ON THE THEORY OF DIFFERENTIAL EQUATIONS IN COORDINATED SPACES

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1 *Preliminary.* Most of ordinary differential equations of infinite dimension have been treated, roughly speaking, in two manners. One manner is mainly analytical as treated by K. P. Persidskii [1, 2][(1)] and other Soviet mathematicians in cases of denumerable systems of differential equations. The other is rather topologico-analytical; namely, there have been investigated differential equations in Banach spaces [for example, Massera, 3]. Investigations in the latter manner may seem apparently to be more general than those of the Soviet mathematicians, but these investigations are concerned mostly, to translate in the terms of topology, with differential equations in the space (m) or in the linear space (s) with the product topology, the latter of which is not a Banach space. Whereas, on the other hand, they are explicitely analytical and somewhat inexact, or ambiguous, to consider from the topological standpoint. To avoid these inexactnesses or ambiguities, we have investigated differential equations in coordinated spaces as a generalization of denumerable systems of differential equations [4, 5]. Namely, given an infinite system of differential equations

$$\frac{dx_i}{dt}=f(t, x_1, x_2, \cdots, x_n, \cdots) \tag{1}$$

$$(i=1, 2, \cdots, n, \cdots),$$

we consider this as a differential equation in a linear space E, and denote it by

$$\frac{dx}{dt}=f(t, x), \tag{2}$$

where x is a sequence $\{x_1, x_2, \cdots, x_n, \cdots\}$ of real numbers x_n, and called a *point* of E.

Here we must give some basic definitions and notations. A coordinated space E is a sequence space, whose element $x=\{x_1, x_2, \cdots, x_n, \cdots\}$, or abbreviated $\{x_n; n=1, 2, \cdots\}$, is a sequence of real numbers x_n and whose topology is locally convex and such that each mapping $x_n(x)$: $x\rightarrow x_n (n=1, 2, \cdots)$ is linear and continuous. For any point $x=\{x_1, x_2, \cdots, x_n, \cdots\}$, the point, which is constructed by equating some of the coordinates of x to zero, is called a *projection* of the point x, and especially that which is constructed by equating the coordinates with indices greater than n to zero is called a *section* of x and denoted by $x^{[n]}$. A coordinated space is called to have the property (P), if, for every neighborhood U of the fundamental system of neighborhoods of the origin $\mathfrak{U}$, the relation $x\in U$ implies that every projection of x also belongs to U. The space E is called to have the property "*Abschnittskonvergenz*" or simply (AK) [5, 6], if for

(1) Numbers in brackets refer to the references at the end of the paper.

every point x, the corresponding sequence of sections $\{x^{[n]}; n=1,2,\cdots\}$ converges to the point x. Most of linear spaces used in practice have the property (P), but the property (AK) except the space (m), the space of all bounded sequences with the norm $\|x\|=\sup_n |x_n|$ and the space (c), the space of all convergent sequences with the same norm.

Let $x(t)$ be a vector function defined on the real interval I, that is, a function on I to the coordinated space E, and $x_n(t)$ its coordinate functions:

$$x(t)=\{x_1(t), x_2(t), \cdots, x_n(t), \cdots\}.$$

If the space is provided with the product topology, the limiting, accordingly the differentiation and the integration of the vector function $x(t)$ are equivalent to those of each coordinate function $x_n(t)$ respectively. Most of differential equations (1) treated by Persidskii and other Soviet mathematicians are mostly, in reality, differential equations (2) in coordinated spaces with the product topology. The present paper is concerned with differential equations in more general coordinated spaces.

We now consider the differential equation

$$\frac{dx}{dt}=f(t, x) \tag{2}$$

in the coordinated space E, where t is a real variable on the interval $I=[t_1, t_2]$ or $[t_1, \infty)$, x is a vector variable in the domain D in the space E, and $f(t, x)$ is a function on $I\times D$ to E. In paragraph 2, our main attention will be directed to the problem of existence, uniqueness and dependency of solutions of the differential equation (2) in the space E with different topologies rather than that in the space E with a fixed topology. In paragraph 3, we shall be concerned with linear, homogeneous differential equations, contrasting them with the case of finite dimension. In paragraph 4, we shall be concerned with relations between the stability of solutions of the differential equations (1) and that of solutions of the truncated differential equations associated with (2);

$$\frac{dx}{dt}=\hat{f}_{[n]}(t, x), \tag{3}$$

$$\frac{dx}{dt}=\tilde{f}_{[n]}(t, x), \tag{4}$$

$$\frac{dx}{dt}=\tilde{\tilde{f}}_{[n]}(t, x), \tag{5}$$

where the functions $\hat{f}_{[n]}(t, x), \tilde{f}_{[n]}(t, x), \tilde{\tilde{f}}_{[n]}(t, x)$ denote the trancated functions $f^{[n]}(t,x)$, $f(t, x^{[n]}), f^{[n]}(t, x^{[n]})$ respectively.

2. *Existence, Uniqueness and Dependency.* We consider the differential equation

$$\frac{dx}{dt}=f(t, x) \tag{2}$$

in a complete coordinated space E, where t is a real variable on the interval $I=[0, T]$ or $[0, \infty)$, x is a vector variable in the domain D in the space E, and $f(t, x)$ is a function on $I\times D$ to E. We suppose in the following that $f(t, x)$ is continuous.

Peano's existence theorem, i. e. the theorem of existence of solution of the equation (2) under the condition of continuity (and boundedness) alone of the function $f(t, x)$ cannot always be valid as was shown by J. Dieudonné [7]. However, we showed that Peano's existence theorem holds in Montel spaces and in their product spaces [4]. As its special case, we state the following theorem.

Theorem 1. Let E be the space (s) with the product topology, and let the function $f(t, x)$ be continuous and bounded in the domain $I\times E$. Then there exists at least one solution of the differential equation (2), which satisfies the initial condition: $x=x^0$ for $t=t^0$ and is defined in the interval I. This solution we shall call a solution through the point (t_0, x^0) $\in I\times E$, and denote by $x(t; t_0, x^0)$.

In other spaces, it will be appropriate to introduce some condition, say, Lipschitz condition to be imposed on the function $f(t, x)$. Lipschitz condition in a linear topological space E is stated as follows. For any neighborhood U of the fundamental system of neighborhoods of the origin $\mathfrak{U}$, the relation $x'-x''\in U$ implies that $f(t, x')-f(t, x'')\in kU$, where k is a positive constant, or equivalently, the relations $\|x'-x''\|_U\leqq 1$ implies that $\|f(t, x')-f(t, x'')\|_U\leqq k$, where $\| \ \|_U$ denotes the semi-norm defined by the neighborhood U. Or, if the space happens to be normed, Lipschitz condition is formulated as used usually:

$$\|f(t, x')-f(t, x'')\|\leqq k\|x'-x''\|.$$

The constant k may be replaced by a positive integrable function $k(t)$ of t, such that $\int_0^T k(t)\, dt<\infty$ (or $\int_0^\infty k(t)\, dt<\infty$).

The space E, as linear space, can be topologized in several manners. Of two topologies $\mathcal{T}_1$ and $\mathcal{T}_2$, let $\mathcal{T}_1$ be stronger (or finer) than $\mathcal{T}_2$ (or equivalently $\mathcal{T}_2$ weaker than $\mathcal{T}_1$). Since the continuity (and the convergence) of a function on I to E for the stronger topology $\mathcal{T}_1$ implies the continuity (and the convergence) for the topology $\mathcal{T}_2$, we have the following theorem.

Theorem 2. If a continuous function $x(t)$ is a solution of the differential equation (2) for a topology $\mathcal{T}_1$, then it is also a solution for a topology $\mathcal{T}_2$ weaker than $\mathcal{T}_1$.

Since, for any coordinated space, the product topology is the weakest topology, we have immediately the following corollary.

Corollary. A solution of the differential equation (2) in any coordinated space E is also a solution of the differential equation (2) for the product topology.

The converse is not always true, as is shown by the following example, given by K. P. Persidskii [1].

Example 1. We consider the differential equation

$$\frac{dx}{dt}=f(t, x),$$

where

$$f_n(t, x) = -x_n + x_{n+1}.$$

Let $\varphi(t)$ be a function defined as follows:

$$\varphi(t) = 0 \qquad \text{for } t = 0,$$
$$= e^{-\frac{1}{t^2}} \qquad \text{for } t \neq 0.$$

Then the vector function $x(t) = \{x_n(t); n = 1, 2, \cdots\}$, where $x_n(t) = e^{-t} \frac{d^{n-1}}{dt^{n-1}} \varphi(t)$, is a solution of the given equation for the product topology. But this function $x(t)$ does not exist for another topology, for example, for the topology of the space (m) or the space $(l^p)(1 \leqq p < \infty)$.

The converse, however, is true in the sense of the following theorem.

Theorem 3. Let the function $f(t, x)$ be continuous and bounded in the domain $I \times D \subset R^1 \times E$ for a topology $\mathcal{T}_1$. Let $x(t)$ be a solution through the point (t_0, x^0) of the differential equation (2) for another topology $\mathcal{T}_2$ weaker than $\mathcal{T}_1$. If the function $x(t)$ is a continuous function on a subinterval I' containing t_0 to the space E for the topology $\mathcal{T}_1$, then it is a solution, in the interval I', of the differential equation (2) also for this topology.

PROOF. By virtue of the well-known mean value theorem, we have, for the topology $\mathcal{T}_2$,

$$\frac{x(t+h) - x(t)}{h} = x'(t + \theta h) \qquad (0 < \theta < 1)$$

for t and $t + h \in I'$. Since $x(t)$ is a solution of the differential equation (2), we have

$$x'(t + \theta h) = f(t + \theta h, x(t + \theta h)),$$

and therefore the equality

$$\frac{x(t+h) - x(t)}{h} - f(t, x(t))$$
$$= f(t + \theta h, x(t + \theta h)) - f(t, x(t))$$

and the continuity of $x(t)$ and $f(t, x)$ for the topology $\mathcal{T}_1$ imply the relation:

$$\lim_{h \to 0} \frac{x(t+h) - x(t)}{h} = f(t, x(t))$$

for the topology $\mathcal{T}_1$, which proves the theorem.

Corollary. Let the function $f(t, x)$ be continuous and bounded in the domain $I \times D \subset R^1 \times E$ for a topology $\mathcal{T}$. Let $x(t)$ be a solution through the point (t_0, x^0) of the differential equation (2) for the product topology. If the function $x(t)$ is a continuous function on a subinterval I' containing t_0 to the space E for the topology $\mathcal{T}$, then it is a solution, in the interval I', of the differential equation (2) also for this topology.

Citing example 1, the function $x(t)$, which is a solution for the product topology, is neither a solution for the (l^2) topology nor for the (m) topology, because it is not a continuous function for either topology. On the other hand, the function $\bar{x}(t)=\left\{e^{-t}\times \sum_{i=0}^{\infty} x_{n+i}^0 \frac{t^i}{i!};\ n=1, 2, \cdots\cdots\right\}$ is evidently a solution through the point (t_0, x^0) for the product topology, where $x^0=\{x_1^0, x_2^0, \cdots, x_n^0, \cdots\}$. In case x^0 is a point of the space (m), the function $\bar{x}(t)$ is a continuous function also for the (m) topology. In fact, at first, we have

$$\|\bar{x}(t)\| = \sup_n \left| e^{-t} \sum_{i=0}^{\infty} x_{n+i}^0 \frac{t^i}{i!} \right|$$
$$\leqq \sup_n \left| e^{-t}\|x^0\| \sum_{i=0}^{\infty} \frac{t^i}{i!} \right| = \|x^0\|.$$

Secondly, we have

$$\|\bar{x}(t+h)-\bar{x}(t)\| \leqq \sup_n \left\{ \left| e^{-(t+h)} \sum_{i=0}^{\infty} x_{n+i}^0 \left(\frac{(t+h)^i}{i!} - \frac{t^i}{i!} \right) \right| \right.$$
$$\left. + \left| (e^{-(t+h)}-e^{-t}) \sum_{i=0}^{\infty} x_{n+i}^0 \frac{t^i}{i!} \right| \right\}$$
$$\leqq \|x^0\| \left\{ e^{-(t+h)} |e^{t+h}-e^t| + |e^{-(t+h)}-e^{-t}| e^t \right\},$$

which shows the continuity of $\bar{x}(t)$ for the (m) topology. Therefore, in case x^0 is a point of the space (m), the function $\bar{x}(t)$ is a solution also for the (m) topology.

Further, we investigate a special case when a solution for the product topology becomes a solution for a topology $\mathcal{T}$. Namely we have the following theorem.

Theorem 4. Let E be a complete coordinated space with the properties (P) and (AK), and let $f(t, x)$ be continuous, bounded and satisfy Lipschitz condition in the domain $I \times D \subset R^1 \times E$. If $x(t)$ is a continuous function such that its section $x^{[n]}(t)$ is a solution $x(t;\, t_0, x^{0[n]})$ of the truncated differential equation

$$\frac{dx}{dt} = \bar{\bar{f}}_{[n]}(t,x), \tag{5}$$

associated with the differential equation (2), then $x(t)$ is a solution $x(t;\, t_0, x^0)$ of the equation (2).

For the proof, we propose a lemma, which we proved previously [5].

Lemma 1. Let $x(t)$ be a continuous function on a bounded interval I to a coordinated space E with the property (AK). Then the sections $x^{[n]}(t)$ converge to $x(t)$ uniformly for $t \in I$, that is, given an arbitrary neighborhood U, there exists a positive integer N (dependent on U, but) independent of t such that $x(t)-x^{[n]}(t) \in U$ for $n \geqq N$.

PROOF OF THE THEOREM. From the supposition on $x^{[n]}(t)$, it is written as follows:

$$x^{[n]}(t) = x^{0[n]} + \int_0^t f^{[n]}(t, x^{[n]}(t))dt.$$

Let $y(t)$ be defined by the relation

$$y(t) = x^0 + \int_0^t f(t, x(t))dt. \tag{6}$$

Then we have

$$y(t)-x^{[n]}(t)=(x^0-x^{0[n]})+\int_0^t[f(t,x(t))-f^{[n]}(t,x(t))]dt$$
$$+\int_0^t[f^{[n]}(t,x(t))-f^{[n]}(t,x^{[n]}(t))]dt. \quad (7)$$

Let U be an arbitrarily given neighborhood of the origin in E. Then there exists a positive integer N, such that

$$x_0-x^{0[n]}\in\frac{1}{3}U \qquad \text{for } n\geqq N, \quad (8)$$

and, by virtue of the above lemma,

$$\int_0^t[f(t,x(t))-f^{[n]}(t,x(t))]dt\in\frac{1}{3}U \qquad \text{for } n\geqq N, \quad (9)$$

and lastly, by the property (P) and Lipschitz condition imposed on $f(t,x)$,

$$\int_0^t[f^{[n]}(t,x(t))-f^{[n]}(t,x^{[n]}(t))]dt\in\frac{1}{3}U \qquad \text{for } n\geqq N. \quad (10)$$

From the relations (8), (9), (10), and (7), we have

$$y(t)-x^{[n]}(t)\in U \qquad \text{for } n\geqq N,$$

whence $x^{[n]}(t)$ converges to $y(t)$ uniformly on I, and therefore $y(t)$ coincides with $x(t)$. Accordingly, from the relation (6), we have

$$x(t)=x^0+\int_0^t f(t,x(t))dt,$$

which proves the theorem.

In case E is a Banach space (that is, a normed and complete space), from the Lipschitz condition on $f(t,x)$, it follows by the method of successive approximations, as is known, Cauchy's theorem of existence and uniqueness, which, for the convenience of formulation, we add as a theorem.

Theorem 5. Let E be a Banach space, and let $f(t,x)$ be continuous, bounded and satisfy Lipschitz condition in the domain $I\times E$. Let $(t_0,x^0)\in I\times E$. Then there exists a unique solution $x(t;t_0,x^0)$ of the differential equation (2).

As regards Lipschitz condition, we give a remark. One cannot find a simple implication relation between Lipschitz condition for a stronger topology and that for a weaker topology. Namely, Lipschitz condition for a stronger (weaker) topology does not imply that for a weaker (stronger) topology as is shown by the following examples.

Example 2. We consider again the differential equation given in example 1, that is,

$$\frac{dx}{dt}=f(t,x),$$

where

$$f_n(t,x)=-x_n+x_{n+1}.$$

In case the space E is provided with the (m) or (l^p) topology $(1 \leq p < \infty)$, a simple calculation shows that

$$\|f(t, x') - f(t, x'')\| \leq 2\|x' - x''\|,$$

which is Lipschitz condition with $k=2$. Whereas, in case E is provided with the product topology, Lipschitz condition does not hold, as will be shown by theorem 8 in the next paragraph. As a matter of fact, in the space (s) with the product topology, the equation has a particular solution

$$\chi(t) = \left\{ e^{-t} \frac{d^{n-1}}{dt^{n-1}} \varphi(t); n=1, 2, \cdots \right\}$$

as is already shown in example 1. Further, through the point (t_0, x^0), there exists a solution

$$\bar{x}(t) = \left\{ \sum_{i=0}^{\infty} x_{n+i}^0 \frac{t^i}{i!} ; n=1, 2, \cdots \right\},$$

as is also already shown. Thus the function

$$x(t) = \bar{x}(t) + c\chi(t),$$

where c is an arbitrary constant, is also a solution through the point (t_0, x^0). Therefore, in the space (s) with the product topology, the differential equation has an infinite number of solutions $x(t; t_0, x^0)$ through the point (t_0, x^0), since c is arbitrary. On the contrary in the space (m) or $(l^p)(1 \leq p < \infty)$, the differential equation has a unique solution $x(t; t_0, x^0) = \bar{x}(t)$ through the point (t_0, x^0).

Example 3.(2) We consider the differential equation

$$\frac{dx}{dt} = f(t, x)$$

where

$$f_n(t, x) = x_1 \qquad \text{for } n=1$$
$$= x_{n-1} \qquad \text{for } n>1.$$

In case the space E is provided with the (m) or the (l^2) topology, it is evident that

$$\|f(t, x') - f(t, x'')\| \leq \|x' - x''\|,$$

which is Lipschitz condition with $k=1$. Whereas, in case E is provided with the direct sum topology (and in this case E is considered as the algebraic direct sum of a denumerable number of real lines $R_n (n=1, 2, \cdots)$), that is, a fundamental system of neighborhoods of the origin $\mathfrak{U}$ is given by

$$U_\varepsilon = \{x; |x_n| < \varepsilon_n, n=1, 2, \cdots\cdots\}$$

where $\varepsilon = \{\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n, \cdots\}$ is any sequence of positive numbers, Lipschitz condition cannot always be valid, since the implication relation:

(2) This example is due to my colleague Y. Kōmura.

$$|x'_{n-1}-x''_{n-1}|<\varepsilon_{n-1} \text{ implies } |x'_{n-1}-x''_{n-1}|<k\varepsilon_n$$

for some fixed positive constant k, cannot always be valid. In fact, for a sequence $\varepsilon=\{\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n, \cdots\}$, such that $\frac{\varepsilon_n}{\varepsilon_{n-1}}\to 0$ for $n\to\infty$, there exists no fixed positive constant k, such that the above implication relation holds. The direct sum topology is strictly stronger than the (m) and (l^2) topologies.

The situation that Lipschitz condition for a weaker topology does not imply that for a stronger topology, is immaterial for the uniqueness of solution. In fact, if the uniqueness of solution holds for a weaker topology $\mathcal{T}_2$, that is, if through the point (t_0, x^0) there exists a unique solution $x(t; t_0, x^0)$ for $\mathcal{T}_2$, then $x(t; t_0, x^0)$ will also be, if there exists any solution through the point (t_0, x^0) for a stronger topology $\mathcal{T}_1$, a unique solution through the point (t_0, x^0) for $\mathcal{T}_1$, because, by virtue of theorem 3, a solution through the point (t_0, x^0) for the topology $\mathcal{T}_1$ is also a solution through the point (t_0, x^0) for the weaker topology $\mathcal{T}_2$, which is unique by the hypothesis. This fact can be formulated as the following theorem.

Theorem 6. If there exists a unique solution $x(t; t_0, x^0)$ through the point (t_0, x^0) in the space E for a topology $\mathcal{T}_2$, then it is also, if there exists any solution through the point (t_0, x^0) for a topology $\mathcal{T}_1$ stronger than $\mathcal{T}_2$, the very unique solution through the point (t_0, x^0) for $\mathcal{T}_1$.

In a previous paper [5], we proved a theorem of continuous dependency of solutions on the function $f(t, x)$ on the right hand side of the differential equation (2). In this connection, we give here a theorem of continuous dependency on the initial condition, whose proof is quite analogous to that of the previous theorem and therefore is omitted.

Theorem 7. Let E be a complete coordinated space, and $f(t, x)$ be continuous, bounded and satisfy Lipschitz condition in the domain $I\times E$, where I here denotes a bounded interval $[0, T]$. Then, t_0 being fixed, the solution $x(t; t_0, x^0)$ through the point (t_0, x^0) depends on the initial value x^0 of x continuously, that is, to express more precisely, for an arbitrarily given neighborhood of the origin U, $x^0-x^{0'}\in U$ implies the relation:

$$x(t; t_0, x^0)-x(t; t_0, x^{0'})\in e^{\int_0^T k(t)dt}U,$$

where $k(t)$ is the Lipschitz function (or constant).

3. *Linear, homogeneous equations.* If the function $f(t, x)$ on the right hand side of the differential equation (2) satisfies the conditions:

$$f(t, x'+x'')=f(t, x')+f(t, x'') \qquad \text{for all } x', x''\in E,$$
$$f(t, \alpha x)=\alpha f(t, x) \qquad \text{for all real } \alpha \text{ and all } x\in E,$$

the function $f(t, x)$ is called *linear, homogeneous*, and the equation (2) also *linear, homogeneous*. If the function $f(t, x)$ is the sum of a linear, homogeneous function and a function of t only, then the equation (2) is called *linear, non-homogeneous*.

Let E be a coordinated space with the property (AK), and then each point x of E can be represented as follows:

$$x=\sum_{i=1}^{\infty} x_i e^i = x^{[n]} + (x - x^{[n]})$$

and

$$x^{[n]}=\sum_{i=1}^{n} x_i e^i,$$

where e^i are basis elements $\{\delta_{in};\ n=1, 2, \cdots\}$ [8. p. 189, theorem 1]. Let $f(t, x)$ be linear, homogeneous and continuous, and then

$$\begin{aligned} f(t, x) &= f(t, x^{[n]}) + f(t, x-x^{[n]}) \\ &= \sum_{i=1}^{n} x_i f(t, e^i) + f(t, x-x^{[n]}). \end{aligned}$$

For $n\to\infty$, $x-x^{[n]}\to 0$, and by virtue of the continuity of $f(t, x)$, we have $f(t, x-x^{[n]})\to 0$. Therefore we have

$$f(t, x)=\sum_{i=1}^{\infty} x_i f(t, e^i).$$

Here let the vector $f(t, e^i)$ be denoted by

$$\{a_{i1}(t), a_{i2}(t), \cdots, a_{in}(t), \cdots\}$$

and then the vector $f(t, x)$ can be represented by as the multiplication of an infinite matrix by the column vector $'x$, the transposed of x considered as row vector.

$$f(t, x) = \begin{pmatrix} a_{11}(t)\ a_{12}(t) \cdots\cdots a_{1n}(t) \cdots\cdots \\ a_{21}(t)\ a_{22}(t) \cdots\cdots a_{2n}(t) \cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_{m1}(t)\ a_{m2}(t) \cdots\cdots a_{mn}(t) \cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \\ \vdots \end{pmatrix}$$

or simply,

$$f(t, x)=A(t)\ 'x,$$

where

$$A(t) = \left(a_{ij}(t)\right),\quad 'x=\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \\ \vdots \end{pmatrix}.$$

Hence we have the following lemma.

Lemma 2. Any differential equation, which is linear, homogeneous and continuous in a coordinated space E with the property (AK), can be written in the form of the multiplication of an infinite matrix by the column vector $'x$, the trnasposed of x as row vector,

$$\frac{dx}{dt}=A(t)\ 'x,$$

where

$$A(t)=\left(a_{ij}(t)\right)=\begin{pmatrix} a_{11}(t) & a_{12}(t) & \cdots\cdots & a_{1n}(t) & \cdots\cdots \\ a_{21}(t) & a_{22}(t) & \cdots\cdots & a_{2n}(t) & \cdots\cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1}(t) & a_{m2}(t) & \cdots\cdots & a_{mn}(t) & \cdots\cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}.$$

Specially, when the coordinated space E is provided with the product topology or the direct sum topology, the following theorem can be easily verified.

Theorem 8. In order that Lipschitz condition should be satisfied for the linear, homogeneous equation

$$\frac{dx}{dt}=A(t)'x$$

for the product topology or the direct sum topology, it is necessary that the infinite matrix $A(t)$ is of the triangular form, or more precisely, $a_{ij}(t)=0$ for $i<j$ in the case of the product topology and $a_{ij}(t)=0$ for $i>j$ in the case of the direct sum topology.(3)

Thus the differential equation in example 1 is represented as follows:

$$\frac{dx}{dt}=A'x,$$

where

$$A=\begin{pmatrix} -1 & 1 & & & 0 \\ & -1 & 1 & & \\ & & -1 & \ddots & \\ 0 & & & & \ddots \end{pmatrix}.$$

For this differential equation, considered for the product topology, Lipschitz condition cannot hold. Since Lipschitz condition is a sufficient condition for the uniqueness of solution of the differential equation also in a coordinated space with the properties (AK) and (P), accordingly in the space with the product topology, it would not be unnatural that the above equation has an infinite number of solutions through the point (t_0, x^0) for the product topology, as is shown in example 2.

Let E be a complete coordinated space with the properties (P) and (AK). Consider the linear, homogeneous differential equation

$$\frac{dx}{dt}=A(t)'x$$

where the right hand side is continuous and satisfies Lipschitz condition. When t_0

(3) In the present paper, for the definiteness, as the fundamental system of neighborhoods of the origin of the product topology and that of the direct sum topology we take the family of the sets $U_{n,m}=\{x;\ \sup_{i\leq n}|x_i|\leqq 1/m\}$, $(n,m=1,2,\cdots\cdots)$ and that of the sets $U_\varepsilon=\{x;\ |x_n|<\varepsilon_n,\ n=1,2,\cdots\cdots\}$, where $\varepsilon=\{\varepsilon_n\}$ is any decreasing sequence of positive numbers ε_n, respectively, unless otherwise stated.

being fixed, by virtue of the property of the uniqueness of solution, the totality of the solutions $x(t; t_0, x^0)$ forms a linear space, denoted by $\mathcal{E}$, which is algebraically isomorphic to the original space E, what can be verified analogously to the case of a finite-dimensional space. We shall show that this isomorphism turns to be also topological.

Let $x^i(t)$ denote the solution $x(t; t_0, e^i)$, that is, the solution passing through the point (t_0, e^i). Then the solution $x(t; t_0, x^{0[n]})$ is represented as follows:

$$x(t; t_0, x^{0[n]}) = x(t; t_0, \sum_{i=1}^{n} x_i^0 e^i) = \sum_{n=1}^{n} x_i^0 x^i(t).$$

For an arbitrarily given neighborhood of the origin U, there exists, by virtue of the property (AK), a positive integer N, such that $n \geqq N$ implies $x^0 - x^{0[n]} \in U$, and therefore, by virtue of theorem 7

$$x(t; t_0, x^0) - x(t; t_0, x^{0[n]}) \in e^{\int_0^T k(t)\,dt} U,$$

where $k(t)$ is the Lipschitz function (or constant). This shows that the solution $x(t; t_0, x^{0[n]})$ converges to the solution $x(t; t_0, x^0)$ and that uniformly on the interval $[0, T]$:

$$\begin{aligned} x(t; t_0, x^0) &= \lim_{n\to\infty} x(t; t_0, x^{0[n]}) \\ &= \lim_{n\to\infty} \sum_{i=1}^{n} x_i^0 x^i(t) = \sum_{i=1}^{\infty} x_i^0 x^i(t). \end{aligned}$$

This fact shows that the set $\{x^i(t); i=1, 2, \cdots\}$ forms an algebraic basis of the derived linear space $\mathcal{E}$. Moreover, the mapping: $x^0 \to x(t; t_0, x^0)$ is not only one-to-one, but, if $\mathcal{E}$ is topologized by the uniform convergence, also continuous and inversely continuous, as is easily seen. Hence we obtain the following theorem.

Theorem 9. Let E be a complete coordinated space with the properties (P) and (AK). Let the right hand side of the linear, homogeneous differential equation

$$\frac{dx}{dt} = A(t)\,'x \tag{11}$$

be continuous and satisfy Lipschitz condition in the domain $I \times E \subset R^1 \times E$. Then, t_0 being fixed, the totality of the solutions $x(t; t_0, x^0)$ forms a linear topological space $\mathcal{E}$ with the topology of uniform convergence, which is topologically isomorphic to the space E, and the solution $x(t; t_0, x^0)$ is represented in the form:

$$x(t; t_0, x^0) = \sum_{i=1}^{\infty} x_i^0 x^i(t), \tag{12}$$

where $x^i(t) = x(t; t_0, e^i)$ $(i=1, 2, \cdots)$.

The set of the solutions $\{x^i(t); i=1,2,\cdots\}$ is called the *fundamental system of solutions* of the linear, homogenuous differential equation (11). Varying the initial value x^0, we shall call the solution of the form (12), the *general solution* of the differential equation (11).

We cite again the linear, homogenuous differential equation in example 3:

$$\frac{dx}{dt} = A\,'x \tag{13}$$

where

$$A=\begin{pmatrix} -1 & 1 & & 0 \\ & -1 & 1 & \\ & & -1 & 1 \\ 0 & & & \ddots \end{pmatrix}.$$

For example, let E be provided with the (l^2) topology, and then the conditions are evidently satisfied. Thus we have the fundamental system of solutions:

$$x^1(t)=e^{-t}\{1, 0, 0, \cdots\cdots, 0, \cdots\cdots\},$$

$$x^2(t)=e^{-t}\left\{\frac{t}{1!}, 1, 0, \cdots\cdots, 0, \cdots\cdots\right\},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$x^n(t)=e^{-t}\left\{\frac{t^{n-1}}{(n-1)!}, \frac{t^{n-2}}{(n-2)!}, \frac{t^{n-3}}{(n-3)!}, \cdots\cdots, 1, 0, \cdots\cdots\right\}.$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

Therefore, if x^0 is a point of the space (l^2), for the solution $x(t;\, t_0, x^0)$ of the differential equation (13), we have the representation:

$$x(t;\, t_0, x^0)=\sum_{i=1}^{\infty} x_i^0 x^i(t). \tag{14}$$

This representation can be rewritten also in the form of coordinate functions as follows:

$$x(t;\, t_0, x^0)=\{x_1(t), x_2(t), \cdots, x_n(t), \cdots\}, \tag{15}$$

where

$$x_n(t)=\sum_{i=1}^{\infty} x_{n+i}^0 \frac{t^i}{i!} \qquad (n=1, 2, \cdots\cdots).$$

If, however, E is provided with the (m) topology, then an immediate application of the theorem is clearly impossible, since the property (AK) is not satisfied in this case. Hence the representation (14) of solution is not valid. But the second representation (15) is valid. In fact, the vector function on the right hand side of (15) is a solution of the differential equation (13) passing through the point (t_0, x^0) for the product topology, and besides continuous for the (m) topology; therefore this vector function (15) is a solution through the point (t_0, x^0), by virtue of the corollary of theorem 3, for the (m) topology. Thus the two representations (14) and (15) are not generally equivalent.

To dissolve this inharmony between these two representations of solution, we introduce another notion of (AK); for example, the space (m) possesses the property (AK) for the $\sigma(m, l)$ topology, that is, the property $\sigma(m, l)$-(AK) [6, p. 191, theorem 2]. Let the sum of a series $\sum^{\infty}$ convergent for the $\sigma(m, l)$ topology be denoted by $\sigma(m, l)$-$\sum$. Then it follows easily that

$$x(t;\, t_0, x^0)=\sigma(m, l)\text{-}\sum_{i=1}^{\infty} x_i^0 x^i(t),$$

that is, the representation (14) is valid for the $\sigma(m, l)$ topology.

This situation can be generalized as follows. Let $\mathcal{J}$ be the proper topology of a complete coordinated space E, and $\mathcal{J}'$ be another topology on E weaker than $\mathcal{J}$. To distinguish the properties (P) and (AK), convergence, limit, and sum $\sum^{\infty}$ for the topology $\mathcal{J}'$ from those for the topology $\mathcal{J}$ respectively, we denote the former by $\mathcal{J}'$-(P) and $\mathcal{J}'$-(AK), $\mathcal{J}'$-convergence, $\mathcal{J}'$-limit, and $\mathcal{J}'$-$\sum^{\infty}$ respectively. Then we shall easily obtain the following theorem.

Theorem 10. Let E be a complete coordinated space with the topology $\mathcal{J}$, and possess the properties $\mathcal{J}'$-(P) and $\mathcal{J}'$-(AK), where the topology $\mathcal{J}'$ is weaker than $\mathcal{J}$. Let the right hand side of the linear, homogeneous differential equation

$$\frac{dx}{dt}=A(t)'x$$

be continuous and satisfy Lipschitz condition in the domain $I\times E\subset R^1\times E$. Then, t_0 being fixed, the totality of the solutions $x(t; t_0, x^0)$ through the point (t_0, x^0) forms a linear topological space $\mathcal{E}$ with the topology of uniform $\mathcal{J}'$-convergence, which is topologically isomorphic to the space E, and the solution $x(t; t_0, x^0)$ is represented as follows:

$$x(t; t_0, x^0)=\mathcal{J}'\text{-}\sum_{i=1}^{\infty} x_i^0 x^i(t),$$

where $x^i(t)=x(t; t_0, e^i)$ $(i=1, 2, \cdots)$.

4. *Stability.* In this paragraph, we assume that $f(t, 0)=0$ identically for t, that is, the differential equation in the linear topological space E

$$\frac{dx}{dt}=f(t, x) \tag{2}$$

has the trivial solution $x=0$. Analogously to the case of finite-dimensional spaces, we define the stability in the sense of Liapounov as follows: the trivial solution $x=0$ of the differential equation (2) is *stable,* if, for an arbitrary pair of a neighbourhood of the origin U and a value $t_0(\geqq 0)$ of t, there exist a value T of t and a neighborhood of the origin V, such that $x^0\in V$ and $t\geqq T$ imply $x(t; t_0, x^0)\in U$, and *unstable* otherwise. The trivial solution $x=0$ is called to be *asymptotically stable,* if it is stable and if, moreover, for any neighborhood of the origin U, there exists a T_U of t, such that $t\geqq T_U$ implies $x(t; t_0, x^0)\in U$. In particular, when the space E is coordinated, furthermore we define another notion of stability as follows; the trivial solution $x=0$ of the equation (2) is *coordinate-wise stable,* if, for an arbitrary triple of coordinate number n, a positive number ε and a value $t_0(\geqq 0)$ of t, there exist a value T of t and a neighborhood of the origin V, such that $x^0\in V$ and $t\geqq T$ imply $|x_n(t; t_0, x^0)|<\varepsilon$, and *coordinate-wise unstable* otherwise The asymptotic coordinate-wise stability is defined similarly. To see from our standpoint, most Soviet mathematicians cited above treated, in reality, the stability for the (m) topology and the asymptotic coordinate-wise stability. In the following, we shall first analyze some relations between stabilities, that is, some implication relations.

In the coordinated space E, since the mapping $x \to x_n$ is continuous, it follows immediately the following theorem.

Theorem 11. In the coordinated space, the (asymptotic) stability implies the (asymptotic) coordinate-wise stability.

The converse, however, is not true, as is shown by the following example.

Example 4. In the space (l^2), we consider the differentiable equation

$$\frac{dx}{dt} = f(x),$$

where

$$\begin{aligned} f_n(x) &= -\lambda x_1 && \text{for } n=1, \\ &= x_{n-1} - \lambda x_n && \text{for } n>1, \end{aligned}$$

and λ is a positive number such that $0<\lambda<\frac{1}{4}$. The function $f(x)$ is bounded and satisfies Lipschitz condition, accordingly is continuous. In fact, a simple calculation shows that

$$\|f(x)\| \leqq (\lambda+1)\|x\|,$$

and consequently

$$\|f(x')-f(x'')\| \leqq (\lambda+1)\|x'-x''\|.$$

The fundamental system of solutions of (2) is given by

$$x^1(t) = e^{-\lambda t}\left(1, \frac{t}{1!}, \frac{t^2}{2!}, \cdots, \frac{t^{n-1}}{(n-1)!}, \cdots\right),$$

$$x^2(t) = e^{-\lambda t}\left(0, 1, \frac{t}{1!}, \cdots, \frac{t^{n-2}}{(n-2)!}, \cdots\right),$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$x^n(t) = e^{-\lambda t}\left(0, 0, 0, \cdots, 1, \frac{t}{1!}, \cdots\right),$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

where $\|x^n(t)\| = e^{-\lambda t}\sqrt{\sum_{i=1}^{\infty}\left(\frac{t^{i-1}}{(i-1)!}\right)^2}$ for all n.

Accordingly, the solution $x(t;\, t_0, x^0)$ of (2) passing through the point $(0, x^0)$ is given by

$$x(t;\, 0,\, x^0) = x_1^0 x^1(t) + x_2^0 x^2(t) + \cdots + x_n^0 x^n(t) + \cdots,$$

or

$$= \{x_1(t), x_2(t), \cdots, x_n(t), \cdots\},$$

where

$$x_1(t) = x_1^0 e^{-\lambda t},$$

$$x_2(t) = \left(x_1^0 \frac{t}{1!} + x_2^0\right) e^{-\lambda t},$$

$$x_3(t)=\left(x_1^0\frac{t^2}{2!}+x_2^0\frac{t}{1!}+x_3^0\right)e^{-\lambda t},$$

....

$$x_n(t)=\left(x_1^0\frac{t^{n-1}}{(n-1)!}+x_2^0\frac{t^{n-2}}{(n-2)!}+\cdots+x_n^0\right)e^{-\lambda t},$$

....

The trivial solution $x=0$ is evidently asymptically coordinate-wise stable. However, it is not stable. In fact, let $x^0=\left\{\varepsilon, \frac{\varepsilon}{2}, \frac{\varepsilon}{2^2}, \cdots, \frac{\varepsilon}{2^{n-1}}\cdots\right\}$, and then

$$\|x^0\|=\frac{2}{\sqrt{3}}\varepsilon$$

$$\|x(t)\|^2=\sum_{n=1}^{\infty}x_n^2(t)$$

$$>\varepsilon^2e^{-2\lambda t}\left[1+\frac{t}{2}\frac{1}{1!}+\left(\frac{t}{2}\right)^2\frac{1}{2!}+\cdots+\left(\frac{t}{2}\right)^{n-1}\frac{1}{(n-1)!}+\cdots\right]$$

$$=\varepsilon^2e^{-2\lambda t}e^{\frac{t}{2}}=\varepsilon^2e^{(\frac{1}{2}-2\lambda)t}.$$

Consequently, for an arbitrarily small ε, $t\to\infty$ implies $\|x(t)\|\to\infty$, which shows the instability of the trivial solution $x=0$.

The general solution of the above equation converges to the trivial solution $x=0$ also for the product topology, which indicates that the (asymptotic) stability for the product topology does not imply the (asymptotic) stability for the (l^2) topology. On the other hand, that the stability for the (l^2) topology does not imply the stability for the product topology, is shown by the following example.

Example 5. Let the function $\varphi(r)$ be defined as follows:

$$\varphi(r)=(r-1) \qquad \text{for } 0\leqq r\leqq 1,$$

$$=\frac{r-1}{r} \qquad \text{for } r\geqq 1.$$

In the space (l^2) we consider the equation

$$\frac{dx}{dt}=\varphi(\|x\|)x.$$

Lipschitz condition is easily verified to be satisfied:

$$\|\varphi(\|x'\|)x'-\varphi(\|x''\|)x''\|\leqq 3\rho\|x'-x''\|$$

in the sphere $\|x\|<\rho$. Hence the solution $x(t)=x(t;\, t_0, x^0)$ passing through the point $(0, x^0)$ is given by

$$\frac{x_1(t)}{x_1^0}=\frac{x_2(t)}{x_2^0}=\cdots=\frac{x_n(t)}{x_n^0}=\cdots,$$

and

$$\|x(t)\| = \frac{1}{1+\left(\frac{1}{\|x^0\|}-1\right)e^t} \qquad \text{for} \quad \|x^0\|<1,$$

$$=1 \qquad \text{for} \quad \|x^0\|=1,$$

$$=1+(\|x^0\|-1)e^t \qquad \text{for} \quad \|x^0\|>1.$$

The trivial solution $x=0$ is asymptotically stable for the (l^2) topology, but unstable for the product topology.

The above two examples show that it is impossible to find a simple implication relation between the stability for a stronger topology and that for a weaker topology. Furthermore, the above last example shows also that the stability for the product topology does not coincide with the coordinate-wise stability. In fact, the trivial solution $x=0$ of the above last example is unstable for the product topology, but stable for the (l^2) topology, accordingly by virtue of theorem 11, coordinate-wise stable. On the other hand, by virtue of the same theorem, the stability for the product topology implies the coordinate-wise stability.

Next, we shall clarify the relation between the stability of the trivial solution $x=0$ of the differential equation (2) and that of the truncated differential equations associated with (2). Here we consider the truncated differential equation

$$\frac{dx}{dt}=\tilde{\tilde{f}}_{[n]}(t, x) \tag{5}$$

only, since similar arguments are available for other truncated differential equations. If $x=0$ is a trivial solution of the differential equation (2), then it is clearly also that of the truncated differential equation (5). It follows immediately that the stability of the trivial solution $x=0$ of the equation (5) for all n does not imply that of the original equation (2). In fact, we cite again example 4, where the trivial solution $x=0$ is unstable for the (l^2) topology as was already verified. Whereas, the solution $\tilde{\tilde{x}}_{[n]}(t;\, t_0, x^0)$ through the point (t_0, x^0) of the truncated equation is

$$\tilde{\tilde{x}}_{[n]}(t;\, t_0, x^0)=\{x_1(t), x_2(t), \cdots, x_n(t), x^0_{n+1}, x^0_{n+2}, \cdots\},$$

where $x_1(t), x_2(t), \cdots, x_n(t)$ are such coordinate functions as are shown in the example. This shows the stability of the trivial solution $x=0$ of the truncated equation for all n.

We now introduce a new notion on stability in order to state a condition for that the stability of solutions of the truncated equations (5) should imply that of the original equation (2). We call the trivial solutions $x=0$ of the truncated equations (5) *stable equally for* n, or simply *equi-stable*, if, for an arbitrary pair of a neighborhorhood of the origin U and a value t_0 of t, there exist a value T of t and a neighborhood of the origin V, both independent of n, such that $x^0 \in V$ and $t \geqq T$ imply $\tilde{\tilde{x}}_{[n]}(t;\, t_0, x^0) \in U$. We obtain the following theorom.

Theorem 12. Let E be a complete coordinated space with the properties (P) and (AK). Let the right hand side $f(t, x)$ of the differential equation

$$\frac{dx}{dt}=f(t, x) \tag{2}$$

be continuous and bounded, and satisfy Lipschitz condition, in each domain $I\times D\subset R^1\times E$, *where* I *is any bounded subinterval of the half line* $[0, \infty)$ *and* D *a fixed domain of* E. *If the trivial solutions* $x=0$ *of the truncated differential equations*

$$\frac{dx}{dt}=\tilde{\tilde{f}}_{[n]}(t, x) \tag{5}$$

associated with (2) *are equi-stable, then the trivial solution* $x=0$ *of the original equation* (2) *is also stable.*

PROOF. Let U be an arbitrary neighborhood of the fundemental system of neighborhoods of the origin. Then, by the hypothesis of the equi-stability, there exist a value T of t and a neighborhood V, independent of n, such that $x^0\in V$ and $t\geqq T$ imply $\tilde{\tilde{x}}_{[n]}(t;\, t_0, x^0)\in U/2$ for all n. Let l be an arbitrary but fixed positive number. By virtue of the théorème de réduite [5, p. 238, theorem 2], there exists a positive integer N such that

$$x(t;\, t_0, x^0)-\tilde{\tilde{x}}_{[N]}(t;\, t_0, x^0)\in U/2 \qquad \text{for } t\in[T, T+l],$$

therefore

$$x(t;\, t_0, x^0)\in U \qquad \text{for } t\in[T, T+l].$$

Since l is arbitrary, this implication holds for all t of the half line $[T, \infty)$, which proves the theorem.

The condition of equi-stability for the stability of the trivial solution of the original equation (2) is only sufficient, but not necessary, which is shown by the following example.

Example 6. In the space (l), we consider the linear, homogenuous differential equation

$$\frac{dx}{dt}=A\,'x,$$

where A is a diagonal matrix of matrixes:

$$A=\begin{pmatrix} B & & & \\ & B & & \\ & & B & \\ & & & \ddots \end{pmatrix}$$

and B is a (2, 2)-matrix of the form $\begin{pmatrix} 1 & -2 \\ 3 & -4 \end{pmatrix}$.

Lipschitz condition is evidently satisfied. The general solution, that is, the solution $x(t;\, t_0, x^0)$ through the point (t_0, x^0) is given by the coordinate functions:

$$x_{2m-1}(t)=(3x^0_{2m-1}-2x^0_{2m})e^{-(t-t_0)}+2(x^0_{2m}-x^0_{2m-1})e^{-2(t-t_0)},$$
$$x_{2m}(t)=(3x^0_{2m-1}-2x^0_{2m})e^{-(t-t_0)}+3(x^0_{2m}-x^0_{2m-1})e^{-2(t-t_0)}$$
$$(m=1, 2, \cdots).$$

Thus the trivial solution $x=0$ of the given equation is clearly (asymptotically) stable, while the trivial solution of the truncated equation is stable for n even and unstable for n odd.

Hence the condition of equi-stability of the theorem may be expected to weaken. The equi-stability above defined is called anew the *equi-stability in the strong sense*, and in case the selection of the neighborhood V in the above definition is independent (uniform) of not all, but of an infinite number of values of n, the equi-stability is called the *equi-stability in the weak sense*. Then, with a slight modification of the proof, the above theorem can be replaced by the following theorem.

Theorem 13. Assume that the supposition of the theorem 12 is fulfilled. If the trivial solutions $x=0$ *of the truncated equations* (5) *are equi-stable in the weak sense, then the trivial solution* $x=0$ *of the original equation* (2) *is stable.*

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