SOME STUDIES ON THE ORTHOGONALITY RELATIONS FOR GROUP CHARACTERS

Kenzo IIZUKA

(Received April 24, 1961)

R. BRAUER, in his paper [1], gave an important refinement of some of the orthogonality relations for group characters.[1] He and M. OSIMA gave independently another refinement, [9].[2] In [10], [5] ([4]) and [6], other refinements were discussed. In the present note, we shall extend the results to a more general case.

The author wishes to thank Professor M. OSIMA and Professor T. NAKAYAMA for their helpful suggestions.

1. Preliminaries.

Let $\mathfrak{G}$ be a group of finite order g and let $p_1, p_2, \cdots, p_r$ be the rational primes dividing g: $g=p_1^{a_1}p_2^{a_2}\cdots p_r^{a_r}, a_i>0$. We shall denote by $K_1, K_2, \cdots, K_n$ the classes of conjugate elements in $\mathfrak{G}$; $\mathfrak{G}$ has n distinct absolutely irreducible (ordinary) characters, $\chi_1, \chi_2, \cdots, \chi_n$.

1. 1. We shall consider the subsets of the set $\{p_1, p_2, \cdots, p_r\}$: Π_0=empty set, $\Pi_1=\{p_1, p_2, \cdots, p_r\}$, $\Pi_2, \cdots, \Pi_{2^r-1}$. If $\bigcup$ and $\bigcap$ respectively mean the set theoretical union and intersection, then the subsets Π_f form a lattice with respect to two operations $\bigcup$ and $\bigcap$, which has the maximum element Π_1 and the minimum element Π_0. We put $\pi_f=\prod_{p_i\in\Pi_f} p_i^{a_i}$ $(1\leqq f\leqq 2^r-1)$ and $\pi_0=1$.

For each Π_f, we can find the maximum one, say $\mathfrak{N}_{\Pi_f}$, among the normal subgroups of $\mathfrak{G}$ whose orders are prime to π_f. The following facts are easily seen:

1) $\mathfrak{N}_{\Pi_0}=\mathfrak{G}$, $\mathfrak{N}_{\Pi_1}=\{1\}$.
2) $\Pi_f\supseteqq\Pi_h$ implies $\mathfrak{N}_{\Pi_f}\subseteqq\mathfrak{N}_{\Pi_h}$.
3) $\mathfrak{N}_{\Pi_f\cup\Pi_h}=\mathfrak{N}_{\Pi_f}\cap\mathfrak{N}_{\Pi_h}$, $\mathfrak{N}_{\Pi_f\cap\Pi_h}\supseteqq\mathfrak{N}_{\Pi_f}\cdot\mathfrak{N}_{\Pi_h}$.

We shall call an element P of $\mathfrak{G}$ a Π_f-element if and only if its order is not divisible by any rational prime not belonging to Π_f, while we shall call an element R of $\mathfrak{G}$ a Π_f-regular element if and only if its order is prime to π_f. As is well known, an element G of $\mathfrak{G}$ can be uniquely expressed as a product of two commutative elements P and R where P is a *Π_f-element*, while R is a *Π_f-regular* element; we shall call P the Π_f-factor of G and R the Π_f-regular factor of G. If Π_f consists of one rational prime p, then "Π_f-" will become "p-".

The *Π_f-section* of a Π_f-element P in $\mathfrak{G}$ is the set of all elements of $\mathfrak{G}$ whose Π_f-factors are conjugate to P in $\mathfrak{G}$; we shall denote by $\mathfrak{S}^{\Pi_f}(G)$ the Π_f-section of $\mathfrak{G}$ represented by an element G. As is easily seen, each Π_f-section of $\mathfrak{G}$ is a collection

1) Cf. also [2].
2) Cf. also [3].

of classes K_ν of $\mathfrak{G}$. We choose a complete system of representatives, $P_1^{\Pi_f}=1, P_2^{\Pi_f}, \cdots, P_{s_f}^{\Pi_f}$, for the classes K_ν of $\mathfrak{G}$ which consist of Π_f-elements. It is easily seen that the elements of $\mathfrak{G}$ are distributed into the s_f Π_f-sections $\mathfrak{S}_\gamma^{\Pi_f}=\mathfrak{S}^{\Pi_f}(P_\gamma^{\Pi_f})$ such that each element of $\mathfrak{G}$ belongs to exactly one $\mathfrak{S}_\gamma^{\Pi_f}$. If Π_f consists of one rational prime p, then the Π_f-sections of $\mathfrak{G}$ are the p-sections of $\mathfrak{G}$. On the other hand, if Π_f consists of $r-1$ rational primes, then the Π_f-sections of $\mathfrak{G}$ are the p-regular sections of $\mathfrak{G}$, where p is the rational prime in Π_1 not belonging to Π_f.

The following facts are easily seen:

4) Only Π_0-section of $\mathfrak{G}$ is $\mathfrak{G}$ itself, while the Π_1-sections of $\mathfrak{G}$ are the classes K_ν of $\mathfrak{G}$.

5) If $\Pi_f \supseteqq \Pi_h$, then each Π_h-section of $\mathfrak{G}$ is a collection of Π_f-sections of $\mathfrak{G}$.

1. 2. The *Π_f-block* $\boldsymbol{B}^{\Pi_f}(\chi_i)$ of irreducible characters of $\mathfrak{G}$ which is represented by an irreducible character χ_i is the set of all irreducible characters χ_j of $\mathfrak{G}$ such that each χ_j is connected to χ_i by a chain of irreducible characters of $\mathfrak{G}$,

$$\chi_i,\ \chi_\lambda,\ \cdots,\ \chi_\rho,\ \chi_j,$$

in which any two consecutive χ_α and χ_β belong to a p-block of $\mathfrak{G}$, where $p=p_{\varphi(\alpha,\beta)}$ is a rational prime in Π_f. It is understood that each irreducible character χ_i of $\mathfrak{G}$ itself forms a Π_0-block of $\mathfrak{G}$. We denote by $\boldsymbol{B}_1^{\Pi_f}, \boldsymbol{B}_2^{\Pi_f}, \cdots, \boldsymbol{B}_{t_f}^{\Pi_f}$ the Π_f-blocks of $\mathfrak{G}$. If Π_f consists of one rational prime p, then the Π_f-blocks of $\mathfrak{G}$ are the p-blocks of $\mathfrak{G}$. On the other hand, if Π_f consists of $r-1$ rational primes and p is the rational prime in Π_1 not belonging to Π_f,then the Π_f-blocks of $\mathfrak{G}$ are the p-complementary blocks of $\mathfrak{G}$.

The following facts are easily seen:

1) If $\Pi_f \supseteqq \Pi_h$, then each Π_f-block of $\mathfrak{G}$ is a collection of Π_h-blocks of $\mathfrak{G}$.

2) $\boldsymbol{B}^{\Pi_f \cup \Pi_h}(\chi_i) \supseteqq \boldsymbol{B}^{\Pi_f}(\chi_i) \cup \boldsymbol{B}^{\Pi_h}(\chi_i)$.

1. 3. Let $\mathfrak{N}$ be a normal subgroup of $\mathfrak{G}$ and let $\mathfrak{B}_1^{\mathfrak{N}}, \mathfrak{B}_2^{\mathfrak{N}}, \cdots, \mathfrak{B}_{u(\mathfrak{N})}^{\mathfrak{N}}$ be the $\mathfrak{N}$-blocks[3] of irreducible characters χ_i of $\mathfrak{G}$. It is well known that the classes of associated irreducible characters θ_λ of $\mathfrak{N}$ in $\mathfrak{G}$ one-one correspond to the $\mathfrak{N}$-blocks of $\mathfrak{G}$; we shall denote by $\mathfrak{U}_\sigma^{\mathfrak{N}}$ the class of associated irreducible characters of $\mathfrak{N}$ which corresponds to $\mathfrak{B}_\sigma^{\mathfrak{N}}$. If we denote by Φ_σ the sum of all irreducible characters θ_λ in $\mathfrak{U}_\sigma^{\mathfrak{N}}$, then for each irreducible character χ_i in $\mathfrak{B}_\sigma^{\mathfrak{N}}$

$$\chi_i(N)=s_{i\sigma}\Phi_\sigma(N) \qquad (N\in\mathfrak{N}),$$

where $s_{i\sigma}$ is a positive rational integer.

The following facts are well known:

1) Two irreducible characters χ_i and χ_j of $\mathfrak{G}$ belong to a same $\mathfrak{N}$-block of $\mathfrak{G}$ if and only if $\chi_i(N)/\chi_i(1)=\chi_j(N)/\chi_j(1)$ holds for all elements N of $\mathfrak{N}$.

2) If we denote by $\phi^{\mathfrak{G}}$ the character of $\mathfrak{G}$ which is induced by a character ϕ of $\mathfrak{N}$, then

$$\theta_\lambda^{\mathfrak{G}}(G)=\sum_{\chi_i\in\mathfrak{B}_\sigma^{\mathfrak{N}}} s_{i\sigma}\chi_i(G) \qquad (G\in\mathfrak{G}),$$

3) Cf. [7]. Cf. also [5] or [4].

where θ_λ is an irreducible character in $\mathfrak{U}_\sigma^{\mathfrak{M}}$.

3) If $\mathfrak{M} \supseteqq \mathfrak{N}$, then every $\mathfrak{N}$-block of $\mathfrak{G}$ is a collection of $\mathfrak{M}$-blocks of $\mathfrak{G}$.

4) If the order of $\mathfrak{N}$ is prime to π_f, then every $\mathfrak{N}$-block of $\mathfrak{G}$ is a collection of $\mathfrak{N}_{\Pi_f}$-blocks of $\mathfrak{G}$.

5) If $\Pi_f \supseteqq \Pi_h$, then every $\mathfrak{N}_{\Pi_f}$-block of $\mathfrak{G}$ is a collection of $\mathfrak{N}_{\Pi_h}$-blocks of $\mathfrak{G}$.

2. Π_f-blocks.

2. 1. Let Ω be the field of g-th roots of unity and let Z be the center of the group ring Γ of $\mathfrak{G}$ over Ω. We denote by e_i the primitive idempotent of Z which is associated with an irreducible character χ_i of $\mathfrak{G}$:

$$e_i = \frac{1}{g} \sum_{\nu=1}^{n} \chi_i(1)\chi_i(G_\nu^{-1})K_\nu,$$

where G_ν is a representative element of K_ν and each class K_ν also denotes the sum of all its elements.

Let p be a rational prime in Π_1 and let $\mathfrak{o}_\mathfrak{p}$ be the ring of all $\mathfrak{p}$-integers in Ω, where $\mathfrak{p}$ is a prime ideal divisor of p in Ω. It is well known that if, for each p-block B_τ of $\mathfrak{G}$, we denote by E_τ the idempotent of Z which is associated with B_τ, then the idempotents E_τ are the primitive idempotents of the center Z_0 of the group ring Γ_0 of $\mathfrak{G}$ over $\mathfrak{o}_\mathfrak{p}$ and that each E_τ is a linear combination of p-regular classes K_ν of $\mathfrak{G}$.[4)]

We denote by $\boldsymbol{E}_\delta^{\Pi_f}$ the idempotent of Z which is associated with a Π_f-block $\boldsymbol{B}_\delta^{\Pi_f}$ of $\mathfrak{G}$:

$$\boldsymbol{E}_\delta^{\Pi_f} = \sum_{\chi_i \in B_\delta^{\Pi_f}} e_i.$$

We set

$$\boldsymbol{E}_\delta^{\Pi_f} = \sum_\nu \beta_{\delta,\nu}^{\Pi_f} K_\nu.$$

[2. 1. A] *The Π_f-blocks $\boldsymbol{B}_\delta^{\Pi_f}$ of $\mathfrak{G}$ are characterized as the minimal sets $\boldsymbol{B}$ of irreducible characters χ_i of $\mathfrak{G}$ such that (a) each $\boldsymbol{B}$ is a collection of q-blocks of $\mathfrak{G}$ for any rational prime q in Π_f, (b) each $\boldsymbol{B}$ is not vacuous.*

[2. 1. B] *$\beta_{\delta,\nu}^{\Pi_f}$ can be different from zero only for Π_f-regular classes K_ν of $\mathfrak{G}$ (i. e. classes K_ν of $\mathfrak{G}$ which consist of Π_f-regular elements). The $\beta_{\delta,\nu}^{\Pi_f}$ multiplied by g/π_f are algebraic integers.*

PROOF. Since $\boldsymbol{B}_\delta^{\Pi_f}$ is a collection of q-blocks of $\mathfrak{G}$ for each rational prime q in Π_f, $\beta_{\delta,\nu}^{\Pi_f}$ can differ from zero only for Π_f-regular classes K_ν of $\mathfrak{G}$. Since, further, the $\beta_{\delta,\nu}^{\Pi_f}$ multiplied by g are algebraic integers, the $g/\pi_f \cdot \beta_{\delta,\nu}^{\Pi_f}$ are algebraic integers.

[2. 1. C] *If $\boldsymbol{B}$ is a set of irreducible characters χ_i of $\mathfrak{G}$ such that the coefficients β_ν of*

$$\alpha \cdot \sum_{\chi_i \in B} e_i = \sum_\nu \beta_\nu K_\nu$$

4) Cf. [8].

are algebraic integers, then $\boldsymbol{B}$ *is a collection of* Π_f*-blocks* $\boldsymbol{B}_\delta^{\Pi_f}$ *of* $\mathfrak{G}$, *where* α *is a product of powers of the rational primes in* Π_1 *not belonging to* Π_f.

PROOF. If the β_ν are algebraic integers for an α, then $\boldsymbol{B}$ is a collection of q-blocks of $\mathfrak{G}$ for each rational prime q in Π_f. Hence, it is easily seen from [2.1.A] that $\boldsymbol{B}$ is a collection of Π_f-blocks of $\mathfrak{G}$.

As a special case, we have

[2.1.D] *Only* Π_1*-block of* $\mathfrak{G}$ *is the set of all irreducible characters* χ_i *of* $\mathfrak{G}$. *Only primitive idempotent of the center of the group ring of* $\mathfrak{G}$ *over the ring of all rational integers is the identity* 1.

2. 2. Let $\Pi_f=\{q_1, q_2, \cdots, q_u\}$ be an arbitrarily given subset of Π_1 and let P be a Π_f-element of $\mathfrak{G}$. We consider the normalizer, $\tilde{\mathfrak{G}}$, of P in $\mathfrak{G}$. If $\tilde{K}_1, \tilde{K}_2, \cdots, \tilde{K}_{\tilde{k}}$ are the Π_f-regular classes of $\tilde{\mathfrak{G}}$ (strictly speaking, the $\tilde{\Pi}_f$-regular classes of $\tilde{\mathfrak{G}}$ where $\tilde{\Pi}_f$ is the set of all rational primes in Π_f which divide the order $\tilde{g}$ of $\tilde{\mathfrak{G}}$) then, for any different $\alpha, \beta\ (1\leqq\alpha, \beta\leqq\tilde{k})$, $P\tilde{K}_\alpha$ and $P\tilde{K}_\beta$ cannot be contained in a same class K_ν of $\mathfrak{G}$. Hence, arranging the classes K_ν of $\mathfrak{G}$ in a suitable order, we may assume that each $P\tilde{K}_\alpha$ is contained in a class K_α of $\mathfrak{G}\ (\alpha=1, 2, \cdots, \tilde{k})$; $K_1, K_2, \cdots, K_{\tilde{k}}$ are the classes of $\mathfrak{G}$ which are contained in the Π_f-section $\mathfrak{S}^{\Pi_f}(P)$ of P in $\mathfrak{G}$.

It is well known that P is uniquely expressed as a product

$$P=Q_1, Q_2, \cdots, Q_u$$

where Q_i is the q_i-factor of $P\ (1\leqq i\leqq u)$. First, for a Π_f-block $\boldsymbol{B}_\delta^{\Pi_f}$ of $\mathfrak{G}$, we consider the collection $\boldsymbol{B}_{(\delta)}^{\Pi_f}(Q_1)$ of q_1-blocks $B_\rho^{Q_1}$ of the normalizer $\mathfrak{G}_1$ of Q_1 in $\mathfrak{G}$ such that each $B_\rho^{Q_1}$ is associated (in Brauer's sense) with a q_1-block of $\mathfrak{G}$ which is contained in $\boldsymbol{B}_\delta^{\Pi_f}$. It is easy to see that $\boldsymbol{B}_{(\delta)}^{\Pi_f}(Q_1)$ is a collection of Π_f-blocks of $\mathfrak{G}_1$[5]. Secondly, if we consider the collection $\boldsymbol{B}_{(\delta)}^{\Pi_f}(Q_1Q_2)$ of q_2-blocks $B_\mu^{Q_2}$ of the normalizer $\mathfrak{G}_2$ of Q_1Q_2 in $\mathfrak{G}$ such that each $B_\mu^{Q_2}$ is associated with a q_2-block of $\mathfrak{G}_1$ contained in $\boldsymbol{B}_{(\delta)}^{\Pi_f}(Q_1)$, then $\boldsymbol{B}_{(\delta)}^{\Pi_f}(Q_1Q_2)$ is a collection of Π_f-blocks of $\mathfrak{G}_2$. Continuing this process, we have finally a collection $\tilde{\boldsymbol{B}}_{(\delta)}^{\Pi_f}=\boldsymbol{B}_{(\delta)}^{\Pi_f}(P)$ of Π_f-blocks $\tilde{\boldsymbol{B}}_\gamma^{\Pi_f}$ of $\tilde{\mathfrak{G}}$.

If we denote by $\tilde{\boldsymbol{E}}_{(\delta)}^{\Pi_f}$ the idempotent of the center $\tilde{Z}$ of the group ring of $\tilde{\mathfrak{G}}$ over Ω which is associated with $\tilde{\boldsymbol{B}}_{(\delta)}^{\Pi_f}$, then we have

[2.2.A] *For* $\alpha=1, 2, \cdots, \tilde{k}$, *we may write*

$$\tilde{K}_\alpha\, \tilde{\boldsymbol{E}}_{(\delta)}^{\Pi_f}=\sum_{\beta=1}^{\tilde{k}} \beta_{\delta,\alpha\beta}^{\Pi_f}\tilde{K}_\beta$$

and

$$K_\alpha \boldsymbol{E}_\delta^{\Pi_f}=\sum_{\beta=1}^{\tilde{k}} \beta_{\delta,\alpha\beta}^{\Pi_f} K_\beta$$

with the same coefficients $\beta_{\delta,\alpha\beta}^{\Pi_f}$.

5) Cf. [6].

2. 3.[6] According to [2.2.A], we obtain the following refinements of some of the orthogonality relations for group characters.

[2.3.A] *If L and M are two elements of $\mathfrak{G}$ which belong to different Π_f-sections of $\mathfrak{G}$, then*

$$\sum_{\chi_i \in B} \chi_i(L)\chi_i(M^{-1})=0$$

for each Π_f-block $\boldsymbol{B}=\boldsymbol{B}_\delta^{\Pi_f}$ of $\mathfrak{G}$.

[2.3.B] *If χ_i and χ_j are two irreducible characters of $\mathfrak{G}$ which belong to different Π_f-blocks of $\mathfrak{G}$, then*

$$\sum_{G \in \mathfrak{S}} \chi_i(G)\chi_j(G^{-1})=0$$

for each Π_f-section $\mathfrak{S}=\mathfrak{S}_\gamma^{\Pi_f}$ of $\mathfrak{G}$.

Combining 5) in 1.1 with Theorem 3 in [8], we obtain

[2.3.C] *If $\boldsymbol{B}$ is a set of irreducible characters χ_i of $\mathfrak{G}$ such that*

$$\sum_{\chi_i \in B} \chi_i(L)\chi_j(M^{-1})=0$$

for any two elements L and M of $\mathfrak{G}$ which belong to different Π_f-sections of $\mathfrak{G}$, then $\boldsymbol{B}$ is a collection of Π_f-blocks of $\mathfrak{G}$.

REMARK. Let $\mathfrak{S}$ be a collection of classes K_ν of $\mathfrak{G}$. $\mathfrak{S}$ is not always a collection of Π_f-sections of $\mathfrak{G}$, if

$$\sum_{G \in \mathfrak{S}} \chi_i(G)\chi_j(G^{-1})=0$$

holds for any two irreducible characters χ_i and χ_j of $\mathfrak{G}$ which belong to different Π_f-blocks of $\mathfrak{G}$.

2. 4. Let X be the character ring of $\mathfrak{G}$ over Ω:

$$X=\Omega\chi_1+\Omega\chi_2+\cdots+\Omega\chi_n.$$

The identity of X is the sum of n mutually orthogonal primitive idempotents $d_1, d_2, \cdots, d_n$ of X:

$$d_\mu(G_\nu)=\begin{cases} 1 & (\mu=\nu) \\ 0 & (\mu\neq\nu) \end{cases}$$

where G_ν is a representative element of K_ν $(\nu=1, 2, \cdots, n)$. As is well known, d_μ is given by

(2)
$$d_\mu=\frac{1}{g}\sum_{i=1}^{n} c_\mu \chi_i(G_\mu^{-1})\chi_i,$$

where c_μ is the number of elements in K_μ.

6) Cf. Remark in [6].

It is well known that if $S_1, S_2, \cdots, S_l$ are the p-regular sections of $\mathfrak{G}$ for a rational prime p, then the idempotents δ_γ of X associated with the p-regular sections S_γ are the mutually orthogonal primitive idempotents of the character ring

$$X_0=\mathfrak{o}_\mathfrak{p}\chi_1+\mathfrak{o}_\mathfrak{p}\chi_2+\cdots+\mathfrak{o}_\mathfrak{p}\chi_n$$

of $\mathfrak{G}$ over the ring $\mathfrak{o}_\mathfrak{p}$ of all $\mathfrak{p}$-integers in Ω, where $\mathfrak{p}$ is a prime ideal divisor of p in Ω.[7]

For each Π_f-section $\mathfrak{S}_\gamma^{\Pi_f}$ of $\mathfrak{G}$, we consider the idempotent $\varepsilon_\gamma^{\Pi_f}$ of X which is associated with $\mathfrak{S}_\gamma^{\Pi_f}$:

$$\varepsilon_\gamma^{\Pi_f}=\sum_{K_\nu\subseteqq\mathfrak{S}_\gamma^{\Pi_f}} d_\nu.$$

〔2.4.A〕 *The Π_f-sections $\mathfrak{S}_\gamma^{\Pi_f}$ of $\mathfrak{G}$ are characterized as the minimal collections $\mathfrak{S}$ of classes K_ν of $\mathfrak{G}$ such that (a) each $\mathfrak{S}$ is a collection of q-regular sections of $\mathfrak{G}$ for any rational prime q not belonging to Π_f, (b) each $\mathfrak{S}$ is not vacuous.*

If we set

$$\varepsilon_\gamma^{\Pi_f}=\sum_i \alpha_{\gamma,i}^{\Pi_f}\chi_i, \tag{3}$$

then we have

〔2.4.B〕 *$\alpha_{\gamma,i}^{\Pi_f}$ can be different from zero only for characters χ_i which belong to the Π_f-block $B_1^{\Pi_f}$ containing the 1-character χ_1. The $\alpha_{\gamma,i}^{\Pi_f}$ multiplied by π_f are algebraic integers.*

〔2.4.C〕 *If $\mathfrak{S}$ is a collection of classes K_ν of $\mathfrak{G}$ such that the coefficients α_i of $\beta\cdot\sum_{K_\nu\subseteqq\mathfrak{S}} d_\nu=\sum_i \alpha_i\chi_i$ are algebraic integers, then $\mathfrak{S}$ is a collection of Π_f-sections $\mathfrak{S}_\gamma^{\Pi_f}$ of $\mathfrak{G}$, where β is a product of powers of the rational primes in Π_f.*

3. Blocks with regard to normal subgroups.

3. 1. Let $\mathfrak{N}$ be a normal subgroup of $\mathfrak{G}$ whose order is prime to π_f. We consider the idempotents $\Delta_\sigma^\mathfrak{N}$ of Z which are associated with the $\mathfrak{N}$-blocks $\mathfrak{B}_\sigma^\mathfrak{N}$ of $\mathfrak{G}$:

$$\Delta_\sigma^\mathfrak{N}=\sum_{\chi_i\in\mathfrak{B}_\sigma^\mathfrak{N}} e_i.$$

We set

$$\Delta_\sigma^\mathfrak{N}=\sum_\nu a_{\sigma,\nu}^\mathfrak{N} K_\nu, \tag{4}$$

where $a_{\sigma,\nu}^\mathfrak{N}\in\Omega$. According to facts mentioned in 1.3, we have

〔3.1.A〕 *$a_{\sigma,\nu}^\mathfrak{N}$ can be different from zero only for classes K_ν which are contained in $\mathfrak{N}$. The $(\mathfrak{N}:1)a_{\sigma,\nu}^\mathfrak{N}$ are algebraic integers.*

〔3.1.B〕 *If $\mathfrak{B}$ is a set of irreducible characters χ_i of $\mathfrak{G}$ such that $\sum_{\chi_i\in\mathfrak{B}} e_i$ is a linear combination of classes K_ν contained in $\mathfrak{N}$, then $\mathfrak{B}$ is a collection of $\mathfrak{N}$-blocks $\mathfrak{B}_\sigma^\mathfrak{N}$ of $\mathfrak{G}$.*

7) Cf. [11], [12]. Cf. also [6].

Combining [3.1.A] with [2.1.C], we have

[3.1.C] *If $\mathfrak{N}$ is a normal subgroup of $\mathfrak{G}$ whose order is prime to π_f, then each $\mathfrak{N}$-block $\mathfrak{B}_\sigma^{\mathfrak{N}}$ of $\mathfrak{G}$ is a collection of π_f-blocks $\boldsymbol{B}_\delta^{\Pi_f}$ of $\mathfrak{G}$.*

We set

$$(5) \qquad K_\mu \varDelta_\sigma^{\mathfrak{N}} = \sum_\nu a_{\sigma,\mu\nu}^{\mathfrak{N}} K_\nu,$$

where $a_{\sigma,\mu\nu}^{\mathfrak{N}} \in \Omega$. If $\mathfrak{M}$ is a normal subgroup of $\mathfrak{G}$ which contains $\mathfrak{N}$, then $a_{\sigma,\mu\nu}^{\mathfrak{N}}$ can differ from zero only when either both K_μ and K_ν are contained in $\mathfrak{M}$ or when both are not contained in $\mathfrak{M}$. Thus we have

[3.1.D] *If $\mathfrak{M}$ is a normal subgroup of $\mathfrak{G}$ which contains $\mathfrak{N}$ and if exactly one of two elements L and M of $\mathfrak{G}$ belongs to $\mathfrak{M}$, then*

$$\sum_{\chi_i \in \mathfrak{B}} \chi_i(L)\chi_i(M^{-1}) = 0$$

for each $\mathfrak{N}$-block $\mathfrak{B} = \mathfrak{B}_\sigma^{\mathfrak{N}}$ of $\mathfrak{G}$.

3. 2. Let P be a Π_f-element of $\mathfrak{G}$ and $\mathfrak{N}$ a normal subgroup of $\mathfrak{G}$ whose order is prime to π_f. We shall use the same notation, for this P, as in 2.2: $\tilde{\mathfrak{G}}$; $\tilde{K}_1, \tilde{K}_2, \cdots, \tilde{K}_{\tilde{k}}$; $K_1, K_2, \cdots, K_{\tilde{k}}$; $\tilde{Z}$; $\tilde{\boldsymbol{B}}_{(\delta)}^{\Pi_f}$ and etc. Since each $\mathfrak{N}$-block $\mathfrak{B}_\sigma^{\mathfrak{N}}$ of $\mathfrak{G}$ is a collection of Π_f-blocks $\boldsymbol{B}_\delta^{\Pi_f}$ of $\mathfrak{G}$, we may define the collection $\tilde{\mathfrak{B}}_{(\sigma)}^{\mathfrak{N}}$ of Π_f-blocks $\tilde{\boldsymbol{B}}_\rho^{\Pi_f}$ of $\tilde{\mathfrak{G}}$ such that each $\tilde{\boldsymbol{B}}_\rho^{\Pi_f}$ is contained in a $\tilde{\boldsymbol{B}}_{(\delta)}^{\Pi_f}$ with $\boldsymbol{B}_\delta^{\Pi_f} \subseteqq \mathfrak{B}_\sigma^{\mathfrak{N}}$. It is easy to see that each $\tilde{\mathfrak{B}}_{(\sigma)}^{\mathfrak{N}}$ is a collection of $\tilde{\mathfrak{N}}$-blocks $\tilde{\mathfrak{B}}_\tau^{\tilde{\mathfrak{N}}}$ of $\tilde{\mathfrak{G}}$, where $\tilde{\mathfrak{N}} = \mathfrak{N} \cap \tilde{\mathfrak{G}}$. We denote by $\tilde{\varDelta}_{(\sigma)}^{\mathfrak{N}}$ the idempotent of $\tilde{Z}$ associated with $\tilde{\mathfrak{B}}_{(\sigma)}^{\mathfrak{N}}$:

$$\tilde{\varDelta}_{(\sigma)}^{\mathfrak{N}} = \sum_{B_\sigma^{\Pi_f} \subseteqq \mathfrak{B}_\delta^{\mathfrak{N}}} \tilde{E}_{(\delta)}^{\Pi_f}.$$

Then, by [2.2.A] and (5), we obtain

[3.2.A] *For $\mu = 1, 2, \cdots, \tilde{k}$, we have*

$$K_\mu \varDelta_\sigma^{\mathfrak{N}} = \sum_{\nu=1}^{\tilde{k}} a_{\sigma,\mu\nu}^{\mathfrak{N}} K_\nu$$

and

$$\tilde{K}_\mu \tilde{\varDelta}_{(\sigma)}^{\mathfrak{N}} = \sum_{\nu=1}^{\tilde{k}} a_{\sigma,\mu\nu}^{\mathfrak{N}} \tilde{K}_\nu$$

with the same coefficients $a_{\sigma,\mu\nu}^{\mathfrak{N}}$.

We shall say that two elements L and M of the Π_f-section $\mathfrak{S}_f^{\Pi}(P)$ of P in $\mathfrak{G}$ belong to a same *Π_f-subsection* of P in $\mathfrak{G}$ with regard to $\mathfrak{N}$ if and only if the following two conditions are satisfied:

(a) For any normal subgroup $\mathfrak{M}$ of $\mathfrak{G}$ which contains $\mathfrak{N}$, "$L \in \mathfrak{M}$" is equivalent to "$M \in \mathfrak{M}$".

(*b*) For any normal subgroup $\tilde{\mathfrak{M}}$ of $\tilde{\mathfrak{G}}$ which contains $\mathfrak{N} \cap \tilde{\mathfrak{G}}$, "$Q \in \tilde{\mathfrak{M}}$" is equivalent to "$R \in \tilde{\mathfrak{M}}$", where Q and R are two Π_f-regular elements of $\tilde{\mathfrak{G}}$ such that L and M are conjugate in $\mathfrak{G}$ to PQ and PR, respectively.

Considering this construction for each $P = P_\gamma^{\Pi_f}$, we can distribute the elements of $\mathfrak{G}$ into a certain number of Π_f-subsections with regard to $\mathfrak{N}$. According to [3.2.A], we can refine [3.1.D] as follows:

[3.2.B] *If L and M are two elements of $\mathfrak{G}$ which belong to different Π_f-subsections of $\mathfrak{G}$ with regard to $\mathfrak{N}$, then*

$$\sum_{\chi_i \in \mathfrak{B}} \chi_i(L)\chi_i(M^{-1}) = 0$$

for each $\mathfrak{N}$-block $\mathfrak{B} = \mathfrak{B}_\sigma^{\mathfrak{N}}$ of $\mathfrak{G}$.

[3.2.C] *If χ_i and χ_j are two irreducible characters of $\mathfrak{G}$ which belong to different $\mathfrak{N}$-blocks of $\mathfrak{G}$, then*

$$\sum_{G \in \mathfrak{S}} \chi_i(G)\chi_j(G^{-1}) = 0$$

for each Π_f-subsection $\mathfrak{S}$ of $\mathfrak{G}$ with regard to $\mathfrak{N}$.

Department of Mathematics,
Faculty of Science,
Kumamoto University

REFERENCES

[1] R. BRAUER, On blocks of characters of groups of finite order II, Proc. Nat. Acad. Sci. U. S. A., **32** (1946), 215–219.
[2] ————, Zur Darstellungstheorie der Gruppen endlicher Ordnung II, Math. Zeitschr., **72** (1959), 25–46.
[3] K. IIZUKA, On Brauer's theorem on sections in the theory of blocks of group characters, Math. Zeitschr., **75** (1961), 299–304.
[4] ————, On Osima's blocks of group characters, Proc. Japan Acad., **36** (1960), 392–396.
[5] ————, On Osima's blocks of groups of finite order, Kumamoto Jour. Sci., **4** (1960), 275–283.
[6] ————, On the blocks and the sections of finite groups, Kumamoto Jour. Sci., **5** (1960), 53–62.
[7] M. OSIMA, On the representations of groups of finite order, Math. Jour. Okayama Univ., **1** (1952), 33–61.
[8] ————, Notes on blocks of group characters, Math. Jour. Okayama Univ., **4** (1955), 175–188.
[9] ————, On some properties of group characters, Proc. Japan Acad., **36** (1960), 18–21.
[10] ————, On some properties of group characters II, Math. Jour. Okayama Univ., **10** (1960), 61–66.
[11] P. ROQUETTE, Arithmetische Untersuchung des Charakterringes einer endlichen Gruppe, Crelles Jour., **190** (1952), 148–168.
[12] K. SHIRATANI, On the characters and the character rings of finite groups, Mem. Fac. Kyusyu Univ., **11** (1957), 99–115.