ON REPRESENTATIONS OF JORDAN ALGEBRAS

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It is known that the axioms of Lie algebras characterize the structure of subspaces of associative algebras which are closed relative to the composition $ab-ba$, on the other hand the axioms of abstract Jordan algebras do not characterize the structure of subspaces of associative algebras which are closed relative to the composition $ab+ba$. With this situation in view, N. Jacobson gave a remarkable definition for the representation of Jordan algebras with an ordinary one (special representation) [9, also see 5][1] and showed that for the study of a Jordan algebra $\mathfrak{J}$ it is important to consider the structure of an associator Lie triple system of $\mathfrak{J}$ [cf. 7, 2]. Then, it seems that it is necessary to generalize the notion of representations of $\mathfrak{J}$ on the foundation of the existence of inner derivations which have meannings from the results of C. Chevalley and R. D. Schafer [4] and N. Jacobson [8].

The purpose of this paper is to define the representations of Jordan algebras in two manners and to construct the cohomology spaces which are associated with these representations. The one stands on the notion of derivations of Jordan algebras and the other stands on the notion of Lie triple derivations of associator Lie triple systems. Recently, B. Harris defined the cohomology space of special Jordan algebras and studied its properties [6].

1. **Preliminaries.**[2] We begin this section with a recalling of the basic definitions of Jordan algebras and their representations by N. Jacobson.

A *Jordan algebra*[3] over a field Φ is a non-associative algebra defined by the following identities:

$$ab=ba, \tag{1.1}$$

$$(a^2b)a=a^2(ba). \tag{1.2}$$

A subspace of an associative algebra A which is closed relative to the composition $ab=a\cdot b+b\cdot a$ is a Jordan algebra relative to the new composition ab, where $a\cdot b$ denotes the associative composition in A. A Jordan algebra isomorphic to one obtained from a subspace of an associative algebra in the above manner is called to be special. Contrary to the theory of Lie algebras, it is known that there exist non-special (exceptional) Jordan algebras.

If the characteristic of the base field Φ is different from 2, then the axioms (1.1) and (1.2) imply

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1) Numbers in brackets refer to the references at the end of the paper.

2) The facts in this section will be found in the papers by A. A. Albert [1] and N. Jacobson [8, 9].

3) Except when the contrary is explicitly stated, throughout this paper we shall assume that the characteristic of the base field Φ is 0 and a Jordan algebra has a finite dimension.

(1.3) $$a((bc)d)+b((ca)d)+c((ab)d)=(ab)(cd)+(bc)(ad)+(ca)(bd).$$

Conversely, if the characteristic of Φ is not 3, then we have (1.2) from (1.1) and (1.3). Therefore, a Jordan algebra may be defined by (1.1) and (1.3) in the case that the characteristic of Φ is not 2 or 3. This leads us to the following two definitions of the representations of Jordan algebras given by N. Jacobson [9].

(I) A linear mapping ρ: $a\to\rho(a)$ of a Jordan algebra $\mathfrak{J}$ into the associative algebra $E(V)$ of linear transformations of a vector space V is called a *special representation* if

(1.4) $$\rho(ab)=\rho(a)\rho(b)+\rho(b)\rho(a),$$

where $(\rho(a)\rho(b))(x)=\rho(a)(\rho(b)x)$.

(II) A linear mapping ρ: $a\to\rho(a)$ of $\mathfrak{J}$ into $E(V)$ is called a *representation* if

(1.5) $$[\rho(a), \rho(bc)]+[\rho(b), \rho(ca)]+[\rho(c), \rho(ab)]=0,$$

(1.6) $$\rho(a)\rho(b)\rho(c)+\rho(c)\rho(b)\rho(a)+\rho(b(ca)) =\rho(ab)\rho(c)+\rho(bc)\rho(a)+\rho(ca)\rho(b),$$

where $[\rho(a), \rho(b)]$ denotes $\rho(a)\rho(b)-\rho(b)\rho(a)$.

It is easy to see that the special representation is a representation in the sense defined in (II). In a Jordan algebra $\mathfrak{J}$, the left multiplication $L(a)$: $x\to ax(=xa)$ satisfies (1.5) and (1.6), hence L is a representation (II) in $\mathfrak{J}$. This representation is called a regular representation.

Combining (1.5) and (1.6) we have

(1.7) $$\rho(a)\rho(b)\rho(c)+\rho(c)\rho(b)\rho(a)+\rho(b(ca)) =\rho(a)\rho(bc)+\rho(b)\rho(ca)+\rho(c)\rho(ab).$$

Therefore, by using the regular representation for $\mathfrak{J}$, we see that it holds the following identity in a Jordan algebra $\mathfrak{J}$:

(1.8) $$a(b(cd))+c(b(ad))+d(b(ca))=a(d(bc))+b(d(ca))+c(d(ab)).$$

Hence, if we put

(1.9) $$[abc]=a(bc)-b(ac),$$

it follows that the linear mapping $\sum_i D_{(a_i,b_i)}$: $x\to\sum_i[a_ib_ix]$ is an inner derivation of a Jordan algebra, that is

(1.10) $$\sum_i D_{(a_i,b_i)}(xy)=(\sum_i D_{(a_i,b_i)}x)y+x(\sum_i D_{(a_i,b_i)}y).$$

Thus, we define a more general representation than (II) as follows:

DEFINITION 1.1. If a linear mapping ρ of a Jordan algebra $\mathfrak{J}$ into the associative algebra of linear transformations of a vector space V satisfies the condition (1.7), then ρ is called a *representation (III)*.

For the representation (III), from (1.7) it follows

$$\rho([abc])=[[\rho(a),\rho(b)]\rho(c)], \tag{1.11}$$

therefore, $\rho(a)$'s are a Lie triple system and $\rho(a)+\sum_i[\rho(b_i),\rho(c_i)]$'s and $\sum_i[\rho(a_i),\rho(b_i)]$'s are Lie algebras, because

$$[[\rho(a),\rho(b)],[\rho(c),\rho(d)]]=[\rho([abc]),\rho(d)]+[\rho(c),\rho([abd])].$$

By using the relation (1.10), we obtain a simple direct proof of the result of N. Jacobson concerning a Lie triple structure of a Jordan algebra.

LEMMA 1.1. (Jacobson) *In a Jordan algebra* $\mathfrak{J}$ *(over a field of characteristic* $\neq 2$*) it holds the following relations:*

$$[aab]=0, \tag{1.12}$$

$$[abc]+[bca]+[cab]=0, \tag{1.13}$$

$$[[abc]de]+[[bad]ce]+[ba[cde]]+[cd[abe]]=0, \tag{1.14}$$

that is, $\mathfrak{J}$ *is a Lie triple system with respect to the composition* $[abc]=a(bc)-b(ac)$.

PROOF. (of (1.14))

$$\begin{aligned}[ab[cde]]&=[abc](de)-[abd](ce)\\ &\qquad +c([abd]e+d[abe])-d([abc]e+c[abe])\\ &=[[abc]de]+[c[abd]e]+[cd[abe]].\end{aligned}$$

2. **Cohomology space of Jordan algebras** (1).[4] In this section we define a cohomology space of Jordan algebras for the representation (III). Hence we base our argument on the identities (1.1) and (1.8).

Let ρ be a representation (III) of a Jordan algebra $\mathfrak{J}$ into a vector space V, and let f be a $2n$-linear mapping of $\underbrace{\mathfrak{J}\times\cdots\times\mathfrak{J}}_{2n\ times}$ into V satisfying

$$f(x_1,x_2,\cdots,x_{2n-2},x,y)-f(x_1,x_2,\cdots,x_{2n-2},y,x)=0.$$

We denote by $C^{2n}(\mathfrak{J},V)\,(n=0,1,2,\cdots)$ the vector space spanned by such $2n$-linear mappings, where $C^0(\mathfrak{J},V)=V$ by definition. Also, we consider the vector space $C^1(\mathfrak{J},V)$ spanned by linear mappings of $\mathfrak{J}$ into V.

Next, we define a linear mapping δ of $C^n(\mathfrak{J},V)$ into $C^{n+1}(\mathfrak{J},V)$ $(n=0,1)$ and of $C^{2n}(\mathfrak{J},V)$ into $C^{2n+2}(\mathfrak{J},V)$ $(n=1,2,3,\cdots)$ as follows:

$$(\delta f)(x)=\rho(x)f \qquad \text{for } f\in C^0(\mathfrak{J},V), \tag{2.1}$$

$$(\delta f)(x_1,x_2)=\rho(x_1)f(x_2)+\rho(x_2)f(x_1)-f(x_1x_2) \qquad \text{for } f\in C^1(\mathfrak{J},V), \tag{2.2}$$

4) cf. [3, 10].

$$
\begin{aligned}
(2.3)\qquad & (\delta f)(x_1, x_2, \cdots, x_{2n+2}) \\
&= (-1)^n\big[\rho(x_{2n+1})\rho(x_{2n-1})f(x_1,\cdots,x_{2n-2},x_{2n},x_{2n+2}) - \rho(x_{2n+1})\rho(x_{2n})f(x_1,\cdots,x_{2n-1},x_{2n+2}) \\
&\quad + \rho(x_{2n+2})\rho(x_{2n-1})f(x_1,\cdots,x_{2n-2},x_{2n},x_{2n+1}) - \rho(x_{2n+2})\rho(x_{2n})f(x_1,\cdots,x_{2n-1},x_{2n+1}) \\
&\quad - \rho(x_{2n-1})f(x_1,\cdots,x_{2n-2},x_{2n},x_{2n+1}x_{2n+2}) + \rho(x_{2n})f(x_1,\cdots,x_{2n-1},x_{2n+1}x_{2n+2}) \\
&\quad + \rho(x_{2n+1})f(x_1,\cdots,x_{2n-1},x_{2n}x_{2n+2}) - \rho(x_{2n+1})f(x_1,\cdots,x_{2n-2},x_{2n},x_{2n-1}x_{2n+2}) \\
&\quad + \rho(x_{2n+2})f(x_1,\cdots,x_{2n-1},x_{2n}x_{2n+1}) - \rho(x_{2n+2})f(x_1,\cdots,x_{2n-2},x_{2n},x_{2n-1}x_{2n+1}) \\
&\quad - f(x_1,\cdots,x_{2n-1},x_{2n}(x_{2n+1}x_{2n+2})) + f(x_1,\cdots,x_{2n-2},x_{2n},x_{2n-1}(x_{2n+1}x_{2n+2}))\big] \\
&\quad + \sum_{k=1}^{n}(-1)^{k+1}[\rho(x_{2k-1}),\rho(x_{2k})]f(x_1,x_2,\cdots,\hat{x}_{2k-1},\hat{x}_{2k},\cdots,x_{2n+2}) \\
&\quad + \sum_{k=1}^{n}\sum_{j=2k+1}^{2n+2}(-1)^k f(x_1,x_2,\cdots,\hat{x}_{2k-1},\hat{x}_{2k},\cdots,[x_{2k-1}x_{2k}x_j],\cdots,x_{2n+2}) \\
&\qquad\qquad\qquad \text{for}\quad f\in C^{2n}(\mathfrak{J},V),\ n=1,2,3,\cdots,
\end{aligned}
$$

where the sign ^ over a letter indicates that this letter is to be omitted.

For instance, if $f\in C^2(\mathfrak{J},V)$, then

$$
\begin{aligned}
&-(\delta f)(x_1,x_2,x_3,x_4) \\
&= \rho(x_3)\rho(x_1)f(x_2,x_4) - \rho(x_3)\rho(x_2)f(x_1,x_4) + \rho(x_4)\rho(x_1)f(x_2,x_3) - \rho(x_4)\rho(x_2)f(x_1,x_3) \\
&\quad - \rho(x_1)f(x_2,x_3x_4) + \rho(x_2)f(x_1,x_3x_4) + \rho(x_3)f(x_1,x_2x_4) - \rho(x_3)f(x_2,x_1x_4) \\
&\quad + \rho(x_4)f(x_1,x_2x_3) - \rho(x_4)f(x_2,x_1x_3) - [\rho(x_1),\rho(x_2)]f(x_3,x_4) \\
&\quad - f(x_1,x_2(x_3x_4)) + f(x_2,x_1(x_3x_4)) + f([x_1x_2x_3],x_4) + f(x_3,[x_1x_2x_4]).
\end{aligned}
$$

If $f\in C^0(\mathfrak{J},V)$, then for every $x_1, x_2\in\mathfrak{J}$

$$(\delta\delta f)(x_1,x_2) = (\rho(x_1)\rho(x_2)+\rho(x_2)\rho(x_1)-\rho(x_1x_2))f,$$

hence $\delta\delta f=0$ for all $f\in C^0(\mathfrak{J},V)$ if and only if the representation ρ reduces to the special representation.

But, we shall prove that $\delta\delta f=0$ for any $f\in C^{2n}(\mathfrak{J},V)(n=1,2,3,\cdots)$ in the sequel. If $f\in C^2(\mathfrak{J},V)$, then this fact follows by a direct computation. In order to prove the general case, we consider the following two operations.

For $a, b\in\mathfrak{J}$, we define a linear mapping $\kappa(a,b)$ of $C^{2n}(\mathfrak{J},V)$ into $C^{2n}(\mathfrak{J},V)$ and a linear mapping $\iota(a,b)$ of $C^{2n}(\mathfrak{J},V)$ into $C^{2n-2}(\mathfrak{J},V)$ by the following formulas respectively

$$(2.4)\qquad (\kappa(a,b)f)(x_1,\cdots,x_{2n}) = [\rho(a),\rho(b)]f(x_1,\cdots,x_{2n}) - \sum_{j=1}^{2n} f(x_1,\cdots,[abx_j],\cdots,x_{2n}),$$

$$(2.5)\qquad (\iota(a,b)f)(x_1,\cdots,x_{2n-2}) = f(a,b,x_1,\cdots,x_{2n-2}),\qquad n=2,3,\cdots.$$

By a direct calculation we have the following two formulas:

$$(2.6)\qquad \iota(a,b)\delta f + \delta\iota(a,b)f = \kappa(a,b)f \qquad \text{for } f\in C^{2n}(\mathfrak{J},V),\quad n=2,3,\cdots,$$

$$(2.7)\qquad [\kappa(a,b),\iota(c,d)]f = \iota([abc],d)f + \iota(c,[abd])f \qquad \text{for } f\in C^{2n}(\mathfrak{J},V),\quad n=2,3,\cdots.$$

Next, it holds the following relation:

(2.8) $$[\kappa(a,b),\kappa(c,d)]f=\kappa([abc],d)f+\kappa(c,[abd])f,$$
for $f\in C^{2n}(\mathfrak{J},V)$, $n=2,3,\cdots$.

Since for $f\in C^4(\mathfrak{J},V)$ we can prove (2.8) directly, we assume that (2.8) holds for all $f\in C^{2n}(\mathfrak{J},V)$ and let $f\in C^{2n+2}(\mathfrak{J},V)$, $n\geqq 2$, then for arbitrary $k,l\in\mathfrak{J}$ we have

$$\begin{aligned}&\iota(k,l)([\kappa(a,b),\kappa(c,d)]-\kappa([abc],d)-\kappa(c,[abd]))f\\&=([\kappa(a,b),\kappa(c,d)]-\kappa([abc],d)-\kappa(c,[abd]))\iota(k,l)f\\&\quad+(\iota([cd[abk]],l)-\iota([ab[cdk]],l)+\iota([[abc]dk],l)+\iota([c[abd]k],l))f\\&\quad+(\iota(k,[[abc]dl])+\iota(k,[c[abd]l])+\iota(k,[cd[abl]])-\iota(k,[ab[cdl]]))f\\&=0,\end{aligned}$$

by (1.14) and (2.7). Therefore, (2.8) holds for all $f\in C^{2n+2}(\mathfrak{J},V)$.

Moreover, it holds that

(2.9) $$\kappa(a,b)\delta f=\delta\kappa(a,b)f,\qquad \text{for } f\in C^{2n}(\mathfrak{J},V),\quad n=2,3,\cdots.$$

If $f\in C^4(\mathfrak{J},V)$, then we obtain (2.9) directly, hence we assume that (2.9) holds for all $f\in C^{2n}(\mathfrak{J},V)$. Then for $f\in C^{2n+2}(\mathfrak{J},V)$, $n\geqq 2$, and every $k,l\in\mathfrak{J}$

$$\begin{aligned}&\iota(k,l)(\kappa(a,b)\delta-\delta\kappa(a,b))f\\&=\kappa(a,b)\kappa(c,d)f-\kappa(c,d)\kappa(a,b)f-\kappa(a,b)\delta\iota(c,d)f+\delta\kappa(a,b)\iota(c,d)f\\&\qquad-\iota([abc],d)\delta f-\delta\iota([abc],d)f-\iota(c,[abd])\delta f-\delta\iota(c,[abd])f\\&=0,\end{aligned}$$

by (2.6). Therefore, (2.9) holds for all $f\in C^{2n+2}(\mathfrak{J},V)$.

Next we see that

(2.10) $$\delta\delta f=0$$

for all $f\in C^{2n}(\mathfrak{J},V)$, $n=1,2,3,\cdots$.

We assume that (2.10) has been proved for all $f\in C^{2n}(\mathfrak{J},V)$, then for every $a,b\in\mathfrak{J}$ and $f\in C^{2n+2}(\mathfrak{J},V)$, $n\geqq 1$, by using (2.6) and (2.9) we have

$$\begin{aligned}\iota(a,b)(\delta\delta f)&=\kappa(a,b)\delta f-\delta\iota(a,b)\delta f\\&=\delta\delta\iota(a,b)f\\&=0.\end{aligned}$$

Thus we obtain the following

THEOREM 2.1. *For the operator δ defined above, it holds that $\delta\delta f=0$ for all $f\in C^{2n}(\mathfrak{J},V)$, $n=1,2,3,\cdots$. This relation holds for all $f\in C^0(\mathfrak{J},V)$ if and only if the representation ρ is a special representation.*

Let $Z^{2n}(\mathfrak{J}, V)$ be a subspace spanned by elements f of $C^{2n}(\mathfrak{J}, V)$ such that $\delta f=0$, and let $B^{2n}(\mathfrak{J}, V)$ be a subspace spanned by elements of $C^{2n}(\mathfrak{J}, V)$ of the form δf, then by Theorem 2.1 $B^{2n}(\mathfrak{J}, V)$ is a subspace of $Z^{2n}(\mathfrak{J}, V)$. Therefore, we can define a cohomology space $H^{2n}(\mathfrak{J}, V)$ of order $2n$ of a Jordan algebra $\mathfrak{J}$ as the factor space $Z^{2n}(\mathfrak{J}, V)/B^{2n}(\mathfrak{J}, V)$, where $n=1, 2, 3, \cdots$.

REMARK 2.1. For $f\in C^1(\mathfrak{J}, V)$ we have $\delta\delta f=0$, because

$$\begin{aligned}
&(\delta\delta f)(x_1, x_2, x_3, x_4)\\
&=(\rho(x_3)\rho(x_2)\rho(x_4)+\rho(x_4)\rho(x_2)\rho(x_3)+\rho(x_2(x_3x_4))\\
&\qquad -\rho(x_2)\rho(x_3x_4)-\rho(x_3)\rho(x_4x_2)-\rho(x_4)\rho(x_2x_3))f(x_1)\\
&\quad -(\rho(x_3)\rho(x_1)\rho(x_4)+\rho(x_4)\rho(x_1)\rho(x_3)+\rho(x_1(x_3x_4))\\
&\qquad -\rho(x_1)\rho(x_3x_4)-\rho(x_3)\rho(x_4x_1)-\rho(x_4)\rho(x_1x_3))f(x_2)\\
&\quad +([[\rho(x_1), \rho(x_2)]\rho(x_4)]-\rho([x_1x_2x_4]))f(x_3)\\
&\quad +([[\rho(x_1), \rho(x_2)]\rho(x_3)]-\rho([x_1x_2x_3]))f(x_4)\\
&\quad -f(x_1(x_2(x_3x_4)))+f(x_2(x_1(x_3x_4)))+f([x_1x_2x_3]x_4)+f(x_3[x_1x_2x_4])\\
&=0.
\end{aligned}$$

REMARK 2.2. For $f\in C^0(\mathfrak{J}, V)$ we can define the coboundary operator δ of $C^0(\mathfrak{J}, V)$ into $C^2(\mathfrak{J}, V)$ as follows:

$$(\delta f)(x_1, x_2)=(\rho(x_1)\rho(x_2)+\rho(x_2)\rho(x_1)-\rho(x_1x_2))f.$$

Then, $\delta\delta f=0$, hence in this case we can define the cohomology space $H^{2n}(\mathfrak{J}, V)$ for all non-negative integer n, and the gap in Theorem 2.1 will be filled.

3. **Cohomology space of Jordan algebras** (2).

In this section we define a cohomology space of Jordan algebras as associator Lie triple systems. For this purpose we generalize the representation (II) of Jordan algebras.

DEFINITION 3.1. Let $\rho: a\to\rho(a)$ be a linear mapping of a Jordan algebra $\mathfrak{J}$ into the associative algebra of linear transformations of a vector space V. This mapping is called a *representation* (*IV*) of $\mathfrak{J}$ if $\rho(a)$ satisfies the following relations:

$$\begin{aligned}
(3.1)\quad &(\rho(cd)-\rho(c)\rho(d))(\rho(ab)-\rho(a)\rho(b))-(\rho(bd)-\rho(b)\rho(d))(\rho(ac)-\rho(a)\rho(c))\\
&=\rho(a[bcd])-\rho(a)\rho([bcd])-[\rho(b), \rho(c)](\rho(ad)-\rho(a)\rho(d)),\\
(3.2)\quad &[[\rho(a), \rho(b)]\ \rho(cd)-\rho(c)\rho(d)]\\
&=\rho([a, b, cd])-\rho([abc])\rho(d)-\rho(c)\rho([abd]).
\end{aligned}$$

Then we have the following

THEOREM 3.1. *Let $\rho: a\to\rho(a)$ be a representation* (II) *of a Jordan algebra $\mathfrak{J}$ into a vector space, then $\rho(a)$ satisfies the conditions* (3.1) *and* (3.2).

PROOF.

$$(\rho(cd)-\rho(c)\rho(d))(\rho(ab)-\rho(a)\rho(b))-(\rho(bd)-\rho(b)\rho(d))(\rho(ac)-\rho(a)\rho(c))$$
$$-\rho(a[bcd])+\rho(a)\rho([bcd])+[\rho(b),\rho(c)](\rho(ad)-\rho(a)\rho(d))$$
$$=\rho(cd)(\rho(ab)-\rho(a)\rho(b))-\rho(bd)(\rho(ac)-\rho(a)\rho(c))-\rho(a[bcd])+\rho(a)\rho([bcd])$$
$$-\rho(c)(\rho(a(bd))-\rho(a)\rho(bd))+\rho(b)(\rho(a(cd))-\rho(a)\rho(cd))$$
$$=\rho(ab)\rho(cd)+\rho(b(cd))\rho(a)+\rho(a(cd))\rho(b)-\rho(ac)\rho(bd)$$
$$-\rho(c(bd))\rho(a)-\rho(a(bd))\rho(c)-\rho(cd)\rho(a)\rho(b)+\rho(bd)\rho(a)\rho(c)$$
$$-\rho(a[bcd])+\rho(c)\rho(a)\rho(bd)-\rho(b)\rho(a)\rho(cd)$$
$$=0,$$

hence, (3.1) was proved. Next, we shall prove (3.2).

$$[\rho(cd)-\rho(c)\rho(d),[\rho(a),\rho(b)]]+\rho([a,b,cd])-\rho([abc])\rho(d)-\rho(c)(\rho[abd])$$
$$=[\rho(cd)[\rho(a),\rho(b)]]+\rho([a,b,cd])$$
$$+\rho(c)([[\rho(a),\rho(b)]\rho(d)]-\rho([abd]))$$
$$+([[\rho(a),\rho(b)]\rho(c)]-\rho([abc]))\rho(d)$$
$$=0.$$

Now, for the representation (IV) $\rho\colon a\to\rho(a)$ of a Jordan algebra $\mathfrak{J}$ into a vector space V, if we put

$$\theta(a,b)=\rho(ab)-\rho(a)\rho(b),$$
$$D(a,b)=\theta(b,a)-\theta(a,b),$$

then the conditions (3.1) and (3.2) can be rewritten as

(3.1)′ $$\theta(c,d)\theta(a,b)-\theta(b,d)\theta(a,c)-\theta(a,[bcd])+D(b,c)\theta(a,d)=0,$$

(3.2)′ $$[D(a,b),\theta(c,d)]=\theta([abc],d)+\theta(c,[abd]).$$

(3.1)′ and (3.2)′ are the conditions for the representation of Lie triple systems in [10], hence by Lemma 1.1 the representation space V becomes a $\mathfrak{T}$-module [10, Definition 2]. Especially, for a regular representation in a Jordan algebra $\mathfrak{J}$, $D(a,b)$ is an inner derivation in $\mathfrak{J}$.

Let $C^n(\mathfrak{J},V)$ be a vector space spanned by n-linear mappings f of $\underbrace{\mathfrak{J}\times\cdots\times\mathfrak{J}}_{n\ times}$ into a $\mathfrak{T}$-module V such that

$$f(x_1,x_2,\cdots,x_{n-3},x,x,x_n)=0$$

and

$$f(x_1,x_2,\cdots,x_{n-3},x,y,z)+f(x_1,x_2,\cdots,x_{n-3},y,z,x)+f(x_1,x_2,\cdots,x_{n-3},z,x,y)=0,$$

where we define $C^0(\mathfrak{J},V)=V$.

A linear mapping δ of $C^n(\mathfrak{J},V)$ into $C^{n+2}(\mathfrak{J},V)$ is defined by the following formulas:

$$(\delta f)(x_1, x_2)=\theta(x_1, x_2)f \qquad \text{for} \quad f\in C^0(\mathfrak{J}, V), \tag{3.3}$$

$$\begin{aligned}(\delta f)&(x_1, x_2,\cdots, x_{2n+1})\\ &=\theta(x_{2n}, x_{2n+1})f(x_1, x_2,\cdots, x_{2n-1})-\theta(x_{2n-1}, x_{2n+1})f(x_1, x_2,\cdots, x_{2n-2}, x_{2n})\\ &\quad+\sum_{k=1}^{n}(-1)^{n+k}D(x_{2k-1}, x_{2k})f(x_1, x_2,\cdots, \hat{x}_{2k-1}, \hat{x}_{2k},\cdots ,x_{2n+1})\\ &\quad+\sum_{k=1}^{n}\sum_{j=2k+1}^{2n+1}(-1)^{n+k+1}f(x_1, x_2,\cdots, \hat{x}_{2k-1}, \hat{x}_{2k},\cdots, [x_{2k-1}x_{2k}x_j],\cdots, x_{2n+1})\\ &\qquad\qquad \text{for } f\in C^{2n-1}(\mathfrak{J}, V),\quad n=1, 2, 3, \cdots,\end{aligned} \tag{3.4}$$

$$\begin{aligned}(\delta f)&(y, x_1, x_2,\cdots, x_{2n+1})\\ &=\theta(x_{2n}, x_{2n+1})f(y, x_1, x_2,\cdots, x_{2n-1})-\theta(x_{2n-1}, x_{2n+1})f(y, x_1, x_2,\cdots, x_{2n-2}, x_{2n})\\ &\quad+\sum_{k=1}^{n}(-1)^{n+k}D(x_{2k-1}, x_{2k})f(y, x_1, x_2,\cdots, \hat{x}_{2k-1}, \hat{x}_{2k},\cdots, x_{2n+1})\\ &\quad+\sum_{k=1}^{n}\sum_{j=2k+1}^{2n+1}(-1)^{n+k+1}f(y, x_1, x_2,\cdots, \hat{x}_{2k-1}, \hat{x}_{2k},\cdots, [x_{2k-1}x_{2k}x_j],\cdots, x_{2n+1})\\ &\qquad\qquad \text{for } f\in C^{2n}(\mathfrak{J}, V),\quad n=1, 2, 3, \cdots,\end{aligned} \tag{3.5}$$

where the sign ^ over a letter indicates that this letter is to be omitted. Then, from [10, Theorem 1] we have $\delta\delta f=0$ for any $f\in C^n(\mathfrak{J}, V)(n=0, 1, 2, \cdots)$, hence we can define a cohomology space $H^n(\mathfrak{J}, V)$ of order n for a Jordan algebra $\mathfrak{J}$ as the factor space $Z^n(\mathfrak{J}, V)/B^n(\mathfrak{J}, V)$, where $Z^n(\mathfrak{J}, V)$ is a subspace of $C^n(\mathfrak{J}, V)$ spanned by f such that $\delta f=0$ and $B^n(\mathfrak{J}, V)=\delta C^{n-2}(\mathfrak{J}, V)$.

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