_0n_1\cdots n_r/2}G^{-1/2}\prod_{d=1}^{r-s}\prod_{U_d\subset R-S}\{G_{(U_d)}\}^{-(n_{u_1}-1)\cdots(n_{u_d}-1)/2} \tag{4.77}$$

$$\cdot\prod_{k=1}^{s}\prod_{I_k\subset S}\prod_{h=0}^{r-s}\prod_{J_h\subset R-S}\{C_{(I_k)}D_{(J_h)}\}^{-(n_{i_1}-1)\cdots(n_{i_k}-1)(n_{j_1}-1)\cdots(n_{j_h}-1)/2}$$

$$\cdot\{\sigma_0^2\}^{-(n_0-1)n_1n_2\cdots n_r/2}$$

$$\cdot exp\left[-\frac{1}{2}\left\{\prod_{j=0}^{r} n_j(\bar{X}-\mu)^2\frac{1}{G}+\sum_{d=1}^{r-s}\sum_{U_d\subset R-S}\frac{T_{(U_d)}}{G_{(U_d)}}\right.\right.$$

$$\left.\left.+\sum_{k=1}^{s}\sum_{I_k\subset S}\sum_{h=0}^{r-s}\sum_{J_h\subset R-S}\frac{S_{(I_kJ_h)}}{C_{(I_k)}D_{(J_h)}}+\frac{S_0}{\sigma_0^2}\right\}\right],$$

where

$$T_{(U_d)}=\prod_{\zeta=0}^{r} n_\zeta^{1-\delta_{U_d}^{\zeta}} \tag{4.78}$$

$$\cdot\sum_{t_{u_1},\cdots,t_{u_d}}\left\{\sum_{\beta=0}^{d}\sum_{L_\beta\subset U_d}(-1)^{d-\beta}\bar{X}_{t_{l_1}t_{l_2}\cdots t_{l_\beta}}-\alpha(u_1,\cdots,u_d;\ t_{u_1},\cdots,t_{u_d})\right\}^2,$$

$$(U_d\subset R-S;\ d=1,2,\cdots,r-s),$$

$$S_{(I_kJ_h)}=\prod_{\zeta=0}^{r} n_\zeta^{1-\delta_{I_kJ_h}^{\zeta}}\sum_{t_{i_1},\cdots,t_{i_k}}\sum_{t_{j_1},\cdots,t_{j_h}}\left\{\sum_{\beta=0}^{k+h}\sum_{L_\beta\subset(I_k\cup J_h)}(-1)^{k+h-\beta}\bar{X}_{t_{l_1}\cdots t_{l_\beta}}\right\}^2, \tag{4.79}$$

$$(I_k\subset S;\ J_h\subset R-S;\ k=1,\cdots,s;\ h=0,\cdots,r-s),$$

(4.80) $$S_0=\sum_{t_0,t_1,\cdots,t_r}(x_{t_0t_1\cdots t_r}-\bar{X}_{t_1t_2\cdots t_r})^2,$$

and

(4.81) $$\bar{X}_{t_{l_1}t_{l_2}\cdots t_{l_\beta}}=\frac{1}{\prod_{\zeta=0}^{r}n_\zeta^{1-\delta^{\zeta}_{L_\beta}}}\sum_{\substack{t_{v_1},\cdots,t_{v_{r-\beta}}\\V_{r-\beta}\subset R-L_\beta}}\sum_{t_0}x_{t_0t_1\cdots t_r},\qquad (L_\beta\subset R,\ \beta=0,\cdots,r),$$

where in (4.81) $\bar{X}_{t_{l_1}\cdots t_{l_\beta}}=\bar{X}$ *when* $\beta=0$.

PROOF. As it is obvious that the type of the density function is the normal distribution, the constant factor in (4.77) is easily derived from Theorem 4.1, and there remains only to derive the quadratic form of $x_{t_0t_1\cdots t_r}$.

Now, let us introduce new variables defined by

(4.82) $$y_{t_0t_1\cdots t_r}=x_{t_0t_1\cdots t_r}-\mu-\sum_{h=1}^{r-s}\sum_{J_h\subset R-S}\alpha(j_1,\cdots,j_h;\ t_{j_1},\cdots,\ t_{j_h}),$$

(4.83) $$Y_{t_{l_1}t_{l_2}\cdots t_{l_\beta}}=\sum_{\substack{t_{v_1},\cdots,t_{v_{r-\beta}}\\V_{r-\beta}\subset R-L_\beta}}\sum_{t_0}y_{t_0t_1\cdots t_r},\qquad (L_\beta\subset R;\ \beta=0,\cdots,r),$$

and

(4.84) $$\bar{Y}_{t_{l_1}t_{l_2}\cdots t_{l_\beta}}=\frac{1}{\prod_{\zeta=0}^{r}n_\zeta^{1-\delta^{\zeta}_{L_\beta}}}Y_{t_{l_1}t_{l_2}\cdots t_{l_\beta}},\qquad (L_\beta\subset R;\ \beta=0,\cdots,r),$$

and when $\beta=0$ we shall use the same convention for $Y_{t_{l_1}t_{l_2}\cdots t_{l_\beta}}$ and $\bar{Y}_{t_{l_1}t_{l_2}\cdots t_{l_\beta}}$ as the one stated in this theorem. Furthermore, we note that (4.84) is expressed in terms of $\bar{X}_{t_{l_1}\cdots t_{l_\beta}}$ and $\alpha(v_1,\cdots,v_p;\ t_{v_1},\cdots,t_{v_p})$ by the assumptions (2.2), as follows

(4.85) $$\bar{Y}_{t_{i_1}\cdots t_{i_k}t_{j_1}\cdots t_{j_h}}=\bar{X}_{t_{i_1}\cdots t_{i_k}t_{j_1}\cdots t_{j_h}}-\mu-\sum_{p=1}^{h}\sum_{V_p\subset J_h}\alpha(v_1,\cdots,v_p;\ t_{v_1},\cdots,t_{v_p}),$$
$$(I_k\subset S;\ J_h\subset R-S;\ k=0,\cdots,s;h=1,\cdots,r-s),$$

(4.86) $$\bar{Y}_{t_{i_1}\cdots t_{i_k}}=\bar{X}_{t_{i_1}\cdots t_{i_k}}-\mu,\qquad (I_k\subset S;\ k=0,\cdots,s).$$

Using the inverse matrix derived in Theorem 4.2 we have the quadratic form in the joint density function:

(4.87) $$S=X_G\Big(\sum_{t_0,t_1,\cdots,t_r}y_{t_0t_1\cdots t_r}\Big)^2$$
$$+\sum_{k=1}^{s}\sum_{I_k\subset S}\sum_{h=0}^{r-s}\sum_{J_h\subset R-S}X_{I_kJ_h}$$
$$\cdot\Big\{\sum_{t_{i_1},\cdots,t_{i_k}}\sum_{t_{j_1},\cdots,t_{j_h}}\Big(\sum_{\substack{t_{p_1},\cdots,t_{p_{s-k}}\\P_{s-k}\subset S-I_k}}\sum_{\substack{t_{q_1},\cdots,t_{q_{r-s-h}}\\Q_{r-s-h}\subset R-S-J_h}}\sum_{t_0}y_{t_0t_1\cdots t_r}\Big)^2\Big\}$$
$$+\sum_{h=1}^{r-s}\sum_{J_h\subset R-S}X_{J_h}\Big\{\sum_{t_{j_1},\cdots,t_{j_h}}\Big(\sum_{\substack{t_{q_1},\cdots,t_{q_{r-s-h}}\\Q_{r-s-h}\subset R-S-J_h}}\sum_{t_0,t_1,\cdots,t_s}y_{t_0t_1\cdots t_s}\Big)^2\Big\}$$

$$+X_0 \sum_{t_0, t_1, \cdots, t_r} y^2_{t_0 t_1 \cdots t_r}$$

$$= \frac{1}{\prod_{\zeta=0}^{r} n_\zeta} \left[\frac{1}{G} + \sum_{\beta=1}^{r-s} \sum_{N\beta \subset R-S} \frac{(-1)^\beta}{G_{(N\beta)}} + \sum_{\alpha=1}^{s} \sum_{M\alpha \subset S} \sum_{\beta=0}^{r-s} \sum_{N\beta \subset R-S} \frac{(-1)^{\alpha+\beta}}{C_{(M\alpha)} D_{(N\beta)}} \right] Y^2$$

$$+ \sum_{k=1}^{s} \sum_{I_k \subset S} \sum_{h=0}^{r-s} \sum_{J_h \subset R-S} \frac{1}{\prod_{\zeta=0}^{r} n_\zeta^{1-\delta^\zeta_{I_k J_h}}}$$

$$\cdot \left[\sum_{\alpha=0}^{s-k} \sum_{M\alpha \subset S-I_k} \sum_{\beta=0}^{r-s-h} \sum_{N\beta \subset R-S-J_h} \frac{(-1)^{\alpha+\beta}}{C_{(I_k M\alpha)} D_{(J_h N\beta)}} \right]$$

$$\cdot \left(\sum_{t_{i_1}, \cdots, t_{i_k}} \sum_{t_{j_1}, \cdots, t_{j_h}} Y^2_{t_{i_1} \cdots t_{i_k} t_{j_1} \cdots t_{j_h}} \right)$$

$$+ \sum_{h=1}^{r-s} \sum_{J_h \subset R-S} \frac{1}{\prod_{\zeta=0}^{r} n_\zeta^{1-\delta^\zeta_{J_h}}} \left[\sum_{\alpha=1}^{s} \sum_{M\alpha \subset S} \sum_{\beta=0}^{r-s-h} \sum_{N\beta \subset R-S-J_h} \frac{(-1)^{\alpha+\beta}}{C_{(M\alpha)} D_{(J_h N\beta)}} \right.$$

$$\left. + \sum_{\beta=0}^{r-s-h} \sum_{N\beta \subset R-S-J_h} \frac{(-1)^\beta}{G_{(J_h N\beta)}} \right] \left(\sum_{t_{j_1}, \cdots, t_{j_h}} Y^2_{t_{j_1} \cdots t_{j_h}} \right)$$

$$+ \frac{1}{\sigma_0^2} \left(\sum_{t_0 t_1 \cdots t_r} y^2_{t_0 t_1 \cdots t_r} \right).$$

The calculation of (4.87) follows the similar line to that of (3.43), which is given by

(4.88) $$S = \prod_{\zeta=0}^{r} n_\zeta \bar{Y}^2 \frac{1}{G}$$

$$+ \sum_{h=1}^{r-s} \sum_{J_h \subset R-S} \prod_{\zeta=0}^{r} n_\zeta^{1-\delta^\zeta_{J_h}} \sum_{t_{j_1}, \cdots, t_{j_h}} \left\{ \sum_{\beta=0}^{h} \sum_{L\beta \subset J_h} (-1)^{h-\beta} \bar{Y}_{t_{l_1} \cdots t_{l\beta}} \right\}^2 \frac{1}{G_{(J_h)}}$$

$$+ \sum_{k=1}^{s} \sum_{I_k \subset S} \sum_{h=1}^{r-s} \sum_{J_h \subset R-S} \prod_{\zeta=0}^{r} n_\zeta^{1-\delta^\zeta_{I_k J_h}}$$

$$\cdot \sum_{t_{i_1}, \cdots, t_{i_k}} \sum_{t_{j_1}, \cdots, t_{j_h}} \left\{ \sum_{\beta=0}^{k+h} \sum_{L\beta \subset (I_k \cup J_h)} (-1)^{k+h-\beta} \bar{Y}_{t_{l_1} \cdots t_{l\beta}} \right\}^2 \frac{1}{C_{(I_k)} D_{(J_h)}}$$

$$+ \sum_{k=1}^{s} \sum_{I_k \subset S} \prod_{\zeta=0}^{r} n_\zeta^{1-\delta^\zeta_{I_k}} \sum_{t_{i_1}, \cdots, t_{i_k}} \left\{ \sum_{\beta=0}^{k} \sum_{L\beta \subset I_k} (-1)^{k-\beta} \bar{Y}_{t_{l_1} \cdots t_{l\beta}} \right\}^2 \frac{1}{C_{(I_k)} D}$$

$$+ \sum_{t_0, t_1, \cdots t_r} \left(y_{t_0 t_1 \cdots t_r} - \bar{Y}_{t_1 t_2 \cdots t_r} \right)^2 \frac{1}{\sigma_0^2}.$$

In virtue of (4.85) and (4.86), (4.88) is equal to

(4.89) $$\prod_{\zeta=0}^{r} n_\zeta (\bar{X} - \mu)^2 \frac{1}{G}$$

$$+ \sum_{h=1}^{r-s} \sum_{J_h \subset R-S} \prod_{\zeta=0}^{r} n_\zeta^{1-\delta^\zeta_{J_h}} \sum_{t_{j_1}, \cdots, t_{j_h}} \frac{1}{G_{(J_h)}}$$

$$\cdot\left[\sum_{\beta=0}^{h}\sum_{L_\beta\subset J_h}(-1)^{h-\beta}\left\{\bar{X}_{t_{l_1}\cdots t_{l_\beta}}-\mu-\sum_{p=1}^{\beta}\sum_{V_p\subset L_\beta}\alpha(v_1,\cdots,v_p;\ t_{v_1},\cdots,t_{v_p})\right\}\right]^2$$

$$+\sum_{k=1}^{s}\sum_{I_k\subset S}\sum_{h=1}^{r-s}\sum_{J_h\subset R-S}\prod_{\zeta=0}^{r}n_\zeta^{1-\delta^{\zeta}_{I_kJ_h}}\sum_{t_{i_1},\cdots,t_{i_k}}\sum_{t_{j_1},\cdots,t_{j_h}}$$

$$\cdot\left[\sum_{\lambda=0}^{k}\sum_{A_\lambda\subset I_k}\sum_{\nu=0}^{h}\sum_{B_\nu\subset J_h}(-1)^{k+h-\lambda-\nu}\left\{\bar{X}_{t_{a_1}\cdots t_{a_\lambda}t_{b_1}\cdots t_{b_\nu}}-\mu\right.\right.$$

$$\left.\left.-\sum_{p=1}^{\nu}\sum_{V_p\subset B_\nu}\alpha(v_1,\cdots,v_p;\ t_{v_1},\cdots,t_{v_p})\right\}\right]^2\frac{1}{C_{(I_k)}D_{(J_h)}}$$

$$+\sum_{k=1}^{s}\sum_{I_k\subset S}\prod_{\zeta=0}^{r}n_\zeta^{1-\delta^{\zeta}_{I_k}}\sum_{t_{i_1},\cdots,t_{i_k}}\left[\sum_{\beta=0}^{k}\sum_{L_\beta\subset I_k}(-1)^{k-\beta}\{\bar{X}_{t_{l_1}\cdots t_{l_\beta}}-\mu\}\right]^2\frac{1}{C_{(I_k)}D}$$

$$+\sum_{t_0,t_1,\cdots,t_r}(x_{t_0t_1\cdots t_r}-\bar{X}_{t_1t_2\cdots t_r})^2\frac{1}{\sigma_0^2}.$$

The three formulas squared in the second, third and forth terms are simplified in (4.90), (4.91) and (4.92), respectively.

(4.90) $$\sum_{\beta=0}^{h}\sum_{L_\beta\subset J_h}(-1)^{h-\beta}\bar{X}_{t_{l_1}\cdots t_{l_\beta}}-\alpha(j_1,\cdots,j_h;\ t_{j_1},\cdots,t_{j_h}).$$

(4.91) $$\sum_{\lambda=0}^{k}\sum_{A_\lambda\subset I_k}\sum_{\nu=0}^{h}\sum_{B_\nu\subset J_h}(-1)^{k+h-\lambda-\nu}\bar{X}_{t_{a_1}\cdots t_{a_\lambda}t_{b_1}\cdots t_{b_\nu}}$$

$$-\sum_{\lambda=0}^{k}\sum_{A_\lambda\subset I_k}(-1)^{k-\lambda}\alpha(j_1,\cdots,j_h;\ t_{j_1},\cdots,t_{j_h})$$

$$=\sum_{\beta=0}^{k+h}\sum_{L_\beta\subset(I_k\cup J_h)}(-1)^{k+h}\bar{X}_{t_{l_1}\cdots t_{l_\beta}}.$$

(4.92) $$\sum_{\beta=0}^{k}\sum_{L_\beta\subset I_k}(-1)^{k-\beta}\bar{X}_{t_{l_1}\cdots t_{l_\beta}}.$$

Finally, as (4.89) is equal to the quadratic formula in (4.77), the theorem is proved.

4.4. Estimation.

Also in this type we need to have the analogue of Lemma 3.3:

LEMMA 4.7. *The joint density function of* $x_{t_0t_1\cdots t_r}$, (4.77), *is equal to*

(4.93) $$f(X)=K\varphi_{(\sigma^2,\alpha,\mu)}$$

$$\cdot\ exp\left[\prod_{j=0}^{r}n_jZ^{(1)}\frac{\mu}{G}\right.$$

$$+\sum_{d=1}^{r-s}\sum_{U_d\subset R-S}\sum_{t_{u_1}=1}^{n_{u_1}-1}\sum_{t_{u_2}=1}^{n_{u_2}-1}\cdots\sum_{t_{u_d}=1}^{n_{u_d}-1}\prod_{\zeta=0}^{r}n_\zeta^{1-\delta^{\zeta}_{U_d}}Z^{(2)}_{(U_d;\ t_{u_1},\cdots,t_{u_d})}\frac{\alpha(u_1,\cdots,u_d;\ t_{u_1},\cdots,t_{u_d})}{G_{(U)}}$$

$$-\frac{1}{2}\prod_{j=0}^{r} n_j (Z^{(1)})^2 \frac{1}{G} - \frac{1}{2}\sum_{d=1}^{r-s}\sum_{U_d \subset R-S} Z^{(3)}_{(U_d)} \frac{1}{G_{(U_d)}}$$

$$-\frac{1}{2}\sum_{k=1}^{s}\sum_{I_k \subset S}\sum_{h=0}^{r-s}\sum_{J_h \subset R-S} S_{(I_k J_h)} \frac{1}{C_{(I_k)} D_{(J_h)}} - \frac{1}{2}\frac{S_0}{\sigma_0^2}\Bigg],$$

where

(4.94) $$Z^{(1)} = \bar{X}$$

(4.95) $$Z^{(2)}_{(U_d;\, t_{u_1}, \cdots, t_{u_d})} = \bar{X}_{t_{u_1}, \cdots, t_{u_d}} + \sum_{\gamma=1}^{d}\sum_{P_\gamma \subset U_d} (-1)^\gamma \bar{X}_{t_{p_1} \cdots t_{p_{d-\gamma}}} n_{p_{d-\gamma+1}} \cdots n_{p_d},$$

$(U_d \subset R-S;\ d=1, \cdots, r-s)$,

(4.96) $$Z^{(3)}_{(U_d)} = \prod_{\zeta=0}^{r} n_\zeta^{1-\delta^{\zeta}_{U_d}} \sum_{t_{u_1}, \cdots, t_{u_d}} \left(\sum_{\beta=0}^{d}\sum_{L_\beta \subset U_d} (-1)^{d-\beta} \bar{X}_{t_{l_1} \cdots t_{l_\beta}}\right)^2,$$

$(U_d \subset R-S;\ d=1, \cdots, r-s)$.

This lemma can be easily proved in virture of (3.53).

Now we shall consider about the functional relationship among (4.94), (4.95) and (4.96).

At first let us observe that $\bar{X}_{n_{u_1}}$ is determined uniquely by determining $Z^{(1)}$ and $\{Z^{(2)}_{(u_1;\, t_{u_1})};\ t_{u_1}=1, 2, \cdots, n_{u_1}-1\}$, which implies the unique determination of $Z^{(3)}_{(u_1)}$ and $\bar{X}_{n_{u_2}}$ is determined uniquely by determining $Z^{(1)}$ and $\{Z^{(2)}_{(u_2;\, t_{u_2})};\ t_{u_2}=1, 2, \cdots, n_{u_2}-1\}$, which implies again the unique determination of $Z^{(3)}_{(u_2)}$, and the analogue holds true for all sets $u_i \subset R-S$.

Secondly we observe that $\bar{X}_{n_{u_1} n_{u_2}}$ is determined uniquely by determining $Z_{(1)}$, $\bar{X}_{n_{u_1}}$, $\bar{X}_{n_{u_2}}$ and $\{Z^{(2)}_{(U_2;\, t_{u_1}, t_{u_2})};\ t_{u_1}=1, \cdots, n_{u_1}-1,\ t_{u_2}=1, \cdots, n_{u_2}-1\}$, which implies also the unique determination of $Z^{(3)}_{(u_1, u_2)}$, and the analogue of them holds true for all sets $(u_i, u_j) \subset R-S$.

In general it holds that $\bar{X}_{n_{j_1} n_{j_2} \cdots n_{j_h}}$ $(J_h \subset R-S)$ is determined uniquely by determining $Z^{(1)}$, $\{\bar{X}_{n_{u_1} n_{u_2} \cdots n_{u_d}};\ U_d \subset J_h,\ d=1, 2, \cdots,\ h-1\}$ and $\{Z^{(2)}_{(J_h;\, t_{j_1}, \cdots, t_{j_h})};\ t_{j_c}=1, \cdots, n_{j_c}-1,\ c=1, 2, \cdots, h\}$, which implies also the unique determination of $Z^{(3)}_{(J_h)}$. Thus it can be seen that $\{Z^{(3)}_{(U_d)};\ U_d \subset R-S,\ d=1, 2, \cdots, r-s\}$ are determined uniquely by determining $Z^{(1)}$ and $\{Z^{(2)}_{(U_d;\, t_{u_1}, \cdots, t_{u_d})};\ U_d \subset R-S,\ d=1, 2, \cdots, r-s,\ t_{u_j}=1, \cdots, n_{u_j}-1,\ j=1, \cdots, d\}$.

Now, in our mixed model of Type II, the random variable X and the sample space R^X are the same as those given in Type I, and the family $\mathfrak{P}^X$ is specified by the parameter $\theta = (\mu, \alpha(j_1, \cdots, j_h;\ t_{j_1}, \cdots, t_{j_h}),\ \sigma^2_{I_k J_h},\ \sigma_0^2;\ I_k \subset S,\ J_h \subset R-S,\ t_{j_c}=1, \cdots, n_{j_c}-1,\ c=1, \cdots, h,\ h=1, \cdots, r-s, k=1, \cdots, s)$, whose space is of $\left(2^r - 2^{r-s} + \prod_{i=s+1}^{r} n_i + 1\right)$-dimension, where $-\infty < \mu$

$<\infty$, $-\infty<\alpha(j_1,\cdots,j_h;\ t_{j_1},\cdots,t_{j_h})<\infty$, $0<\sigma^2_{I_kJ_h}<\infty$, $0<\sigma_0^2<\infty$.

We shall consider the transformation of the original parameter such that

$$\tau^{(1)}=\prod_{j=0}^{r} n_j \frac{\mu}{G}, \tag{4.97}$$

$$\tau^{(2)}(U_d;\ t_{u_1},\cdots,\ t_{u_d})=\prod_{\zeta=0}^{r} n_\zeta^{1-\delta^{\zeta}_{U_d}} \frac{\alpha(u_1,\cdots,u_d;\ t_{u_1},\cdots,t_{u_d})}{G_{(U_d)}}, \quad \begin{pmatrix} U_d\subset R-S;\ t_{u_c}=1,\cdots,n_{u_c}-1, \\ c=1,\cdots,d;\ d=1,\cdots,r-s, \end{pmatrix}, \tag{4.98}$$

$$\tau^{(3)}_{(I_kJ_h)}=-\frac{1}{2C_{(I_k)}D_{(J_h)}}, \quad \begin{pmatrix} I_k\subset S;\ J_h\subset R-S, \\ k=1,\cdots,s;\ h=1,\cdots,r-s \end{pmatrix}, \tag{4.99}$$

$$\tau^{(4)}=-\frac{1}{2\sigma_0^2}. \tag{4.100}$$

After observing the independency of the class of parametric functions $\{C_{(I_k)}D_{(J_h)};\ I_k\subset S,\ J_h\subset R-S,\ k=1,\cdots,s,\ h=1,\cdots,r-s\}$, it is seen that the transformation (4.97), $\cdots$, (4.100) from θ to $\tau=(\tau^{(1)},\ \tau^{(2)}_{(U_d;t_{u_1},\cdots,t_{u_d})},\ \tau^{(3)}_{(I_kJ_h)},\ \tau^{(4)};\ I_k\subset S,\ J_h\subset R-S,\ U_d\subset R-S,\ t_{u_c}=1,\cdots,\ n_{u_c}-1,\ c=1,\cdots,\ d,\ d=1,\cdots,\ r-s,\ k=1,\cdots,\ s,\ h=1,\cdots,\ r-s)$ is one-to-one. There fore we can say that $\mathfrak{B}^X$ is specified by τ, where $-\infty<\tau^{(1)}<\infty$, $-\infty<\tau^{(2)}_{(U_d;t_{u_1},\cdots,t_{u_d})}<\infty$, $-\infty<\tau^{(4)}<\tau^{(3)}_{(M)}<\tau^{(3)}_{(N)}<0$ for any pair $(M,\ N)$ such that $M\supset N$, $M\subset R$, $N\subset R$. (We should notice that G and $G_{(U_d)}$ in (4.93) are the functions of $C_{(I_k)}D_{(J_h)}$: see Definition 4.1).

Then, under the new parameter τ, by using the above-mentioned result about $Z^{(3)}_{(U_d)}$ we obtain the probability density function of X as follows,

$$\begin{aligned} K\varphi_{(\tau)}exp\Big[&\tau^{(1)}Z^{(1)}+\sum_{d=1}^{r-s}\sum_{U_d\subset R-S}\sum_{t_{u_1}=1}^{n_{u_1}-1}\cdots\sum_{t_{u_d}=1}^{n_{u_d}-1}\tau^{(2)}_{(U_d;t_{u_1},\cdots,t_{u_d})}Z^{(2)}_{(U_d;\ t_{u_1},\cdots,\ t_{u_d})} \\ &+\sum_{k=1}^{s}\sum_{I_k\subset S}\sum_{h=0}^{r-s}\sum_{J_h\subset R-S}\tau^{(3)}_{(I_kJ_h)}S(I_kJ_h)+\tau^{(4)}S_0 \\ &+\sum_{i=1}^{2^{r-s}}g_i(\tau^{(3)}_{(M)})\,h_i(Z^{(1)},\ Z^{(2)}_{(N)})\Big]. \end{aligned} \tag{4.101}$$

Hence the sufficient statistic for $\mathfrak{B}^X$ is $U=(Z^{(1)},\ Z^{(2)}_{(U_d;\ t_{u_1},\cdots,\ t_{u_d})},\ S_{(I_hJ_k)},\ S_0;\ I_k\subset S,\ J_h\subset R-S,\ U_d\subset R-S,\ k=1,2,\cdots,\ s,\ h=1,2,\cdots,\ r-s,\ t_{u_c}=1,2,\cdots,\ n_{u_c}-1,\ c=1,2,\cdots,\ d,\ d=1,2,\cdots,\ r-s)$. As for completeness the result reviewed in Section 3 cannot be applied directly to the family of our probability density (4.102). In order to show $\mathfrak{B}^U$ in this type be complete, we need to generalize a result due to Gautschi [3], which is given by Lemma 4.8.

Let $U^{(k)}$ be a k-dimensional Euclidean space with the point $u^{(k)}=(u_1,\ u_2,\cdots,\ u_k)$. We shall write the first j components as $u^{(j)}$ and the remaining components $u^{((j))}$ so that we write $u^{(k)}$ in the following different fashion $u^{(k)}=(u^{(j)},\ u^{((j))})=(u^{(k-1)},\ u_k)$ etc. And let $U^{(j)}$ be a j-dimensional Euclidean space with the point $u^{(j)}=(u_1,\ u_2,\cdots,\ u_j)$, U_j be the j-th component space of $U^{(k)}$.

In the following lemma, $T^{(k)}$ denotes a k-dimensional Euclidean space with the point $\tau^{(k)}=(\tau_1, \tau_2, \cdots, \tau_k)$, and the notations such as $\tau^{(j)}$, T_j etc. are to be understood in the way above stated. Further, L with a subscript or a superscript denotes the Lebesgue measure, where the subscript j indicates the space U_j or T_j on which the measure is taken, and the superscript (j) indicates the space $U^{(j)}$ or $T^{(j)}$ on which the measure is taken.

LEMMA 4.8. *Let*

$$\mathfrak{B}^{u^{(k)}}=\{P_{\tau^{(k)}}^{u^{(k)}} \mid \tau^{(k)}\in\omega\},$$

where ω is a Borel set in an Euclidean space containing a non-degenerate k-dimensional interval, be the family of measures $P_{\tau^{(k)}}^{u^{(k)}}$ on the additive family of subsets in the space R^U of point U, having the density

$$p_{\tau^{(k)}}(u^{(k)})=C(\tau^{(k)})\, h(u^{(k)})\ exp\left[\sum_{i=1}^{k} \tau_i u_i+g(\tau^{(s)}, u^{((s))})\right] \tag{4.102}$$

with respect to Lebesgue measure. Then $\mathfrak{B}^{u^{(k)}}$ is strongly complete.

PROOF. We shall assume, without loss of generality, $p_{\tau^{(k)}}(u^{(k)})$ be defined for all $\tau^{(k)}$ and $u^{(k)}$ in the k-dimensional Euclidean spaces $T^{(k)}$ and $U^{(k)}$ respectively such that if it is not originally defined it is assumed to be equal to zero. Let $f(u^{(k-1)}, u_k)$ be $\mathfrak{B}$-measurable and integrable with respect to the Lebesgue measure.

Suppose that

$$I\equiv\int_{U^{(k-1)}\times U_k} f(u^{(k-1)}, u_k)\, p_{\tau^{(k)}}(u^{(k)})\, d(m^{(k-1)}\times m_k)=0 \tag{4.103}$$

$$(a.e.\ L^{(k-1)}\times L_k=L^{(k)}).$$

Let $N^{(1)}$ be the set of parameter points $(\tau^{(k-1)}, \tau_k)$ for which $I\neq 0$ and $N_{(k-1)}^{(1)}$ be the $\tau^{(k-1)}$-section of $N^{(1)}$. Then $L_k(N_{(k-1)})=0$ except for $\tau_{(k-1)}\in N_0^{(1)}$, where $L^{(k-1)}(N_0^{(1)})=0$.

Hence, by Fubini's theorem, we have

$$I=\int_{U_k} e^{\tau_k u_k}\left\{\int_{U^{(k-1)}} f(u^{(k-1)}, u_k)\, C(\tau^{(k)})\, h(u^{(k)}) \cdot exp\left[\sum_{i=1}^{k-1} \tau_i u_i+g(\tau^{(s)}, u^{((s))})\right] dm^{(k-1)}\right\} dm_k=0 \quad (a.e.\ L_k) \tag{4.104}$$

holds true for all $\tau^{(k-1)}\notin N_0^{(1)}$, where integral of inside exists except for $u_k\in M_k$ such that $L_k(M_k)=0$.

By the unicity of the Laplace transform, we obtain

$$I_1(\tau^{(k-1)}, u_k)=\int_{U^{(k-1)}} f(u^{(k-1)}, u_k)\, h(u^{(k)})\ exp\left[\sum_{i=1}^{k-1} \tau_i u_i+g(\tau^{(s)}, u^{((s))})\right] dm^{(k-1)} \tag{4.105}$$

$$=0 \qquad (a.e.\ L_k),\ \tau^{(k-1)}\notin N_0^{(1)}.$$

If $S^{(1)}$ denotes the set of points $(\tau^{(k-1)}, u_k)$ for which $I_1(\tau^{(k-1)}, u_1)$ is neither exists nor equal to zero, almost every $\tau^{(k-1)}$-section of $S^{(1)}$ has L_k-measure zero. Hence $L^{(k)}(S^{(1)})$

$=0$. As this implies that almost every u_k-section of $S^{(1)}$ has $L^{(k-1)}$-measure zero, it follows that

(4.106) $$I_1(\tau^{(k-1)}, u_k)=0 \qquad (a.e.\ L^{(k-1)}),\ u_k \not\in M_k.$$

(4.106) is written by

(4.107) $$\begin{aligned} &I_1(\tau^{(k-1)}, u_k) \\ &= \int_{U^{(k-2)}\times U_{k-1}} f(u^{(k-2)}, u_{k-1}, u_k) h(u^{(k)})\, exp\left[\sum_{i=1}^{k-1} \tau_i u_i + g(\tau^{(s)}, u^{((s))})\right] d(m^{(k-2)}\times m_{k-1}) \\ &= 0 \qquad (a.e.\ L^{(k-2)}\times L_{k-1}=L^{(k-1)}),\ u_k \not\in M_k. \end{aligned}$$

Let $N^{(2)}$ be the set of parameter points $(\tau^{(k-2)}, \tau_{k-1})$ for which $I_1(\tau^{(k-1)}, u_k) \neq 0$, and $N^{(2)}_{(k-2)}$ be the $\tau^{(k-2)}$-section of $N^{(2)}$. Then $L_{k-1}(N^{(2)}_{(k-2)})=0$ except for $\tau^{(k-2)}\in N^{(2)}_0$, where $L^{(k-2)}(N^{(2)}_0)=0$.

Hence, again by Fubini's theorem, we have

(4.108) $$\begin{aligned} &I_1(\tau^{(k-1)}, u_k) \\ &= \int_{U_{k-1}} e^{\tau_{k-1}u_{k-1}} \left\{ \int_{U^{(k-2)}} f(u^{(k-2)}, u_{k-1}, u_k)\, h(u^{(k)}) \right. \\ &\qquad \left. \cdot\ exp\left[\sum_{i=1}^{k-2} \tau_i u_i + g(\tau_{(s)}, u^{((s))})\right] dm^{(k-2)} \right\} dm_{k-1} \\ &= 0 \qquad (a.e.\ L_{k-1}),\ u_k \not\in M_k \end{aligned}$$

holds true for all $\tau^{(k-2)} \not\in N^{(2)}_0$, where integral of inside exists except for $u_{k-1}\in M_{k-1}$ such that $L_{k-1}(M_{k-1})=0$.

By the unicity of the Laplace transform, we obtain

(4.109) $$\begin{aligned} &I_2(\tau^{(k-2)}, u_{k-1}, u_k) \\ &= \int_{U^{(k-2)}} f(u^{(k-2)}, u_{k-1}, u_k)\, h\,(u^{(k)})\, exp\left[\sum_{i=1}^{k-2} \tau_i u_i + g(\tau^{(s)}, u^{((s))})\right] dm^{(k-2)} \\ &= 0 \qquad (a.e.\ L_{k-1}),\ \tau^{(k-2)} \not\in N^{(2)}_0,\ u_k \not\in M_k. \end{aligned}$$

If $S^{(2)}_{(u_k)}$ denotes the set of points $(\tau^{(k-2)}, u_{k-1})$ for which $I_2(\tau^{(k-2)}, u_{k-1}, u_k)$, corresponding to some fixed $u_k \not\in M_k$, neither exists nor is equal to zero, almost every $\tau^{(k-2)}$-section of $S^{(2)}_{(u_k)}$ has L_{k-1}-measure zero.

Hence we have

$$L^{(k-1)}(S^{(2)}_{(u_k)})=0, \qquad u_k \not\in M_k.$$

And we obtain

(4.110) $$I_2(\tau^{(k-2)}, u_{k-1}, u_k)=0 \qquad (a.e.\ L^{(k-2)}),\ u_{k-1}\not\in M_{k-1},\ u_k \not\in M_k.$$

In this way we can proceed inductively and obtain finally

(4.111) $$I_{k-s}(\tau^{(s)}, u^{((s))})$$

$$= e^{g(\tau^{(s)}, u^{((s))})} \int_{U^{(s)}} f(u^{(s)}, u^{((s))})\, h(u^{(s)}, u^{((s))})\ exp\left[\sum_{i=1}^{s} \tau_i u_i\right] dm^{(s)}$$

$$= 0 \qquad (a.e.\ L^{(s)}),\ u_{s+1} \notin M_{s+1},\ u_{s+2} \notin M_{s+2}, \cdots, u_k \notin M_k,$$

where $L^j(M_j) = 0 \quad (j = s+1, \cdots, k).$

On the other hand, as was mentioned in Section 3, the family of measures whose density is given in the form $h'(u^{(s)})\ exp\left[\sum_{i=1}^{s} \tau_i u_i\right]$ is strongly complete. Therefore (4.111) implies that

(4.112) $$f(u^{(s)}, u^{((s))}) = 0 \qquad (a.e. \mathfrak{B}^{u^{(s)}}),\ u_j \notin M_j\ (j = s+1, \cdots, k),$$

which again implies

(4.113) $$f(u^{(s+1)}, u^{((s+1))}) = 0 \qquad (a.e.\ \mathfrak{B}^{u^{(s+1)}}),\ u_j \notin M_j\ (j = s+2, \cdots, k).$$

In this way we finally obtain

(4.114) $$f(u^{(k)}) = 0 \qquad (a.e.\ \mathfrak{B}^{u^{(k)}}),$$

which completes the proof.

Now, as the probability density function of the sufficient statistic U for $\mathfrak{B}^X$ in our case is written in the form (4.102) in Lemma 4.8, also in the case of Type II we have the same conclusion about the estimation problem for the unknown parameters as we obtained in Theorem 3.4 in Section 3.

5. Remark on the random effect model.

At first it should be remarked here that the result of the previous paper of the author concerning the point estimation of the variance components in random effect model can be improved to the level of the generality of this paper by applying the Lemma 4.8 of this paper. The author would like to mention the work of L. H. Herbach [5] concerning the problem of testing in the same model, which seems to be restricted to the case of one and two way layout only and about the testing of variance components belonging to a certain group can be generalized to the case of multi-way layout by making use of the joint density function derived in the previous paper of the author, while there exists no exact F-test of them belonging to another group.

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