A NOTE ON REPRESENTATIONS OF ALGEBRAS AS SUBALGEBRAS OF $C(X)$ FOR X COMPACT

Yukio KŌMURA and Isamu NAKAHARA

(Received May 15, 1962)

In a recent paper Frank W. Anderson and Robert L. Blair gave some characterizations of $C(X)$ for X an arbitrary completely regular space [1][1].

In the paper, the theorem 4.1 states that *if A is a regular algebra, the condition $\mathfrak{M}_A=\mathfrak{R}_A$ is sufficient that A is isomorphic to a regular point-determining subalgebra of $C(X)$ for some topologically unique compact space X.*

But the converse is left open. We shall give in this note an example answering the question negatively.

§ 1. Some preliminary notions and notations.

We adopt the same notions and notations as in [1], but recite here some of them.

Let A be a subset of $C(X)$, A *is regular* in case (i) A contains the identity e of $C(X)$ and (ii) whenever $x\in X$ and U is an open neighborhood of x, there is an $f\in A$ such that $f(x)=0$ and $f(y)=1$ for all $y\in U$.

If A is any ring, then an ideal I in A is said to be *real* in case A/I is isomorphic to R real.

We shall denote by $\mathfrak{M}_A$, $\mathfrak{R}_A$ the set of all maximal ideals of A, and the set of all real maximal ideals of A respectively.

If X is a topological space and A is a subring of $C(X)$ then we shall set

$$M_x=\{f\in A;\ f(x)=0\}.$$

We say that A is point-determining in case $M\in\mathfrak{M}_A$ if and only if $M=M_x$ for some unique $x\in X$.

§ 2. A counter example to the converse of the theorem 4.1.

We define a compact set X and algebras A_0, A_1, A as follows.

X : $[0,1]$

A_0: the algebra of all functions $f\in C[0,1]$ such that for finite partition $0=a_0<a_1<\cdots<a_n=1$, $f(x)=p_i(x)/q_i(x)$ for $x\in[a_i, a_{i+1}]$, where p_i and q_i are polynomials. (The partition depends on f, and $q_i(x)\neq 0$ for $x\in[a_i, a_{i+1}]$, $p_i(a_i)/q_i(a_i)=p_{i+1}(a_{i+1})/q_{i+1}(a_{i+1})$.)

(1)[1] Frank W. Anderson and Robert L. Blair: Characterizations of the algebra of all real-valued continuous functions on a completely regular space. Illinois Jour. of Math. vol. 3 (1959) pp. 121–133.

A_1: the algebra generated by A_0 and e^x.

A : the algebra of all functions $f(x)$ of the form $g(x)/P(e^x)$, where g is an element of A_1 and $P(e^x)$ is a function of the form $\prod_{k=1}^{n}(e^x-\alpha_k)$, $\alpha_k \in A_0$, $e^x-\alpha_k \neq 0$ on $[0,1]$.

Under these preparations we have following propositions.

(1) *A_0 is a regular subset of $C(X)$.*

Proof. For any $X_1 \in X$ and any $\varepsilon>0$, the function defined by

$$f(x)=\begin{cases} 1 & \text{for } |x-x_1|\geqq\varepsilon, \\ |x-x_1|/\varepsilon & \text{for } |x-x_1|\leqq\varepsilon, \end{cases}$$

is evidently contained in A_0.

(2) *A is a regular subset of $C(X)$.*

This is evident from (1).

(3) *Let I be a real ideal of A. Then $I_1=A_1\cap I$ and $I_0=A_0\cap I$ are real ideals.*

Proof. $A_0/I_0\subset A_1/I_1\subset A/I$. On the other hand, $1\in A_0$ and $\notin I$. Since A_0 is an algebra, $R\subset A_0/I_0$. Hence $R\subset A_0/I_0\subset A_1/I_1\subset A/I=R$, which implies that $R=A_0/I_0=A_1/I_1$.

(4) *For the above ideal I_0, there exists a point $x_0\in X$ such that $I_0=\{f\in A;\ f(x_0)=0\}$.*

Proof. Let $g\in A_0$. Then g has no zero point in X if and only if $1/g\in A_0$. Hence if $g\in I_0$, then g has at least one zero point. If $g, h\in I_0$ such that g and h have no common zero point, then $g^2+h^2\in I_0$ which has no zero point. This is a contradiction. Since X is a compact set, the ideal I_0 has a common zero point. Since I_0 is a real ideal, the common zero point are at most one.

(5) *The ideal I_1 contains $e^x-\alpha$ for some α, where $\log\alpha\in X$.*

Proof. $a_nu^n+a_{n-1}u^{n-1}+\cdots+a_0+b\cdot Q(u)\in I_1$, where $u=e^x$, $a_i\in A_0$ and $a_i\notin I_0$, $b\in I_0$, $Q(u)\in A_1$.

Since $I_1\supsetneqq A_1I_0$, we may assume $n\geqq 1$.
$a_nu^n+a_{n-1}u^{n-1}+\cdots+a_0+b\cdot Q(u)\equiv a_n(x_0)u^n+a_{n-1}(x_0)u^{n-1}+\cdots+a_0(x_0)$ (mod. A_1I_0). Hence $a_n(x_0)u^n+a_{n-1}(x_0)u^{n-1}+\cdots+a_0(x_0)\in I_1$, $a_n(x_0)\neq 0$.

If $F(X)=a_n(x_0)X^n+a_{n-1}(x_0)X^{n-1}+\cdots+a_0(x_0)$ is irreducible, and if $n>1$, then a root θ of $F(X)=0$, which is not real, is contained in A_1/I_1. This is a contradiction. Hence $n=1$, that is, $a_1(x_0)u+a_0(x_0)\in I_1$. Since $a_1(x_0)\neq 0$, $I\in u+a_0(x_0)/a_1(x_0)=u-\alpha$, where $\alpha=-a_0(x_0)/a_1(x_0)$.
If $\log\alpha\notin X$, then $u-\alpha$ is a unit of A, so $u-\alpha\notin I$. Therefore $u-\alpha\in I_1\subset I$ implies $\log\alpha\in X$.

(6) $\log\alpha=x_0$. *That is, $I_1=\{f\in A_1;\ f(x_0)=0\}$.*

Proof. Suppose $\log\alpha\neq x_0$, for example, $x_0>\log\alpha$. Then there exists $g\in I_0$ such that $g\geqq 0$, and $g(x)>\alpha$ for $x\leqq\log\alpha$. $I_1\ni e^x-\alpha+g>0$. Hence a unit $e^x-\alpha+g$ of A is contained in I, which is a contradiction.

(7) *The algebra* $\boldsymbol{A}$ *is point-determining.*

Proof. For a real ideal $\boldsymbol{I}$, $\boldsymbol{I}_1=\boldsymbol{I}\cap\boldsymbol{A}_1$ determines a point $x_0\in\boldsymbol{X}$, Hence $\boldsymbol{I}=\boldsymbol{A}\boldsymbol{I}_1$ determines the point x_0.

(8) *The maximal ideal of* $\boldsymbol{A}$ *containing* $e^{2x}+1$ *is not real, that is* $\mathfrak{M}_A\neq\mathfrak{R}_A$.

Therefore we have the conclusion:

If $\boldsymbol{A}$ *is a regular algebra,* $\mathfrak{M}_A=\mathfrak{R}_A$ *is the sufficient but not the necessary condition that* $\boldsymbol{A}$ *is isomorphic to a regular point-determining subalgebra of* $C(\boldsymbol{X})$ *for some topologically unique compact space* $\boldsymbol{X}$.

Quite similarlly, the converse of the theorem 4.5 in [1] does not hold.

Department of Mathematics
Faculty of Science,
Kumamoto University