NOTE ON MALCEV ALGEBRAS

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A Malcev algebra is an anti-commutative algebra defined by the Malcev condition $(xy)(zx)+(xy\cdot z)x+(yz\cdot x)x+(zx\cdot x)y=0$. In an anti-commutative algebra A, the Jacobi condition $(xy)z+(yz)x+(zx)y=0$ says that a mapping $x\to ax$ is a derivation in A. In this note we show that this situation holds for the Malcev condition with a suitable modification. We next show that any Malcev algebra can be made into a certain subspace of a Lie algebra satisfying some conditions. For this purpose it is important to consider a trilinear composition $[xyz]=x(yz)-y(xz)+(xy)z$ with the original composition xy.

1. Axioms of Malcev algebras. An anti-commutative algebra A over a field Φ is a non-associative algebra satisfying

(1.1) $$x^2=0 \qquad \text{for all } x\in A.$$

A *Malcev algebra* M or a *Moufang-Lie algebra* [2][1] is an anti-commutative algebra satisfying

(1.2) $$(xy)(zx)+(xy\cdot z)x+(yz\cdot x)x+(zx\cdot x)y=0 \qquad \text{for all } x,y,z\in M.$$

Any Lie algebra is a Malcev algebra. Let C be a Cayley-Dickson algebra with a multiplication xy. An algebra derived from C with a new multiplication $[x,y]=xy-yx$ is a Malcev algebra but not a Lie algebra.

Let M_1 be an anti-commutative algebra defined by the identity:

(1.3) $$[x,y,zw]=[xyz]w+z[xyw],$$

where

(1.4) $$[xyz]=x(yz)-y(xz)+(xy)z.$$

(1.3) shows that a linear mapping $z\to\sum_i[x_iy_iz]$ is a derivation in M_1. Then we have the following

THEOREM 1.1. *In an anti-commutative algebra over a field Φ of characteristic not 2 or 3, the Malcev condition* (1.2) *is equivalent to* (1.3).

That a Malcev algebra over a field Φ of characteristic different from 2 is an algebra M_1 is proved by Sagle [3, Prop. 8.3]. Hence we prove the converse under

1) Numbers in brackets refer to the references at the end of the paper.

the assumption that the characteristic of Φ is not 2 or 3. In an anti-commutative algebra, put

$$g(x,y,z,w)=[x,y,zw]+[z,w,xy]. \tag{1.5}$$

Clearly $g(x,x,y,z)=g(x,y,z,z)=0$. Also, put

$$J(x,y,z)=(xy)z+(yz)x+(zx)y, \tag{1.6}$$

then J is a skew-symmetric function of its arguments.

LEMMA 1.1. *An algebra M_1 of characteristic not 2 is a Malcev algebra if and only if the function g is skew-symmetric with respect to its variables.*

PROOF. Let M_1 be a Malcev algebra. The identity $g(x,y,z,w)+J(x,y,zw)+J(xy,z,w)=0$ and the result of Kleinfeld [1] imply that g is skew-symmetric with respect to its arguments. Conversely, assume that g is a skew-symmetric function in an algebra M_1. In an anti-commutative algebra it holds the identity: $[x,y,zw]-[xyz]w-z[xyw]=(xy)(zw)+x(z\cdot wy)+w(x\cdot yz)+y(w\cdot zx)+z(y\cdot xw)+g(x,y,z,w)+g(y,w,x,z)$. Hence we have $(xy)(zw)+x(z\cdot wy)+w(x\cdot yz)+y(w\cdot zx)+z(y\cdot xw)=0$[2], which implies (1.2) by putting $w=x$. q. e. d.

Next, by using the identity

$$J(x,y,z)+[xyz]=2(xy)z \tag{1.7}$$

in an anti-commutative algebra and that $z\to[xyz]$ is a derivation in an algebra M_1, we have $J(x,y,zw)=2(xy)(zw)-[xyz]w-z[xyw]$ in M_1, hence by (1.7) we obtain

$$J(x,y,zw)+2J(xy,z,w)=J(x,y,z)w+zJ(x,y,w).$$

In this relation, if we interchange x with z and y with w respectively, we have

$$J(z,w,xy)+2J(zw,x,y)=J(z,w,x)y+xJ(z,w,y).$$

Adding these two relations and using that J is a skew-symmetric function, we have

$$3g(x,y,z,w)+J(x,y,z)w+zJ(x,y,w)+J(z,w,x)y+xJ(z,w,y)=0.$$

Since the characteristic of Φ is not 3, we obtain $g(x,y,y,z)=0$ and g is skew-symmetric with respect to its variables. Hence Theorem 1.1 is proved from Lemma 1.1.

2. Relation between Malcev algebras and Lie algebras. In this section we prove the following

THEOREM 2.1. *For a Malcev algebra M over a field Φ of characteristic not 2, there exists a Lie algebra $\mathfrak{L}$ such that $\mathfrak{L}=M\oplus\mathfrak{D}$ (a vector space direct sum) and $[\mathfrak{D},\mathfrak{D}]\subseteqq\mathfrak{D}$, $[M,\mathfrak{D}]\subseteqq M$ and where the product xy in M is an M-component of a product $[x,y]$ in $\mathfrak{L}$.*

2) We remark that this relation is equivalent to (1.2) by [3, Prop. 2.21].

PROOF. We have the following identities in M.

$$[xxy]=0, \tag{2.1}$$

$$[xyz]+[yzx]+[zxy]+(xy)z+(yz)x+(zx)y=0, \tag{2.2}$$

and

$$[xy,z,w]+[yz,x,w]+[zx,y,w]=0. \tag{2.3}$$

In fact, put $F(x,y,z)=(xy)(zx)+(xy\cdot z)x+(yz\cdot x)x+(zx\cdot x)y$, then since the characteristic of Φ is not 2, $F(x+w,y,z)+F(y+w,z,x)+F(z+w,x,y)=0$ implies (2.3). Applying the derivation $D(x,y)$: $z\rightarrow[xyz]$ for the ternary product $[zvw]$ we have

$$[xy[zvw]]=[[xyz]vw]+[z[xyv]w]+[zv[xyw]], \tag{2.4}$$

from this

$$[D(x,y),\ D(z,w)]=D([xyz],w)+D(z,[xyw]). \tag{2.5}$$

Let $\mathfrak{D}(M)$ be a vector space over Φ spanned by $\sum_i D(x_i,y_i)$'s, then $\mathfrak{D}(M)$ is a Lie algebra by (2.5). Let $\mathfrak{L}$ be a vector space direct sum $M\oplus\mathfrak{D}(M)$, then the element of $\mathfrak{L}$ is of the form $x+\sum_i D(y_i,z_i)$. A multiplication in $\mathfrak{L}$ is defined as follows:

$$\begin{aligned}&[x+\sum_i D(y_i,z_i),\ u+\sum_i D(v_i,w_i)]\\ &\quad=xu+\sum_i[y_iz_iu]-\sum_i[v_iw_ix]+D(x,u)\\ &\qquad+\sum_i D([y_iz_iv_i],w_i)+\sum_i D(v_i,[y_iz_iw_i]),\end{aligned}$$

then from (1.1), (1.3), (2.1),$\cdots$, (2.4) $\mathfrak{L}$ is a Lie algebra such that $[M,\mathfrak{D}(M)]\subseteq M$ and xy is an M-component of $[x,y]$ for $x,y\in M$ [6]. Therefore the theorem is proved.

COROLLARY 2.1. *That a Malcev algebra M reduces to a Lie algebra relative to the original composition xy is equivalent to that M is a Lie triple system*[3] *relative to the ternary composition* $[xyz]=x(yz)-y(xz)+(xy)z$.

COROLLARY 2.2. *If a Malcev algebra M satisfies $[MMM]=0$, then M is a Lie algebra.*

REMARK. If M is finite-dimensional, then the Lie algebra $\mathfrak{L}=M\oplus\mathfrak{D}(M)$ is also finite-dimensional, since $\dim\mathfrak{L}\leqq n(n+1)/2$ where $n=\dim M$. Suppose that a vector space T has a binary composition xy and a ternary composition $[xyz]$ satisfying (1.1), (1.3), (2.1),$\cdots$, (2.4), then T is called a *general Lie triple system* [6]. Therefore a Malcev algebra has a structure of general Lie triple system relative to the com-

3) A Lie triple system is a vector space with a trilinear composition $[xyz]$ satisfying $[xxy]=0$, $[xyz]+[yzx]+[zxy]=0$ and $[xy[zvw]]=[[xyz]vw]+[z[xyv]w]+[zv[xyw]]$. See [5].

positions xy and $[xyz]=x(yz)-y(xz)+(xy)z$.

Let $\mathfrak{L}$ be a Lie algebra over $\boldsymbol{R}$ with a basis $X_1=\frac{\partial}{\partial x}$, $X_2=x\frac{\partial}{\partial x}$, $X_3=x\frac{\partial}{\partial y}$, $X_4=y\frac{\partial}{\partial x}$, $X_5=\frac{\partial}{\partial y}$, $X_6=y\frac{\partial}{\partial y}$ and let M and $\mathfrak{D}$ be the subspaces of $\mathfrak{L}$ with bases X_1, X_2, X_3, X_4 and X_5, X_6 respectively. Then $\mathfrak{L}$ is the vector space direct sum $M\oplus\mathfrak{D}$ and $[\mathfrak{D},\mathfrak{D}]\subseteqq\mathfrak{D}$, $[M,\mathfrak{D}]\subseteqq M$. For X_1, X_2, X_3, X_4 define

$$X_iX_j=[X_i, X_j]_M,$$
$$[X_iX_jX_k]=[[X_i, X_j]_{\mathfrak{D}}, X_k],$$

where $[X_i, X_j]_M$ and $[X_i, X_j]_{\mathfrak{D}}$ denote the M-component of $[X_i, X_j]$ and the $\mathfrak{D}$-component of $[X_i, X_j]$ respectively. Then, M is a general Lie triple system relative to XY and $[XYZ]$ but not a Malcev algebra relative to XY, in fact for $X=X_1+X_2$, $Y=X_3$, $Z=X_4$ $(XY)(ZX)+(XY\cdot Z)X+(YZ\cdot X)X+(ZX\cdot X)Y=-2X_1\neq 0$.

PROPOSITION 2.1. *In a general Lie triple system T with compositions xy and $[xyz]$, the Malcev condition* (1.2) *is equivalent to the following condition:*

$$[x, y, zx]+[y, zx, x]+[zx, x, y]+[xyz]x+[yzx]x+[zxy]x=0.$$

PROOF. From (2.2) we have the identity: $(xy)(zx)+(xy\cdot z)x+(yz\cdot x)x+(zx\cdot x)y+[x, y, zx]+[y, zx, x]+[zx, x, y]+[xyz]x+[yzx]x+[zxy]x=0$, which proves the proposition.

Let N be a subalgebra of a Malcev algebra M and let $\mathfrak{D}(N, N)$ be a Lie algebra generated by $\sum_i D(x_i, y_i)$'s, $x_i, y_i\in N$. From Theorem 2.1 the vector space direct sum $N\oplus\mathfrak{D}(N, N)$ becomes a Lie algebra. A subalgebra $\mathfrak{A}$ of a Lie algebra $\mathfrak{L}$ is called to be subinvariant in $\mathfrak{L}$ if there exists a finite sequence of subalgebras $\mathfrak{L}=\mathfrak{A}_0, \mathfrak{A}_1, \cdots, \mathfrak{A}_r=\mathfrak{A}$ such that $\mathfrak{A}_i$ is an ideal in $\mathfrak{A}_{i-1}, i=1, 2, \cdots, r$ [4]. Let N be an ideal of a Malcev algebra M, then the next proposition shows that the Lie algebra $N\oplus\mathfrak{D}(N, N)$ is subinvariant in the Lie algebra $M\oplus\mathfrak{D}(M, M)$.

PROPOSITION 2.2. *Let N be an ideal of a Malcev algebra M. Then, the Lie algebra $N\oplus\mathfrak{D}(N, N)$ is an ideal of the Lie algebra $N\oplus\mathfrak{D}(N, M)$ and $N\oplus\mathfrak{D}(N, M)$ is an ideal of the Lie algebra $M\oplus\mathfrak{D}(M, M)$.*

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