ON THE COHOMOLOGY SPACE OF LIE TRIPLE SYSTEM

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It is well known that a Lie group can be characterized locally by a Lie algebra. More generally, the algebraic system which characterizes locally a totally geodesic subspace in a group space or a symmetric space is a Lie triple system [1, 9, 10][1]. A Lie algebra and a special Jordan algebra are the typical examples which may be a Lie triple system and the systematic study of this system was done by N. Jacobson [6] and W. G. Lister [7]. In this paper, we give a method defining a cohomology space of a Lie triple system and a relation between a cohomology space of order 3 and an extension of a Lie triple system. Next, we prove for a non-degenerate Lie triple system an analogue of the Casimir theorem. We see that an identity (3), which is called the Ricci formula in the differential geometry, plays a fundamental role in this study.

Recently, Professor B. Harris studied on the cohomology of Lie triple system independently of us [3], his detailed results will appear in [4]. The author wishes to express his sincere thanks for his kind informations.

1. We begin with the definition of the Lie triple system.

DEFINITION 1. A *Lie triple system* (L. t. s.) is a vector space $\mathfrak{T}$ over a field Φ[2], which is closed with respect to a trilinear multiplication $[abc]$ and satisfying

$$(1)\qquad [aab]=0,$$

$$(2)\qquad [abc]+[bca]+[cab]=0,$$

$$(3)\qquad [[abc]de]+[[bad]ce]+[ba[cde]]+[cd[abe]]=0.$$

PROPOSITION 1.[3] In L. t. s. it holds the following identities:

$$(4)\qquad [[abc]de]+[[bad]ce]+[[cda]be]+[[dcb]ae]=0,$$

$$(5)\qquad \begin{aligned}&[[[abc]de]fg]+[[[bac]df]eg]+[[[abd]cf]eg]+[[[bad]ce]fg]\\&+[[[cde]fa]bg]+[[[dce]fb]ag]+[[[cdf]eb]ag]+[[[dcf]ea]bg]\\&+[[[efa]bc]dg]+[[[fea]bd]cg]+[[[efb]ad]cg]+[[[feb]ac]dg]=0.\end{aligned}$$

PROOF. Interchanging pairs (a,b) and (c,d) in (3), we have

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1) Numbers in brackets refer to the references at the end of the paper.

2) Throughout this paper we shall assume that the characteristic of the base field Φ is 0 and L. t. s. has a finite dimension. See [6, 7, 10] as to the terminologies for L. t. s. in this paper.

3) These identities were first stated by N. Jacobson [5] and W. G. Lister first pointed out that (1), (2), (3) imply (4), (5), but he did not publish. This is derived also from [5, § 3] and [10, Theorem 2. 1].

$$(3)' \quad [[cda]be]+[[dcb]ae]+[dc[abe]]+[ab[cde]]=0.$$

The addition of (3) and (3)′ implies (4). For a proof of (5) we use twice (3).

$$\begin{aligned}
&\mathfrak{S}\{[[[abc]de]fg]+[[[bac]df]eg]+[[[abd]cf]eg]+[[[bad]ce]fg]\}\\
&=\mathfrak{S}\{[[abc]d[efg]]-[ef[[abc]dg]]+[[abd]c[feg]]-[fe[[abd]cg]]\}\\
&=\mathfrak{S}\{[ab[cd[efg]]]-[cd[ab[efg]]]-[ef[ab[cdg]]]+[ef[cd[abg]]]\}\\
&=0,
\end{aligned}$$

where $\mathfrak{S}$ denotes the summation obtained by cyclic permutations of the pairs (a,b), (c,d), (e,f).

DEFINITION 2. Let $\mathfrak{T}$ be a L. t. s. and let V be a vector space over Φ. Suppose that there exists a bilinear mapping θ: $(a,b)\rightarrow\theta(a,b)$ of $\mathfrak{T}\times\mathfrak{T}$ into an associative algebra of linear transformations of V. Then, V is called a *$\mathfrak{T}$-module* if θ satisfies the following conditions:

$$(6) \qquad \theta(c,d)\theta(a,b)-\theta(b,d)\theta(a,c)-\theta(a,[bcd])+D(b,c)\theta(a,d)=0,$$

$$(7) \qquad \theta(c,d)D(a,b)-D(a,b)\theta(c,d)+\theta([abc],d)+\theta(c,[abd])=0,$$

where

$$(8) \qquad D(a,b)=\theta(b,a)-\theta(a,b).$$

From (7) we obtain

$$(9) \qquad D(c,d)D(a,b)-D(a,b)D(c,d)+D([abc],d)+D(c,[abd])=0,$$

hence the vector space spanned by $\sum_i D(a_i,b_i)$, $a_i, b_i\in\mathfrak{T}$ is a subalgebra of $\mathfrak{Gl}(V)$.

In a L. t. s. $\mathfrak{T}$, let $\theta(a,b)$ be a linear mapping $x\rightarrow[xab]$ of $\mathfrak{T}$ into itself, a,b being in $\mathfrak{T}$, then we can prove that $\mathfrak{T}$ is a $\mathfrak{T}$-module by using (3), and in this case $D(a,b)$ becomes a linear mapping $x\rightarrow[abx]$ by (2) (inner derivation). An ideal of L. t. s. $\mathfrak{T}$ is an invariant subspace of the mappings $\theta(a,b)$ for all a,b in $\mathfrak{T}$.

REMARK. Let $a\rightarrow R_a$ be a linear mapping of L. t. s. $\mathfrak{T}$ into an associative algebra of linear transformations of a vector space V and satisfies $R_{[abb]}=[[R_aR_b]R_b]$ for all a,b in $\mathfrak{T}$, where $[R_aR_b]\equiv R_aR_b-R_bR_a$. Then, from $R_{[a\,b+c\,b+c]}=[[R_a\,R_b+R_c]\,R_b+R_c]$, it follows $R_{[abc]}+R_{[acb]}=[[R_aR_b]R_c]+[[R_aR_c]R_b]$. Hence $R_{[bac]}+R_{[bca]}=[[R_bR_a]R_c]+[[R_bR_c]R_a]$. By using (2) and the Jacobi identity, from the last two relations we have $R_{[abc]}=[[R_aR_b]R_c]$. If we put $\theta(a,b)=R_bR_a$, then $D(a,b)=[R_aR_b]$. Since these operators satisfy (6) and (7), it follows that V is a $\mathfrak{T}$-module.

W. G. Lister [7] defined a representation of L. t. s. in a natural sense as a L. t. s. homomorphism of a L. t. s. into a L. t. s. of linear transformations of a vector space. Therefore, the mapping θ may be considered as a representation of L. t. s. in a general sense.

Let V be a $\mathfrak{T}$-module defined by a bilinear mapping θ and let f be an n-linear mapping of $\underbrace{\mathfrak{T}\times\cdots\times\mathfrak{T}}_{n\ times}$ into V satisfying

$$f(x_1, x_2, \cdots, x_{n-3}, x, x, x_n)=0$$

and

$$f(x_1, x_2, \cdots, x_{n-3}, x, y, z)+f(x_1, x_2, \cdots, x_{n-3}, y, z, x)+f(x_1, x_2, \cdots, x_{n-3}, z, x, y)=0.$$

We denote the vector space spanned by such n-linear mappings by $C^n(\mathfrak{T}, V)$, $(n=0, 1, 2, \cdots)$, where we define $C^0(\mathfrak{T}, V)=V$.

Next, we define a linear mapping δ of $C^n(\mathfrak{T}, V)$ into $C^{n+2}(\mathfrak{T}, V)$ by the following formulas:

$$\delta f(x_1, x_2)=\theta(x_1, x_2)f \qquad \text{for } f\in C^0(\mathfrak{T}, V), \tag{10}$$

$$\begin{aligned}
&\delta f(x_1, x_2, \cdots, x_{2n+1})\\
&=\theta(x_{2n}, x_{2n+1})f(x_1, x_2, \cdots, x_{2n-1})-\theta(x_{2n-1}, x_{2n+1})f(x_1, x_2, \cdots, x_{2n-2}, x_{2n})\\
&\quad+\sum_{k=1}^{n}(-1)^{n+k}D(x_{2k-1}, x_{2k})f(x_1, x_2, \cdots, \hat{x}_{2k-1}, \hat{x}_{2k}, \cdots, x_{2n+1})\\
&\quad+\sum_{k=1}^{n}\sum_{j=2k+1}^{2n+1}(-1)^{n+k+1}f(x_1, x_2, \cdots, \hat{x}_{2k-1}, \hat{x}_{2k}, \cdots, [x_{2k-1}x_{2k}x_j], \cdots, x_{2n+1})\\
&\qquad\qquad\text{for } f\in C^{2n-1}(\mathfrak{T}, V), n=1, 2, 3, \cdots,
\end{aligned} \tag{11}$$

$$\begin{aligned}
&\delta f(y, x_1, x_2, \cdots, x_{2n+1})\\
&=\theta(x_{2n}, x_{2n+1})f(y, x_1, x_2, \cdots, x_{2n-1})-\theta(x_{2n-1}, x_{2n+1})f(y, x_1, x_2, \cdots, x_{2n-2}, x_{2n})\\
&\quad+\sum_{k=1}^{n}(-1)^{n+k}D(x_{2k-1}, x_{2k})f(y, x_1, x_2, \cdots, \hat{x}_{2k-1}, \hat{x}_{2k}, \cdots, x_{2n+1})\\
&\quad+\sum_{k=1}^{n}\sum_{j=2k+1}^{2n+1}(-1)^{n+k+1}f(y, x_1, x_2, \cdots, \hat{x}_{2k-1}, \hat{x}_{2k}, \cdots, [x_{2k-1}x_{2k}x_j], \cdots, x_{2n+1})\\
&\qquad\qquad\text{for } f\in C^{2n}(\mathfrak{T}, V), n=1, 2, 3, \cdots,
\end{aligned} \tag{12}$$

where the sign ^ over a letter indicates that this letter is to be omitted. The operator δ is as follows for lower orders:

If $n=1$, $\delta f(x_1, x_2, x_3)=\theta(x_2, x_3)f(x_1)-\theta(x_1, x_3)f(x_2)+D(x_1, x_2)f(x_3)-f([x_1x_2x_3])$,
$n=2$, $\delta f(x_1, x_2, x_3, x_4)=\theta(x_3, x_4)f(x_1, x_2)-\theta(x_2, x_4)f(x_1, x_3)+D(x_2, x_3)f(x_1, x_4)$
$\qquad -f(x_1, [x_2x_3x_4])$.

Now, if $f\in C^0(\mathfrak{T}, V)$, then

$$\begin{aligned}
&\delta\delta f(x_1, x_2, x_3, x_4)\\
&=\theta(x_3, x_4)\delta f(x_1, x_2)-\theta(x_2, x_4)\delta f(x_1, x_3)+D(x_2, x_3)\delta f(x_1, x_4)-\delta f(x_1, [x_2x_3x_4])\\
&=(\theta(x_3, x_4)\theta(x_1, x_2)-\theta(x_2, x_4)\theta(x_1, x_3)+D(x_2, x_3)\theta(x_1, x_4)-\theta(x_1, [x_2x_3x_4]))f\\
&=0
\end{aligned}$$

by (6). Similarly, $\delta\delta f=0$ for $f\in C^1(\mathfrak{T}, V)$ by (3), (6), (7), (9).

For $a, b\in\mathfrak{T}$ we define a linear mapping $\kappa(a, b)$ of $C^{2n-1}(\mathfrak{T}, V)$ into $C^{2n-1}(\mathfrak{T}, V)$ and a linear mapping $\iota(a, b)$ of $C^{2n-1}(\mathfrak{T}, V)$ into $C^{2n-3}(\mathfrak{T}, V)$ as follows:

$$(13)\qquad (\kappa(a, b)f)(x_1, \cdots, x_{2n-1}) \\ =(-1)^{n+1}\left(D(a, b)f(x_1, \cdots, x_{2n-1})-\sum_{j=1}^{2n-1}f(x_1, \cdots, [abx_j], \cdots, x_{2n-1})\right),$$

$$(14)\qquad (\iota(a, b)f)(x_1, \cdots, x_{2n-3})=f(a, b, x_1, \cdots, x_{2n-3}),$$

$$n=2, 3, \cdots$$

Then we have the following relations.

LEMMA 1. *For* $a, b, c, d\in\mathfrak{T}$ *and* $f\in C^{2n-1}(\mathfrak{T}, V)$ $(n=2, 3, \cdots)$

(i) $(\iota(a, b)\delta-\delta\iota(a, b))f=\kappa(a, b)f,$

(ii) $(\kappa(a, b)\iota(c, d)+\iota(c, d)\kappa(a, b))f=(-1)^n(\iota([abc], d)+\iota(c, [abd]))f,$

(iii) $(\kappa(a, b)\kappa(c, d)-\kappa(c, d)\kappa(a, b))f=(-1)^{n+1}(\kappa([abc], d)+\kappa(c, [abd]))f,$

(iv) $(\delta\kappa(a, b)+\kappa(a, b)\delta)f=0.$

PROOF. Since it is easy to prove (i) and (ii), we shall prove (iii) and (iv).

(iii): If $f\in C^3(\mathfrak{T}, V)$, then it follows easily (iii). Hence, we assume (iii) holds for $f\in C^{2n-3}(\mathfrak{T}, V)$. Then for $f\in C^{2n-1}(\mathfrak{T}, V)$ and arbitrary $k, l\in\mathfrak{T}$

$$\begin{aligned}
&\iota(k, l)(\kappa(a, b)\kappa(c, d)-\kappa(c, d)\kappa(a, b)+(-1)^n\kappa([abc], d)+(-1)^n\kappa(c, [abd]))f\\
&=-\kappa(a, b)\iota(k, l)\kappa(c, d)f+(-1)^n\iota([abk], l)\kappa(c, d)f+(-1)^n\iota(k, [abl])\kappa(c, d)f\\
&\quad+\kappa(c, d)\iota(k, l)\kappa(a, b)f-(-1)^n\iota([cdk], l)\kappa(a, b)f-(-1)^n\iota(k, [cdl])\kappa(a, b)f\\
&\quad+(-1)^n\iota(k, l)\kappa([abc], d)f+(-1)^n\iota(k, l)\kappa(c, [abd])f\\
&=(\kappa(a, b)\kappa(c, d)-\kappa(c, d)\kappa(a, b)-(-1)^n\kappa([abc], d)-(-1)^n\kappa(c, [abd]))\iota(k, l)f\\
&\quad+(\iota([cd[abk]], l)-\iota([ab[cdk]], l)+\iota([[abc]dk], l)-\iota([[abd]ck], l))f\\
&\quad+(\iota(k, [cd[abl]])-\iota(k, [ab[cdl]])+\iota(k, [[abc]dl])-\iota(k, [[abd]cl]))f\\
&=0,
\end{aligned}$$

by (ii) and (3). Therefore, (iii) holds for $f\in C^{2n-1}(\mathfrak{T}, V)$.

(iv): For $f\in C^3(\mathfrak{T}, V)$ we obtain (iv). Therefore, we assume that (iv) holds for all $f\in C^{2n-3}(\mathfrak{T}, V)$. Then, in the case $f\in C^{2n-1}(\mathfrak{T}, V)$, by using (i), (ii), (iii) for arbitrary $c, d\in\mathfrak{T}$

$$\begin{aligned}
&\iota(c, d)\delta\kappa(a, b)f+\iota(c, d)\kappa(a, b)\delta f\\
&=\delta\iota(c, d)\kappa(a, b)f+\kappa(c, d)\kappa(a, b)f-\kappa(a, b)\iota(c, d)\delta f\\
&\qquad\qquad+(-1)^{n+1}\iota([abc], d)\delta f+(-1)^{n+1}\iota(c, [abd])\delta f\\
&=-(\delta\kappa(a, b)+\kappa(a, b)\delta)\iota(c, d)f\\
&\qquad-(\kappa(a, b)\kappa(c, d)-\kappa(c, d)\kappa(a, b)+(-1)^n\kappa([abc], d)+(-1)^n\kappa(c, [abd]))f\\
&=0,
\end{aligned}$$

by the inductive assumption. Hence, (iv) holds for $f\in C^{2n-1}(\mathfrak{T}, V)$.

For every $a, b\in\mathfrak{T}$ and $f\in C^{2n-1}(\mathfrak{T}, V)$ ($n=2, 3, \cdots$), by using Lemma 1 and the induction we obtain

$$\begin{aligned}\iota(a, b)(\delta\delta f) &= \delta\iota(a, b)\delta f+\kappa(a, b)\delta f\\ &= \delta\delta\iota(a, b)f+\delta\kappa(a, b)f+\kappa(a, b)\delta f\\ &= 0,\end{aligned}$$

hence $\delta\delta f=0$ for $f\in C^{2n-1}(\mathfrak{T}, V)$ ($n=1, 2, \cdots$). Then it follows immediately that $\delta\delta f=0$ for $f\in C^{2n}(\mathfrak{T}, V)$ ($n=1, 2, \cdots$).

Thus we have the following main theorem.

THEOREM 1. *For the operator δ defined above, it holds that $\delta\delta f=0$ for any $f\in C^{n}(\mathfrak{T}, V)$, $n=0, 1, 2, \cdots$*

The mapping $f\in C^{n}(\mathfrak{T}, V)$ is called a *cocycle* of order n if $\delta f=0$. We denote by $Z^{n}(\mathfrak{T}, V)$ a subspace spanned by cocycles of order n. The element of $B^{n}(\mathfrak{T}, V) \equiv \delta C^{n-2}(\mathfrak{T}, V)$ is a *coboundary*. From Theorem 1, $B^{n}(\mathfrak{T}, V)$ is a subspace of $Z^{n}(\mathfrak{T}, V)$. Therefore we can define a *cohomology space* $H^{n}(\mathfrak{T}, V)$ of order n of $\mathfrak{T}$ as the factor space $Z^{n}(\mathfrak{T}, V)/B^{n}(\mathfrak{T}, V)$, ($n=0, 1, 2, \cdots$).

2.[4] DEFINITION 3. Let $\mathfrak{T}, \mathfrak{U}, \mathfrak{m}$ be L.t.s. over the same base field. $\mathfrak{T}$ is an *extension of* $\mathfrak{U}$ *by* $\mathfrak{m}$ if there exists an exact sequence of L. t. s.:

$$0\longrightarrow\mathfrak{m}\stackrel{\iota}{\longrightarrow}\mathfrak{T}\stackrel{\pi}{\longrightarrow}\mathfrak{U}\longrightarrow 0.$$

Two extensions $\mathfrak{T}$ and $\mathfrak{T}'$ are said to be *equivalent* if the following diagram is commutative:

$$\begin{array}{ccccccccc}0&\longrightarrow&\mathfrak{m}&\longrightarrow&\mathfrak{T}&\longrightarrow&\mathfrak{U}&\longrightarrow&0\\ &&\downarrow 1&&\downarrow&&\downarrow 1&&\\ 0&\longrightarrow&\mathfrak{m}&\longrightarrow&\mathfrak{T}'&\longrightarrow&\mathfrak{U}&\longrightarrow&0\end{array}.$$

As a special case of a solvable ideal in a L. t. s. defined by W. G. Lister [7], we define an abelian ideal $\mathfrak{m}$ in $\mathfrak{T}$ as an ideal such that $[\mathfrak{T}\mathfrak{m}\mathfrak{m}]=(0)$. We consider the case that $\mathfrak{T}$ is an extension of $\mathfrak{U}$ by abelian ideal $\mathfrak{m}$ in $\mathfrak{T}$, that is $[\mathfrak{T}\iota(\mathfrak{m})\iota(\mathfrak{m})]=(0)$. Then, for elements $u=x+p, v=y+q$ ($x, y\in\mathfrak{T}, p, q\in\mathfrak{m}$), $\theta(u, v)m\equiv[muv]=[mxy]$. Therefore, $\mathfrak{m}$ is an $\mathfrak{U}$-module by defining

$$\theta(u, v)m=[mtt'] \qquad \text{for any } t, t' \text{ in } \mathfrak{T} \text{ such that } \pi(t)=u, \pi(t')=v.$$

Let l be a section of the extension $\mathfrak{T}$ of $\mathfrak{U}$ by an abelian ideal in $\mathfrak{T}$, that is, l is a linear mapping of $\mathfrak{U}$ into $\mathfrak{T}$ such that $\pi l=1$. Next, we put

$$(15)\qquad f(x_1, x_2, x_3)=[l(x_1)l(x_2)l(x_3)]-l([x_1x_2x_3]) \qquad x_i\in\mathfrak{U}\ (i=1, 2, 3),$$

4) In this section, we follow the method in [2].

then, f is a trilinear mapping of $\mathfrak{U}\times\mathfrak{U}\times\mathfrak{U}$ into $\mathfrak{m}$, since π is a homomorphism of $\mathfrak{T}$ onto $\mathfrak{U}$, and f belongs to $C^3(\mathfrak{U}, \mathfrak{m})$. We identify $\mathfrak{m}\times\mathfrak{U}$ and $\mathfrak{T}$ as vector spaces by $(m, x)\to m+l(x)$. In $\mathfrak{T}$ the following relation holds:

$$\begin{aligned}&[m_1+l(x_1)\ m_2+l(x_2)\ m_3+l(x_3)]\\ &\quad=[m_1 l(x_2)l(x_3)]-[m_2 l(x_1)l(x_3)]+[l(x_1)l(x_2)m_3]+f(x_1, x_2, x_3)+l([x_1x_2x_3]).\end{aligned}$$

Hence we can define a Lie triple product on $\mathfrak{m}\times\mathfrak{U}$ by

$$\begin{aligned}(16')\qquad &[(m_1, x_1)(m_2, x_2)(m_3, x_3)]\\ &\quad=(\theta(x_2, x_3)m_1-\theta(x_1, x_3)m_2+D(x_1, x_2)m_3+f(x_1, x_2, x_3),\ [x_1x_2x_3]).\end{aligned}$$

From this we obtain

$$[(m_1, x_1)(m_1, x_1)(m_2, x_2)]=(f(x_1, x_1, x_2), 0),$$

$$\begin{aligned}&[(m_1, x_1)(m_2, x_2)(m_3, x_3)]+[(m_2, x_2)(m_3, x_3)(m_1, x_1)]+[(m_3, x_3)(m_1, x_1)(m_2, x_2)]\\ &\quad=(f(x_1, x_2, x_3)+f(x_2, x_3, x_1)+f(x_3, x_1, x_2), 0),\end{aligned}$$

$$\begin{aligned}&[[(m_1, x_1)(m_2, x_2)(m_3, x_3)](m_4, x_4)(m_5, x_5)]\\ &+[[(m_2, x_2)(m_1, x_1)(m_4, x_4)](m_3, x_3)(m_5, x_5)]\\ &+[(m_2, x_2)(m_1, x_1)[(m_3, x_3)(m_4, x_4)(m_5, x_5)]]\\ &+[(m_3, x_3)(m_4, x_4)[(m_1, x_1)(m_2, x_2)(m_5, x_5)]]\end{aligned}$$

$$=\left(\begin{array}{l}(\theta(x_4, x_5)\theta(x_2, x_3)-\theta(x_3, x_5)\theta(x_2, x_4)-\theta(x_2, [x_3x_4x_5])+D(x_3, x_4)\theta(x_2, x_5))m_1\\ -(\theta(x_4, x_5)\theta(x_1, x_3)-\theta(x_3, x_5)\theta(x_1, x_4)-\theta(x_1, [x_3x_4x_5])+D(x_3, x_4)\theta(x_1, x_5))m_2\\ +(\theta(x_4, x_5)D(x_1, x_2)+D(x_2, x_1)\theta(x_4, x_5)-\theta([x_2x_1x_4], x_5)+\theta(x_4, [x_1x_2x_5]))m_3\\ +(\theta(x_3, x_5)D(x_2, x_1)-D(x_2, x_1)\theta(x_3, x_5)-\theta([x_1x_2x_3], x_5)-\theta(x_3, [x_1x_2x_5]))m_4\\ +(D(x_2, x_1)D(x_3, x_4)+D(x_3, x_4)D(x_1, x_2)+D([x_1x_2x_3], x_4)+D([x_2x_1x_4], x_3))m_5,\\ +\theta(x_4, x_5)f(x_1, x_2, x_3)-\theta(x_3, x_5)f(x_1, x_2, x_4)-D(x_1, x_2)f(x_3, x_4, x_5)\\ +D(x_3, x_4)f(x_1, x_2, x_5)+f([x_1x_2x_3], x_4, x_5)+f(x_3, [x_1x_2x_4], x_5)\\ +f(x_3, x_4, [x_1x_2x_5])-f(x_1, x_2, [x_3x_4x_5])\end{array}\quad 0\right)$$

$$=(\delta f(x_1, x_2, x_3, x_4, x_5), 0),$$

in which we used (3), (6), (7), (9). Therefore f is a cocycle of order 3.

Conversely, let $\mathfrak{m}$ be a $(\mathfrak{U}, \theta)$-module and abelian L. t. s. and let f be a cocycle of order 3. We define a ternary product on a vector space $\mathfrak{m}\times\mathfrak{U}$ by (16), then the vector space $\mathfrak{T}=\mathfrak{m}\times\mathfrak{U}$ becomes a L. t. s. with respect to this composition. Next we define the exact sequence:

$$0\longrightarrow\mathfrak{m}\xrightarrow{\ \iota\ }\mathfrak{T}\xrightarrow{\ \pi\ }\mathfrak{U}\longrightarrow 0$$

by $\iota(m)=(m, 0)$ and $\pi(m, x)=x$. Since ι and π are homomorphism, $\mathfrak{T}$ is an extension of $\mathfrak{U}$ by $\mathfrak{m}$, and it is easy to see that $\iota(\mathfrak{m})$ is an abelian in $\mathfrak{T}$. For a special section l: $l(x)=(0, x)$ $(x\in\mathfrak{U})$

$$[l(x_1)l(x_2)l(x_3)]-l([x_1x_2x_3])=(f(x_1, x_2, x_3), 0),$$

hence f is a cocycle defined by this extension.

If there exists another section l', then $g(x)\equiv l'(x)-l(x)$ in $\mathfrak{m}$, and $f'(x_1, x_2, x_3)$ $=[l'(x_1)l'(x_2)l'(x_3)]-l'([x_1x_2x_3])=f(x_1, x_2, x_3)+\theta(x_2, x_3)g(x_1)-\theta(x_1, x_3)g(x_2)+D(x_1, x_2)g(x_3)-g([x_1x_2x_3])=f(x_1, x_2, x_3)+\delta g(x_1, x_2, x_3)$, therefore f and f' belong to the same cohomology class.

Summarizing above results we have the following

THEOREM 2. *An extension* $\mathfrak{T}$ *of* $\mathfrak{U}$ *by an abelian ideal* $\mathfrak{m}$ *in* $\mathfrak{T}$ *defines an element of* $H^3(\mathfrak{U}, \mathfrak{m})$. *Conversely, if* $\mathfrak{m}$ *is abelian, an extension* $\mathfrak{T}$ *of* $\mathfrak{U}$ *by* $\mathfrak{m}$ *corresponds to any element of* $H^3(\mathfrak{U}, \mathfrak{m})$ *and* $\mathfrak{m}$ *becomes abelian in* $\mathfrak{T}$.

The extension: $0\longrightarrow\mathfrak{m}\xrightarrow{\iota}\mathfrak{T}\xrightarrow{\pi}\mathfrak{U}\longrightarrow 0$ is said to be inessential if there exists a subsystem $\mathfrak{T}'$ such that $\mathfrak{T}$ is a vector direct sum of $\iota(\mathfrak{m})$ and $\mathfrak{T}'$. Then

COROLLARY. *An extension* $\mathfrak{T}$ *of* $\mathfrak{U}$ *by an abelian ideal* $\mathfrak{m}$ *in* $\mathfrak{T}$ *is inessential if and only if* $H^3(\mathfrak{U}, \mathfrak{m})=(0)$.

3.[5] Let $\Theta(a, b)$ be a linear mapping $x\rightarrow[xba]$ of L. t. s. $\mathfrak{T}$, $a, b\in\mathfrak{T}$. Put $\phi(a, b)$ $=Tr\Theta(a, b)$, and call this form ϕ a Killing form of $\mathfrak{T}$.

An 1-to-1 mapping A of $\mathfrak{T}$ is called an automorphism of $\mathfrak{T}$ if $A[xyz]=[AxAyAz]$ for all $x, y, z\in\mathfrak{T}$. A derivation D of $\mathfrak{T}$ is a linear mapping of $\mathfrak{T}$ such that $D[xyz]$ $=[(Dx)yz]+[x(Dy)z]+[xy(Dz)]$ for all $x, y, z\in\mathfrak{T}$.

LEMMA 2. *Let* $\phi(a, b)$ *be a Killing form of L. t. s.* $\mathfrak{T}$, *then*

(i) $\phi(Ax, Ay)=\phi(x, y)$ *for automorphism* A *of* $\mathfrak{T}$,

(ii) $\phi(Dx, y)+\phi(x, Dy)=0$ *for derivation* D *of* $\mathfrak{T}$,

i. e. ϕ *is an invariant of* D.

PROOF. From the definition of an automorphism, we have $A\Theta(x, y)=\Theta(Ax, Ay)A$, hence $\phi(Ax, Ay)=Tr\Theta(Ax, Ay)=Tr(A\Theta(x, y)A^{-1})=Tr\Theta(x, y)=\phi(x, y)$. The proof is similar for the derivation, therefore we shall omit it.

Since the mapping: $x\rightarrow\sum_i[a_ib_ix]$ is an (inner) derivation of $\mathfrak{T}$, we have the following

COROLLARY. *A Killing form is an invariant of an inner derivation.*

Let (V, θ) be a $\mathfrak{T}$-module for L. t. s. $\mathfrak{T}$ and let $X_1, X_2, \cdots, X_n$ be a base of $\mathfrak{T}$. We call $\mathfrak{T}$ *non-degenerate* if

$$\det\begin{vmatrix}\phi(X_1, X_1), \cdots, \phi(X_1, X_n)\\ \vdots \qquad\qquad \vdots\\ \phi(X_n, X_1), \cdots, \phi(X_n, X_n)\end{vmatrix}\neq 0.$$

5) In this section, we follow the method in [8].

Then, we may define a linear operator C of V as

$$C=\sum_{i,j=1}^{n}\pi_{ji}\Theta(X_i, X_j),$$

where (π_{ij}) is an inverse matrix of $(\phi(X_i, X_j))$ and $\Theta(X_i, X_j)=\theta(X_j, X_i)$. We call this operator C a *Casimir operator* of θ. If we put $Y_i=\sum_{j=1}^{n}\pi_{ji}X_j$ $(i=1, 2, \cdots, n)$, $(Y_1, Y_2, \cdots, Y_n)$ is a base of $\mathfrak{T}$ and $\phi(X_i, Y_k)=\delta_{ik}$, and $C=\sum_{i=1}^{n}\Theta(X_i, Y_i)$.

Let $(X_1, \cdots, X_n)$ and $(X'_1, \cdots, X'_n)$ be bases of $\mathfrak{T}$ and let $\phi(X_i, X_j)$ and $\phi(X'_i, X'_j)$ be Killing forms with inverse matrix (π_{ij}) and (π'_{ij}) respectively. Denote by C and C' Casimir operators corresponding to bases (X_i) and (X'_i) respectively. Then, putting $X'_i=\sum_{j=1}^{n}a_{ij}X_j$, $Y'_i=\sum_{j=1}^{n}b_{ij}Y_j$, $C'=\sum_{i=1}^{n}\Theta(X'_i, Y'_i)=\sum_{i,s,t}a_{is}b_{it}\Theta(X_s, Y_t)=\sum_{s}\Theta(X_s, Y_s)=C$, since $\sum_{s}a_{is}b_{ks}=\sum_{s,t}a_{is}b_{kt}\delta_{st}=\sum_{s,t}a_{is}b_{kt}\phi(X_s, Y_t)=\phi(X'_i, Y'_k)=\delta_{ik}$. Hence, the Casimir operator is independent to the base of $\mathfrak{T}$.

THEOREM 3. *Let (V, θ) be a $\mathfrak{T}$-module of a non-degenerate L. t. s. $\mathfrak{T}$. Then the Casimir operator C of θ commutes with $D(x, y)$ for all x, y in $\mathfrak{T}$, where $D(x, y)=\theta(y, x)-\theta(x, y)$.*

PROOF. From the fact that V is a $\mathfrak{T}$-module using (7) we have

$$\begin{aligned} D(x, y)C-CD(x, y)&=\sum_{i}\{D(x, y)\Theta(X_i, Y_i)-\Theta(X_i, Y_i)D(x, y)\}\\ &=\sum_{i}\{\Theta([xyX_i], Y_i)+\Theta(X_i, [xyY_i])\}, \end{aligned}$$

where $\Theta(X_i, Y_i)=\theta(Y_i, X_i)$. Putting $[xyX_i]=\sum_{j}a_{ij}X_j$ and $[xyY_i]=\sum_{j}b_{ij}Y_j$, it follows that $a_{ij}+b_{ji}=0$, because a Killing form is an invariant of an inner derivation. Hence

$$D(x, y)C-CD(x, y)=\sum_{i,k}\{a_{ik}\Theta(X_k, Y_i)+b_{ik}\Theta(X_i, Y_k)\}=0.$$

This proves the theorem.

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