# ON THE BLOCKS AND THE SECTIONS OF FINITE GROUPS

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**1.** Let $\mathfrak{G}$ be a group of finite order $g$ and let $p$ be a fixed rational prime; $g = p^a g'$, $(p, g')=1$. Let us assume that $a>0$, for otherwise our results will not give any new information. We denote by $\chi_1, \chi_2, \cdots, \chi_n$ the irreducible characters[1)] of $\mathfrak{G}$ and by $K_1, K_2, \cdots, K_n$ the classes of conjugate elements in $\mathfrak{G}$; we select a complete system of representatives $G_1, G_2, \cdots, G_n$ for the classes $K_1, K_2, \cdots, K_n$. It is well known that

$$\sum_{i=1}^{n} \chi_i(G_\lambda)\chi_i(G_\mu^{-1})=0 \qquad (\lambda \neq \mu) \tag{1.1}$$

and

$$\sum_{\nu=1}^{n} c_\nu \chi_i(G_\nu)\chi_j(G_\nu^{-1})=0 \qquad (i \neq j) \tag{1.1'}$$

where $c_\nu$ denotes the number of elements in $K_\nu$, $\nu=1, 2, \cdots, n$.

Let $B_1, B_2, \cdots, B_t$ be the $p$-blocks[2)] of $\mathfrak{G}$ and let $P_1=1, P_2, \cdots, P_l$ be a full system of $p$-elements[3)] of $\mathfrak{G}$ such that every $p$-element of $\mathfrak{G}$ is conjugate in $\mathfrak{G}$ to exactly one element $P_\delta$ of the system. It is well known that if we denote by $\mathfrak{S}(P)$ the $p$-section[4)] of a $p$-element $P$ in $\mathfrak{G}$, then the elements of $\mathfrak{G}$ are distributed into $l$ $p$-sections $\mathfrak{S}(P_1)$, $\mathfrak{S}(P_2), \cdots, \mathfrak{S}(P_l)$. R. BRAUER, in his paper [1], gave the following refinement of some of the orthogonality relations (1.1):[5)]

[1. A] *If $L$ and $M$ are two elements of $\mathfrak{G}$ which belong to different $p$-sections of $\mathfrak{G}$, then*

$$\sum_{\chi_i \in B_\tau} \chi_i(L)\chi_i(M^{-1})=0 \tag{1.2}$$

*for each $p$-block $B_\tau$ of $\mathfrak{G}$.*

Recently, R. BRAUER and M. OSIMA have given independently a refinement of some of the orthogonality relations (1.1′):

[1. B] *If $\chi_i$ and $\chi_j$ are two irreducible characters of $\mathfrak{G}$ which belong to different*

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1) The term "irreducible character" will always mean absolutely irreducible ordinary character.
2) Cf. §9 in [3].
3) An element of $\mathfrak{G}$ is called a $p$-element if its order is a power of $p$.
4) An element $G$ of $\mathfrak{G}$ can be expressed uniquely as a product $PR$ of two commutative elements where $P$ is a $p$-element, while $R$ is a $p$-regular element. We shall call $P$ the *$p$-factor* of $G$ and $R$ the *$p$-regular factor* of $G$. The *$p$-section* $\mathfrak{S}(P)$ of a $p$-element $P$ in $\mathfrak{G}$ is the set of all elements of $\mathfrak{G}$ whose $p$-factors are conjugate to $P$ in $\mathfrak{G}$.
5) This result was announced in [1] and its proof has been printed in [2].

*$p$-blocks of $\mathfrak{G}$, then*

(1.3)
$$\sum_{K_\nu \subseteq \mathfrak{S}(P_\delta)} c_\nu \chi_i(G_\nu)\chi_j(G_\nu^{-1})=0$$

*for each $p$-section $\mathfrak{S}(P_\delta)$ of $\mathfrak{G}$.* ([9])

Let $\Omega$ be the field of $g$-th roots of unity and let $\mathfrak{p}$ be an arbitrarily fixed prime ideal divisor of $p$ in $\Omega$; $\mathfrak{o}_\mathfrak{p}$ will denote the ring of all $\mathfrak{p}$-integers in $\Omega$. In [11], P. ROQUETTE considered arithmetically the blocks of the character ring of $\mathfrak{G}$ over the ring of all $\mathfrak{p}$-addic integers; K. SHIRATANI has shown in [12] that, essentially, we have only to consider the character ring $X_\mathfrak{p}$ of $\mathfrak{G}$ over $\mathfrak{o}_\mathfrak{p}$. The object of this note is to consider the duals of some results on $p$-blocks and $p$-sections, especially the duals of [1. A] and [1. B], from a certain standpoint that (1.1) and (1.1′) are dual. In the duality, the $p$-regular sections[6] of $\mathfrak{G}$ will correspond to the $p$-blocks of $\mathfrak{G}$. In § 3, we shall define a new kind of blocks ($p$-complementary blocks) of group characters which will correspond to $p$-sections.

2. In this section, we shall sketch an outline of some well known results[7] on $X_\mathfrak{p}$ which will correspond to some fundamental properties[8] of the primitive idempotents of the center $Z_\mathfrak{p}$ of the group ring of $\mathfrak{G}$ over $\mathfrak{o}_\mathfrak{p}$. In the first place, in order to see "dual", we shall refer to the definition of the primitive idempotents of $Z_\mathfrak{p}$.

We denote by $Z$ the center of the group ring of $\mathfrak{G}$ over $\Omega$. The primitive idempotent $e_i$ of $Z$ belonging to $\chi_i$ is given by

(2.1)
$$e_i=\frac{1}{g}\sum_{\nu=1}^{n} x_i\chi_i(G_\nu^{-1})K_\nu^{9)},$$

where $x_i=\chi_i(1)$. After M. OSIMA [8], we set

(2.2)
$$E_\tau=\sum_{\chi_i\in B_\tau} e_i=\sum_{\nu=1}^{n} b_\nu^\tau K_\nu \qquad (\tau=1,2,\cdots,t).$$

It is well known that all $E_\tau$ belong to $Z_\mathfrak{p}$ and that if, for a set $B$ of irreducible characters $\chi_i$ of $\mathfrak{G}$, $\sum_{\chi_i\in B} e_i$ belongs to $Z_\mathfrak{p}$, then $B$ is a collection of $p$-blocks $B_\tau$ of $\mathfrak{G}$. Thus we have the following:

[2. A] *In the center $Z_\mathfrak{p}$ of the group ring of $\mathfrak{G}$ over $\mathfrak{o}_\mathfrak{p}$,*

$$1=E_1+E_2+\cdots+E_t$$

*is the decomposition of the identity 1 into the sum of mutually orthogonal primitive idempotents.* ([8])

6) The term "$p$-regular section" has the same meaning as "$\mathfrak{p}$-Oberklasse" in [11]: The $p$-regular section $S(R)$ of a $p$-regular element $R$ in $\mathfrak{G}$ is the set of all elements of $\mathfrak{G}$ whose $p$-regular factors (See footnote 4) are conjugate to $R$ in $\mathfrak{G}$.

7) Cf. [11], [12].

8) Cf. [8].

9) Each class $K_\nu$ is interpreted very often as the sum of all its elements.

We now consider the character ring $X$ of $\mathfrak{G}$ over $\Omega$. Let $d_1, d_2, \cdots, d_n$ be the primitive idempotents of $X$: $d_\mu(G_\nu)=1$ or 0 according as $\mu=\nu$ or not. If we denote by $\xi_\mu$ the linear character of $X$ belonging to $d_\mu$, then $\xi_\mu(\chi_i)=\chi_i(G_\mu)$ for all $\chi_i$. As is easily seen, each $d_\mu$ is expressed as

$$d_\mu=\frac{1}{g}\sum_{i=1}^{n} c_\mu \chi_i(G_\mu^{-1})\chi_i. \tag{2.1'}$$

Evidently the elements $d_1, d_2, \cdots, d_n$ form a $\Omega$-basis of $X$: $X=\Omega d_1+\Omega d_2+\cdots+\Omega d_n$. Further we shall consider four subrings of $X$:

$$X_{\mathbf{P}}=\mathbf{P}\chi_1+\mathbf{P}\chi_2+\cdots+\mathbf{P}\chi_n,$$
$$X_p=X_{\mathbf{P}}\cap X_{\mathfrak{p}}=\mathbf{I}_p\chi_1+\mathbf{I}_p\chi_2+\cdots+\mathbf{I}_p\chi_n,$$
$$X_{\mathbf{I}}=\mathbf{I}\chi_1+\mathbf{I}\chi_2+\cdots+\mathbf{I}\chi_n,$$
$$\Xi_{\mathbf{I}}=\mathbf{I}d_1+\mathbf{I}d_2+\cdots+\mathbf{I}d_n,$$

where $\mathbf{P}$ is the field of all rational numbers, $\mathbf{I}$ is the ring of all rational integers and $\mathbf{I}_p$ is the ring of all rational $p$-integers.

Let $R_1=1, R_2, \cdots, R_k$ be a full system of $p$-regular elements of $\mathfrak{G}$ such that every $p$-regular element of $\mathfrak{G}$ is conjugate in $\mathfrak{G}$ to exactly one element $R_\gamma$ of the system. If we denote by $S(R)$ the $p$-regular section of a $p$-regular element $R$ in $\mathfrak{G}$, then the elements of $\mathfrak{G}$ are distributed into $k$ $p$-regular sections $S(R_1), S(R_2), \cdots, S(R_k)$. We set

$$\delta_\gamma=\sum_{K_\nu\subseteq S(R_\gamma)} d_\nu=\sum_{i=1}^{n}\alpha_i^\gamma\chi_i, \tag{2.2'}$$

where

$$\alpha_i^\gamma=\frac{1}{g}\sum_{K_\nu\subseteq S(R_\gamma)} c_\nu\chi_i(G_\nu^{-1}). \tag{2.3}$$

As is easily seen, all $\alpha_i^1$ are rational numbers: $\delta_1\in X_{\mathbf{P}}$. Moreover $\delta_1$ *belongs to* $X_p$, as is shown in the following.

Let $\mathfrak{P}$ be a $p$-Sylow subgroup of $\mathfrak{G}$ and let $\theta_1$ be the 1-character of $\mathfrak{P}$. Denoting by $\theta_1^*$ the character of $\mathfrak{G}$ induced by $\theta_1$, we have $\theta_1^*\in X_{\mathbf{I}}\cap\Xi_{\mathbf{I}}$ and

$$\theta_1^*(G)\begin{cases}\equiv\theta_1^*(1)=g'\not\equiv 0 \pmod p & (G\in S(1)),\\ =0 & (G\notin S(1)).\end{cases}$$

Therefore

$$(\theta_1^*)^{\varphi(p^\alpha)}\equiv\delta_1 \pmod{p^\alpha\Xi_{\mathbf{I}}} \qquad (\alpha=1,2,\cdots),$$

where $\varphi$ denotes the Euler's function. Hence, by Lemma 2 in [12], we have

$$(\theta_1^*)^{\varphi(p^\alpha)}-\delta_1\in X_{\mathbf{P}}\cap X_{\mathfrak{p}}=X_p$$

for sufficient large $\alpha$. Since $\theta_1^*\in X_{\mathbf{I}}\subseteq X_p$, we see that $\delta_1\in X_p$.

In consideration of (2.2′) and (2.3), $\delta_1 \in X_p$ yields the congruences

$$\sum_{P \in S(1)} \chi_i(P) \equiv 0 \pmod{p^a} \qquad (i=1, 2, \cdots, n) \tag{2.4}$$

in **I**.

For an arbitrarily fixed $p$-regular element $R=R_\gamma\ (1\leqq\gamma\leqq k)$, we consider the normalizer $\tilde{\mathfrak{G}}=\mathfrak{N}(R)$ of $R$ in $\mathfrak{G}$. We denote by $\tilde{g}$ the order of $\tilde{\mathfrak{G}}$ and by $\tilde{\chi}_1, \tilde{\chi}_2, \cdots, \tilde{\chi}_{\tilde{n}}$ the irreducible characters of $\tilde{\mathfrak{G}}$; $\tilde{\omega}_j$ will denote the linear character of the center $\tilde{Z}$ of the group ring of $\tilde{\mathfrak{G}}$ over $\Omega$ belonging to $\tilde{\chi}_j, j=1, 2, \cdots, \tilde{n}$. The idempotent $\tilde{\delta}_R$ of the character ring $\tilde{X}$ of $\tilde{\mathfrak{G}}$ over $\Omega$ associated with the $p$-regular section of $R$ in $\tilde{\mathfrak{G}}$ is given by

$$\tilde{\delta}_R=\frac{1}{\tilde{g}}\sum_{j=1}^{\tilde{n}}\sum_{P\in\tilde{S}(1)}\tilde{\chi}_j(R^{-1}P^{-1})\tilde{\chi}_j,$$

where $\tilde{S}(1)$ is the $p$-regular section of 1 in $\tilde{\mathfrak{G}}$. If $\psi$ is an element of $\tilde{X}$, we denote by $\psi^*$ the element of $X$ induced by $\psi$: $\psi^*(G)=\frac{1}{\tilde{g}}\sum_{X\in\mathfrak{G}}\psi_0(X^{-1}GX)$ for $G\in\mathfrak{G}$, where $\psi_0$ is the extension of $\psi$ to $\mathfrak{G}$ obtained by putting $\psi_0(G)=0$ for $G\notin\tilde{\mathfrak{G}}$. By Frobenius' theorem on induced characters, we have

$$\delta_\gamma=\tilde{\delta}_R^*=\left\{\sum_{j=1}^{\tilde{n}}\tilde{\omega}_j(R^{-1})\tilde{\alpha}_j^1\chi_j\right\}^*, \tag{2.5}$$

where

$$\tilde{\alpha}_j^1=\frac{1}{\tilde{g}}\sum_{P\in\tilde{S}(1)}\tilde{\chi}_j(P^{-1}). \tag{2.6}$$

Since all $\tilde{\alpha}_j^1$ are rational $p$-integers, $\delta_\gamma$ belongs to $X_{\mathfrak{p}}$. Conversely if, for a collection $S$ of classes $K_\nu$ of $\mathfrak{G}$, $\delta=\sum_{K_\nu\subseteq S} d_\nu$ belongs to $X_{\mathfrak{p}}$, then $S$ is a collection of $p$-regular sections $S(R_\gamma)$ of $\mathfrak{G}$. This can be seen as follows:

Suppose that $S\cap S(R_\gamma)$ is not vacuous and that $S\not\supseteq S(R_\gamma)$. Then we can select two classes $K_\alpha$ and $K_\beta$ in $S(R_\gamma)$ such that $K_\alpha\subseteq S$ but $K_\beta\not\subseteq S$. It is easily seen that $\xi_\alpha(\delta)=1$, while $\xi_\beta(\delta)=0$. On the other hand, in general, if $K_\lambda$ and $K_\mu$ are contained in the same $p$-regular section of $\mathfrak{G}$, then $\xi_\lambda(\chi_i)\equiv\xi_\mu(\chi_i) \pmod{\mathfrak{p}}$ for all $\chi_i$. Hence we have $\xi_\alpha(\delta)\equiv\xi_\beta(\delta) \pmod{\mathfrak{p}}$, which yields a contradiction. Therefore $S$ is a collection of $p$-regular sections of $\mathfrak{G}$.

We thus obtain the following:

[2. A′] *In the character ring* $X_{\mathfrak{p}}$ *of* $\mathfrak{G}$ *over* $\mathfrak{o}_{\mathfrak{p}}$,

$$1=\delta_1+\delta_2+\cdots+\delta_k$$

*is the decomposition of the identity* 1 *into the sum of mutually orthogonal primitive idempotents.* ([11], [12])

Since every $\delta_\gamma$ belongs to $X_{\mathfrak{p}}$, there exist $k$ but not more than $k$ irreducible characters $\chi_i$ of $\mathfrak{G}$ which are linearly independent (mod $\mathfrak{p}$) ([5]). We see also that two classes $K_\alpha$ and $K_\beta$ are contained in the same $p$-regular section $S(R_\gamma)$ of $\mathfrak{G}$ if and only if $\xi_\alpha(\chi_i)\equiv\xi_\beta(\chi_i)$ (mod $\mathfrak{p}$) for all irreducible characters $\chi_i$ of $\mathfrak{G}$.

**3.** Let $q_0=p, q_1, q_2, \cdots, q_f$ be the rational primes included in the group order $g$. In the first place, in order to introduce the concept of $p$-complementary blocks mentioned in §1, we shall give some characterizations of the $p$-sections $\mathfrak{S}(P_\delta)$ of $\mathfrak{G}$.

[3. A] *Two classes $K_\alpha$ and $K_\beta$ of $\mathfrak{G}$ are contained in the same $p$-section $\mathfrak{S}(P_\delta)$ of $\mathfrak{G}$ if and only if there exists a chain of classes*

$$K_\alpha, K_\kappa, \cdots, K_\rho, K_\beta \tag{3.1}$$

*of $\mathfrak{G}$ such that any two consecutive classes $K_\mu$ and $K_\nu$ of the chain are contained in a $q_{r(\mu,\nu)}$-regular section of $\mathfrak{G}$, $1\leqq r(\mu,\nu)\leqq f$.*

PROOF. As is well known, an element $G$ of $\mathfrak{G}$ is expressed as a product $Q_0Q_1Q_2\cdots Q_f$, where $Q_r$ is the $q_r$-factor of $G, r=0, 1, 2, \cdots, f$. It is evident that $Q_0Q_1Q_2\cdots Q_{r-1}$ and $Q_0Q_1Q_2\cdots Q_r$ belong to the $q_r$-regular section of $Q_0Q_1Q_2\cdots Q_{r-1}$ in $\mathfrak{G}, r=1, 2, \cdots, f$. Hence if we denote by $K(Y)$ the class $K_\lambda$ of $\mathfrak{G}$ represented by an element $Y$, then the classes $K(G)$ and $K(Q_0)$ are connected by a chain as (3.1). Therefore if $K_\alpha$ and $K_\beta$ are contained in the same $p$-section of $\mathfrak{G}$, then they are connected by a chain as (3.1).

Conversely, if $K_\mu$ and $K_\nu$ are contained in a $q_r$-regular section of $\mathfrak{G}$ $(1\leqq r\leqq f)$, then the $p$-factors of them are conjugate in $\mathfrak{G}$. Therefore if $K_\alpha$ and $K_\beta$ are connected by a chain as (3.1), then they are contained in the same $p$-section of $\mathfrak{G}$. This completes the proof.

We also have the following characterization of $p$-sections:

[3. B] *The $p$-sections $\mathfrak{S}(P_\delta)$ of $\mathfrak{G}$ are characterized as the minimal sets $\mathfrak{S}$ of elements of $\mathfrak{G}$ such that (a) every $\mathfrak{S}$ is not vacuous, (b) every $\mathfrak{S}$ is a collection of $q$-regular sections of $\mathfrak{G}$ for each rational prime $q$, different from $p$.*

Now we define the $p$-complementary blocks of $\mathfrak{G}$. We shall say that two irreducible characters $\chi_i$ and $\chi_j$ of $\mathfrak{G}$ belong to the same *$p$-complementary block* of $\mathfrak{G}$ if and only if there exists a chain of irreducible characters

$$\chi_i, \chi_h, \cdots, \chi_m, \chi_j \tag{3.1'}$$

of $\mathfrak{G}$ such that any two consecutive characters $\chi_u$ and $\chi_v$ of the chain belong to a $q_{r(u,v)}$-block of $\mathfrak{G}$, $1\leqq r(u,v)\leqq f$. Then we can distribute the irreducible characters $\chi_1, \chi_2, \cdots, \chi_n$ of $\mathfrak{G}$ into a certain number of $p$-complementary blocks $\mathfrak{B}_1, \mathfrak{B}_2, \cdots, \mathfrak{B}_s$ of $\mathfrak{G}$.

[3. B′] *The $p$-complementary blocks $\mathfrak{B}_\sigma$ of $\mathfrak{G}$ are characterized as the minimal sets $\mathfrak{B}$ of irreducible characters $\chi_i$ of $\mathfrak{G}$ such that (a) every $\mathfrak{B}$ is not vacuous, (b) every $\mathfrak{B}$ is a collection of $q$-blocks of $\mathfrak{G}$ for each rational prime $q$, different from $p$.*

We shall consider the idempotents of $\boldsymbol{X}$ associated with the $p$-sections of $\mathfrak{G}$ and the idempotents of $\boldsymbol{Z}$ associated with the $p$-complementary blocks of $\mathfrak{G}$. We set

$$\varepsilon_\delta = \sum_{K_\nu \subseteqq \mathfrak{S}(P_\delta)} d_\nu = \sum_{i=1}^{n} \beta_i^\delta \chi_i, \tag{3.2}$$

where

$$\beta_i^\delta = \frac{1}{g} \sum_{K_\nu \subseteqq \mathfrak{S}(P_\delta)} c_\nu \chi_i(G_\nu^{-1}). \tag{3.3}$$

By [2. A] and [3. B], we have the following:

[3. C] *All* $p^a \beta_i^\delta$ *are algebraic integers.*[10] *Conversely, if* $\mathfrak{S}$ *is a collection of classes* $K_\nu$ *of* $\mathfrak{G}$ *such that all coefficients* $\beta_i$ *of*

$$p^{a'} \sum_{K_\nu \subseteqq \mathfrak{S}} d_\nu = \sum_{i=1}^{n} \beta_i \chi_i$$

*are algebraic integers, then* $\mathfrak{S}$ *is a collection of* $p$*-sections* $\mathfrak{S}(P_\delta)$ *of* $\mathfrak{G}$, *where* $a'$ *is a rational integer.*

Similarly we set

$$\Delta_\sigma = \sum_{\chi_i \in \mathfrak{B}_\sigma} e_i = \sum_{\nu=1}^{n} a_\nu^\sigma K_\nu, \tag{3.2'}$$

where

$$a_\nu^\sigma = \frac{1}{g} \sum_{\chi_i \in \mathfrak{B}_\sigma} x_i \chi_i(G_\nu^{-1}). \tag{3.3'}$$

We then have the following:

[3. C′] *All* $p^a a_\nu^\sigma$ *are algebraic integers. Conversely, if* $\mathfrak{B}$ *is a set of irreducible characters* $\chi_i$ *of* $\mathfrak{G}$ *such that all coefficients* $a_\nu$ *of*

$$p^{a'} \sum_{\chi_i \in \mathfrak{B}} e_i = \sum_{\nu=1}^{n} a_\nu K_\nu$$

*are algebraic integers, then* $\mathfrak{B}$ *is a collection of* $p$*-complementary blocks* $\mathfrak{B}_\sigma$ *of* $\mathfrak{G}$, *where* $a'$ *is a rational integer.*

Corresponding to (2. 4), by [3. C] we have the congruences

$$\sum_{R \in \mathfrak{S}(1)} \chi_i(R) \equiv 0 \pmod{g'} \qquad (i=1, 2, \cdots, n) \tag{3.4}$$

in **I**.

Let $\hat{\mathfrak{G}}$ be the normalizer $\mathfrak{N}(P)$ of a $p$-element $P=P_\delta$ in $\mathfrak{G}$, $1 \leqq \delta \leqq l$. If $\hat{g}$, $\hat{\chi}_h$, $\hat{\omega}_h$, $\hat{n}$, $\hat{\varepsilon}_P$, $\hat{\varepsilon}_P^*$, $\hat{\mathfrak{S}}(1)$ for $\hat{\mathfrak{G}}$ correspond to $\tilde{g}$, $\tilde{\chi}_j$, $\tilde{\omega}_j$, $\tilde{n}$. $\tilde{\delta}_R$, $\tilde{\delta}_R^*$, $\tilde{S}(1)$ for $\tilde{\mathfrak{G}}=\mathfrak{N}(R)$ in § 2, then corresponding to (2. 5) we have

$$\varepsilon_\delta = \hat{\varepsilon}_P^* = \left\{ \sum_{h=1}^{\hat{n}} \hat{\omega}_i(P^{-1}) \hat{\beta}_h^1 \hat{\chi}_h \right\}^*, \tag{3.5}$$

10) Cf. [8].

where

$$\hat{\beta}^1_h = \frac{1}{\hat{g}} \sum_{R \in \hat{\mathfrak{S}}(1)} \hat{\chi}_h(R^{-1}). \tag{3.6}$$

REMARK. Every Osima's block[11] of $\mathfrak{G}$ for $p$ is a collection of $p$-complementary blocks of $\mathfrak{G}$. If $\mathfrak{G}$ has a normal $p$-Sylow subgroup, then the Osima's blocks of $\mathfrak{G}$ for $p$ and the $p$-complementary blocks of $\mathfrak{G}$ are identical as a whole. But, in general, both concepts of blocks are not identical.

**4.** In the first place, we shall refer a result on the primitive idempotents $E_\tau$ of $Z_\mathfrak{p}$, in order to see "dual" and to use it.

Let $P$ be an arbitrarily given $p$-element of $\mathfrak{G}$ and let $\hat{\mathfrak{G}}$ denote the normalizer $\mathfrak{N}(P)$ of $P$ in $\mathfrak{G}$. We denote by $\hat{B}^{(\tau)}$ the collection of $p$-blocks $\hat{B}_\rho$ of $\hat{\mathfrak{G}}$ such that each $\hat{B}_\rho$ determines a given $p$-block $B_\tau$ of $\mathfrak{G}$ in BRAUER's sence[12]; $\hat{E}^{(\tau)}$ will denote the idempotent of the center $\hat{Z}_\mathfrak{p}$ of the group ring of $\hat{\mathfrak{G}}$ over $\mathfrak{o}_\mathfrak{p}$ associated with $\hat{B}^{(\tau)}$, i.e. the sum of all primitive idempotents $\hat{E}_\rho$ of $\hat{Z}_\mathfrak{p}$ such that each $\hat{E}_\rho$ is associated with a $p$-block $\hat{B}_\rho$ of $\hat{\mathfrak{G}}$ contained in $\hat{B}^{(\tau)}$. Let $\hat{R}_1, \hat{R}_2, \cdots, \hat{R}_{\hat{k}}$ be a complete system of representatives for the $p$-regular classes of $\hat{\mathfrak{G}}$. We denote by $K(G)$ the class $K_\mu$ of $\mathfrak{G}$ represented by an element $G$ and by $\hat{K}(\hat{R}_\gamma)$ the class of conjugate elements in $\hat{\mathfrak{G}}$ represented by $\hat{R}_\gamma$. As is well known, the $p$-section $\mathfrak{S}(P)$ of $P$ in $\mathfrak{G}$ is the collection of classes $K(P\hat{R}_1)$, $K(P\hat{R}_2), \cdots, K(P\hat{R}_{\hat{k}})$. We then have the following:

[4. A] *For* $\alpha = 1, 2, \cdots, \hat{k}$, *we have*

$$K(P\hat{R}_\alpha)E_\tau = \sum_{\beta=1}^{\hat{k}} b^\tau_{\alpha\beta} K(P\hat{R}_\beta)$$

*and*

$$\hat{K}(\hat{R}_\alpha)\hat{E}^{(\tau)} = \sum_{\beta=1}^{\hat{k}} b^\tau_{\alpha\beta} \hat{K}(\hat{R}_\beta)$$

*with the same coefficients* $b^\tau_{\alpha\beta}$. ([5])

Let $q$ be an arbitrary rational prime different from $p$ and let $Q$ be an arbitrarily given $q$-element of $\mathfrak{G}$. We select a complete system of representatives $T_1, T_2, \cdots, T_m$ for the $q$-regular classes of the normalizer $\mathfrak{N}(Q)$ of $Q$ in $\mathfrak{G}$. We denote by $\mathfrak{B}^{(\sigma)}(Q)$ the collection of $q$-blocks of $\mathfrak{N}(Q)$ such that each $q$-block of the collection determines a $q$-block of $\mathfrak{G}$ contained in a given $p$-complementary block $\mathfrak{B}_\sigma$ of $\mathfrak{G}$; $\Delta^{(\sigma)}(Q)$ will denote the idempotent of the center $Z(Q)$ of the group ring of $\mathfrak{N}(Q)$ over $\Omega$ associated with $\mathfrak{B}^{(\sigma)}(Q)$. By (4. A] we may write

$$K(Q)\Delta_\sigma = \sum_{\beta=1}^{m} a^\sigma_\beta(Q) K(QT_\beta)$$

11) Cf. [7], [6].

12) Cf. [1], [2]. If a $p$-block $B_\tau$ of $\mathfrak{G}$ is determined by a $p$-block $\hat{B}_\rho$ of $\hat{\mathfrak{G}}$, then $B_\tau$ is denoted by $\hat{B}^{\mathfrak{G}}_\rho$ in the notations of [2].

and

$$\Delta^{(\sigma)}(Q)=\sum_{\beta=1}^{m} a_{\beta}^{\sigma}(Q)K(Q;\,T_{\beta})$$

with the same coefficients $a_{\beta}^{\sigma}(Q)$, where $K(Q;\,T_{\beta})$ denotes the class of conjugate elements in $\mathfrak{N}(Q)$ represented by $T_{\beta}$. Since all $p^{a}a_{\beta}^{\sigma}(Q)$ are algebraic integers, from [3. C′] we see that every $\mathfrak{B}^{(\sigma)}(Q)$ is a collection of $p$-complementary blocks of $\mathfrak{N}(Q)$.

Let $R$ be an arbitrarily given $p$-regular element of $\mathfrak{G}$ and let $Q_r$ be the $q_r$-regular factor of $R, r=1,2,\cdots,f$. Let further $\tilde{P}_1,\tilde{P}_2,\cdots,\tilde{P}_{\tilde{l}}$ be a complete system of representa tives for the classes of conjugate elements in the normalizer $\tilde{\mathfrak{G}}=\mathfrak{N}(R)$ of $R$ in $\mathfrak{G}$ which are contained in the $p$-regular section $\tilde{S}(1)$ of 1 in $\tilde{\mathfrak{G}}$; $\tilde{K}(\tilde{P}_{\delta})$ will denote the class of conjugate elements in $\tilde{\mathfrak{G}}$ represented by $\tilde{P}_{\delta}$. For each $p$-complementary block $\mathfrak{B}_{\sigma}$ of $\mathfrak{G}$, we can define the collection $\tilde{\mathfrak{B}}^{(\sigma)}$ of $p$-complementary blocks $\tilde{\mathfrak{B}}_{\rho}$ of $\tilde{\mathfrak{G}}$ such that each $\tilde{\mathfrak{B}}_{\rho}$ *determines* $\mathfrak{B}_{\sigma}$, in the same way as in § 3 of [7]. In consequence of this, we have the following theorems.

THEOREM 1. *For $\alpha=1,2,\cdots,\tilde{l}$, we have*

$$K(R\tilde{P}_{\alpha})\Delta_{\sigma}=\sum_{\beta=1}^{\tilde{l}} a_{\alpha\beta}^{\sigma}K(R\tilde{P}_{\beta})$$

*and*

$$\tilde{K}(\tilde{P}_{\alpha})\tilde{\Delta}^{(\sigma)}=\sum_{\beta=1}^{\tilde{l}} a_{\alpha\beta}^{\sigma}\tilde{K}(\tilde{P}_{\beta})$$

*with the same coefficients $a_{\alpha\beta}^{\sigma}$, where $\tilde{\Delta}^{(\sigma)}$ is the idempotent of the center $\tilde{Z}$ of the group ring of $\tilde{\mathfrak{G}}$ over $\Omega$ associated with $\tilde{\mathfrak{B}}^{(\sigma)}$.*

THEOREM 2. *If $L$ and $M$ are two elements of $\mathfrak{G}$ which belong to different $p$-regular sections of $\mathfrak{G}$, then*

$$\sum_{\chi_i\in\mathfrak{B}_{\sigma}}\chi_i(L)\chi_i(M^{-1})=0$$

*for each $p$-complementary block $\mathfrak{B}_{\sigma}$ of $\mathfrak{G}$.*[13]

[4. B] *If*

$$\sum_{\nu=1}^{n} a_{\nu}K_{\nu}\Delta_{\nu}=0 \qquad (a_{\nu}\in\Omega),$$

*then*

$$\sum_{K_{\nu}\subseteqq S(R_{\gamma})} a_{\nu}K_{\nu}\Delta_{\nu}=0$$

*for each $p$-regular section $S(R_{\gamma})$ of $\mathfrak{G}$.*

13) This is the refinement mentioned in footnote 6) of [6].

THEOREM 3. *If $\chi_i$ and $\chi_j$ are two irreducible characters of $\mathfrak{G}$ which belong to different $p$-complementary blocks of $\mathfrak{G}$, then*

$$\sum_{K_\nu \subseteqq S(R_\gamma)} c_\nu \chi_i(G_\nu)\chi_j(G_\nu^{-1})=0$$

*for each $p$-regular section $S(R_\gamma)$ of $\mathfrak{G}$.*

It was shown in [8] that each $E_\tau$ is a linear combination of the $p$-regular classes $K_\nu$ of $\mathfrak{G}$, i. e. the classes $K_\nu$ of $\mathfrak{G}$ contained in $\mathfrak{S}(1)$. Moreover, [4. A] shows that if $K_\alpha$ is contained in a $p$-section $\mathfrak{S}(P_\delta)$, then $K_\alpha E_\tau$ is a linear combination of the classes $K_\beta$ contained in $\mathfrak{S}(P_\delta)$. (1. 3) shows that if $\chi_i$ belongs to a $p$-block $B_\tau$ of $\mathfrak{G}$, then $\chi_i\varepsilon_\delta$ is a linear combination of the characters $\chi_j$ belonging to $B_\tau$; especially $\varepsilon_\delta$ is a linear combination of the characters $\chi_j$ which belong to the $p$-block $B_1$ of $\mathfrak{G}$ containing the 1-character $\chi_1$. Theorems 1 and 3 imply the corresponding results for the $p$-complementary blocks and the $p$-regular sections of $\mathfrak{G}$.

REMARK. Let $\Pi$ be a set of rational primes included in $g$ and let $\mathbf{B}_1, \mathbf{B}_2, \cdots, \mathbf{B}_u$ be the minimal sets of irreducible characters $\chi_i$ of $\mathfrak{G}$ such that $(a)$ every $\mathbf{B}_\lambda$ is not vacuous, $(b)$ every $\mathbf{B}_\lambda$ is a collection of $q$-blocks of $\mathfrak{G}$ for each $q\in\Pi$ and let $\mathbf{S}_1, \mathbf{S}_2, \cdots, \mathbf{S}_v$ be the minimal collections of classes $K_\nu$ of $\mathfrak{G}$ such that $(a)$ every $\mathbf{S}_\mu$ is not vacuous, $(b)$ every $\mathbf{S}_\mu$ is a collection of $q$-regular sections of $\mathfrak{G}$ for each rational prime $q$ outside of $\Pi$ (which is included in $g$); we shall call the sets $\mathbf{B}_\lambda$ of $\chi_i$ the *$\Pi$-blocks* of $\mathfrak{G}$ and the collections $\mathbf{S}_\mu$ of $K_\nu$ the $\Pi$-sections of $\mathfrak{G}$. By making use of the $\Pi$-blocks and $\Pi$-sections of $\mathfrak{G}$, we can generalize Theorems 2 and 3.

**5.** As an application of (2. 2′) and (2. 3), we shall determine the primitive idempotents of the character ring $\mathbf{X}_p$ of $\mathfrak{G}$ over $\mathbf{I}_p$.[14)]

For an arbitrarily given $p$-regular element $R$ of $\mathfrak{G}$, we shall use again the notations $\tilde{\mathfrak{G}}, \tilde{n}, \tilde{\chi}_j, \tilde{\omega}_j, \tilde{\alpha}_j^1$ used in § 2. Let $C_{(R)}$ be the $p$-elementary class attached to $R$, i. e. the collection of $p$-regular sections $S(R_\lambda)$ of $\mathfrak{G}$ such that each $R_\lambda$ is conjugate in $\mathfrak{G}$ to $R^y$ for a rational integer $y$ prime to the order $r$ of $R$. We may assume that each $R_\lambda$ with $S(R_\lambda)\subseteqq C_{(R)}$ is a power of $R$; $\tilde{\mathfrak{G}}$ is the normalizer of each $R_\lambda$ with $S(R_\lambda)\subseteqq C_{(R)}$ in $\mathfrak{G}$. We see from (2. 3) that if $S(R_\lambda)\subseteqq C_{(R)}$, then

$$\alpha_i^\lambda=\sum_{j=1}^{\tilde{n}} r_{ij}\tilde{\alpha}_j^1\tilde{\omega}_j(R_\lambda^{-1}) \tag{5.1}$$

where the $r_{ij}$ are defined by

$$\chi_i(\tilde{G})=\sum_{j=1}^{\tilde{n}} r_{ij}\tilde{\chi}_j(\tilde{G}) \qquad (\tilde{G}\in\tilde{\mathfrak{G}}). \tag{5.2}$$

Hence if we set

$$\delta_{(R)}=\sum_{S(R_\lambda)\subseteqq C_{(R)}} \delta_\lambda=\sum_{i=1}^{n} \alpha_i^{(R)}\chi_i, \tag{5.3}$$

then

14) K. SHIRATANI has determined the primitive idempotents of $X_p$ in his paper [12].

$$(5.4)\qquad \alpha_i^{(R)}=\sum_{j=1}^{\tilde{n}} r_{ij}\tilde{\alpha}_j^1\left\{\sum_{S(R_\lambda)\subseteqq C(R)} \tilde{\omega}_j(R_\lambda^{-1})\right\}.$$

We denote by $G$ the galois group of the field of $g'$-th roots of unity over $\mathbf{P}$. If $\sigma\in G$, then for each $R_\lambda$ with $S(R_\lambda)\subseteqq C_{(R)}$ there exists a rational integer $y_\sigma$ prime to $r$ such that $\sigma(\tilde{\omega}_j(R_\lambda))=\tilde{\omega}_j(R^{y_\sigma})$ for $j=1, 2, \cdots, \tilde{n}$. By the above assumption, $R^{y_\sigma}$ is written as $R_{\lambda(\sigma)}$ ; $\lambda(\sigma)$ will be determined uniquely. From (5.1) we see that $\sigma(\alpha_i^\lambda)=\alpha_i^{\lambda(\sigma)}$ for all $\chi_i$.

We consider similarly for each $p$-elementary class of $\mathfrak{G}$. Then the substitutions $\lambda\rightarrow\lambda(\sigma)$ define a permutations group on $k$ linearly independent $n$-dimensional vectors

$$\mathfrak{a}_1=(\alpha_1^1, \alpha_2^1, \cdots, \alpha_n^1),\ \mathfrak{a}_2=(\alpha_1^2, \alpha_2^2, \cdots, \alpha_n^2),\ \cdots,\ \mathfrak{a}_k=(\alpha_1^k, \alpha_2^k, \cdots, \alpha_n^k)$$

by putting $\sigma(\mathfrak{a}_\lambda)=\mathfrak{a}_{\lambda(\sigma)}$, $\sigma$ running over all galois substitutions of $G$. For each $p$-elementary class $C_{(R)}$ of $\mathfrak{G}$, the vectors $\mathfrak{a}_\lambda$ associated with the $p$-regular sections $S(R_\lambda)$ with $S(R_\lambda)\subseteqq C_{(R)}$ form a set of transitivity. Combining this fact with [2. A′], we see that for a collection $C$ of classes $K_\nu$ of $\mathfrak{G}$, $\sum_{K_\nu\subseteqq C} d_\nu$ belongs to $X_p$ if and only if $C$ is a collection of $p$-elementary classes of $\mathfrak{G}$. Therefore if we assume that $C_{(R_1)}, C_{(R_2)}, \cdots, C_{(R_e)}$ are the $p$-elementary classes of $\mathfrak{G}$, then we have the following:

[5. A] *In the character ring* $X_p$ *of* $\mathfrak{G}$ *over the ring* $\mathbf{I}_p$ *of rational* $p$*-integers,*

$$1=\delta_{(R_1)}+\delta_{(R_2)}+\cdots+\delta_{(R_e)}$$

*is the decomposition of the identity* 1 *into the sum of mutually orthogonal primitive idempotents.* ([12])

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