148

ON LINEAR TOPOLOGICAL SPACES

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This is a collection of results on linear topological spaces, particularly on tonnelé or bornologic spaces. The most part of the work was done during 1956–1959, at the University of Tokyo. The author wishes to express his hearty thanks to Professors K. Yosida and S. Irie for their kind advices.

1. Embedding in tonnelé or bornologic spaces.

N. Bourbaki introduced in [1] the notion of tonnelé spaces and remarked that an arbitrary locally convex complete space can be considered as a closed subspace of some tonnelé space. In this section we show more generally that an arbitrary locally convex (not necessarily complete) space is a closed subspace of some tonnelé space, and show an analogous property for bornologic spaces under some restrictions.

LEMMA 1.1 (*Mackey-Dieudonné*) *Every subspace F with co-dimension 1 of a tonnelé space E is also tonnelé.*

PROOF. Let U be an arbitrary closed convex circular absorbing set of F. When $\overline{U}$(=the closure of U in E) contains some point $x_0 \notin F$, $\overline{U}$ is a neighbourhood of zero in E, since $\overline{U}$ is a closed convex circular absorbing set of the tonnelé space E. In this case $\overline{U}=U\cap F$ is a neighbourhood of zero in F. When $\overline{U}\subset F$, we have $U=\overline{U}\cap F=\overline{U}$. In this case, for an element x_0 of E such that $x_0 \notin F$, the set $\widetilde{U}=\{\lambda x_0+y;\ |\lambda|\leq 1,\ y\in U\}$ is a closed convex circular absorbing set of E. Hence $\widetilde{U}$ is a neighbourhood of zero in E, and $U=\widetilde{U}\cap F$ is a neighbourhood of zero in F.

Thus in every case U is a neighbourhood of zero in F.

LEMMA 1.2 *An arbitrary complete locally convex space E is a closed subspace of some product space ΠE_α of Banach spaces E_α.*

This is a well known fact and we show outline of the proof. Pick up a fundamental semi-norm system $\{P_\alpha\}$. For each semi-norm P_α, we define a normed space $F_\alpha=E/N_\alpha$, with the norm induced by P_α, where $N_\alpha=\{x\in E;\ P_\alpha(x)=0\}$. Then the space E is embedded in natural way into the product space of the completions of F_α's.

THEOREM 1.1 *An arbitrary locally convex space E is a closed subspace of some tonnelé space.*

PROOF. Let F be a tonnelé space containing the completion of E, and $\{e_\alpha\}$ be a Hamel base (=maximal linearly independent set) of a compliment space F^C of F.

For any index α, let F_α be the subspace of F generated by E and $\{e_\beta\}_{\beta\neq\alpha}$. Since F_α is of co-dimension 1, it is tonnelé.

The space E is naturally embedded in ΠF_α, which is a tonnelé space. Precisely, $x \in E$ is identified to $(x_\alpha) \in \Pi F_\alpha$ if and only if $x_\alpha = x$ for any index α.

If some directed set $(x_\alpha)^\varphi = (x_\alpha^\varphi)$ in E converges to $(y_\alpha) \in \Pi F_\alpha$, then for any α, x_α^φ converges to y_α. Since x_α^φ is independent of α, y_α is independent of α. Thus $y_\alpha = y \in \bigcap_\alpha F_\alpha = E$, which means the closedness of E in ΠF_α. q. e. d.

Since the dual F' of a tonnelé space F is semi-reflexive with respect to the Mackey topology $\tau(F', F)$, we have the following corollary dual of the theorem.

COROLLARY. *An arbitrary locally convex space E is a quotient space of some semi-reflexive space.*

PROOF. Without losing the generality we may suppose that the topology of E is equal to $\sigma(E, E')$. By virtue of the theorem 1.1 there exists a tonnelé space F which contains E' with the topology $\sigma(E', E)$ as a closed subspace. Then the space $E \cong$ (*the dual of E' with respect to* $\tau(E', E)) \cong F'/E'^{\perp}$. q. e. d.

Concerning bornologic spaces, the same lemma as lemma 1.1 holds good. As to lemma 1.2, the product space ΠE_α of Banach spaces E_α is bornologic if the power of the index set is less than some cardinal number d. (See Köthe [6]). We denote by $\aleph$ the dimension of E (= the power of a maximal linearly independent set). Then, we have the relation (*the power of* $\{\alpha\}) \leq 2^{\aleph}$, since a semi-norm P_α is uniquely determined by a subset $\{x \in E;\ P_\alpha(x) < 1\}$ of E. Suppose that $\aleph < d$. By virtue of the theorem of Mackey-Ulam (see Köthe [6]), $\aleph < d$ implies $2^{\aleph} < d$. Therefore, we have the following lemma.

LEMMA 1.3. *An arbitrary complete locally convex space E with the dimension $\aleph < d$ is a closed subspace of some bornologic space F.*

For the bornologic space F, dim $F < \aleph^{\aleph}$, hence dim $F < d$. Therefore the space E is naturally embedded as a closed subspace in some product space ΠF_α of bornologic spaces F_α, in a same way as the proof of the theorem 1.1. The power of the index set $\{\alpha\}$ is $\leq$ dim F, hence the space ΠF_α is bornologic. Thus we have the following theorem.

THEOREM 1.2. *An arbitrary locally convex space with the dimension $< d$ is a closed subspace of some bornologic space.*

In particular, *an arbitrary locally convex separable space is a closed subspace of some bornologic space,* since the dimension of such a space is not larger than $2^{2^{\aleph_0}}$.

Dually we have a corollary.

COROLLARY. *A separable tonnelé space is a quotient space of some complete space.*

More precisely, a locally convex space E, with the topology $\beta(E, F)$ for some subspace F of E', is a quotient space of some complete space. The author cannot tell whether any locally convex space with Mackey topology is a quotient space of some complete space or not.

2. Tonnelé topology defined by locally convex topology.

Let E be a fixed linear space and τ be a locally convex topology on E. We consider the set $\{\tau_\alpha\}$ of all tonnelé topology which are stronger than τ. The limit inductive topology $\bar{\tau}=\bigcap\tau_\alpha$ of $\{\tau_\alpha\}$, that is, the strongest locally convex topology on E weaker than every τ_α, is uniquely defined. The topology $\bar{\tau}$ is the weakest among the tonnelé topologies stronger than τ.

DEFINITION 2.1. For a locally convex topology τ (on E), the weakest tonnelé topology $\bar{\tau}$ stronger than τ is called *the tonnelé topology defined by* τ.

We denote by $E(\tau)$ (or E_τ) the linear space E with the topology τ. Then the strong topology $\beta(E, E(\tau)')$ is the topology generated by a fundamental neighbourhood system of zero which are all τ-closed convex absorbing sets. We denote by τ^1 the strong topology $\beta(E, E(\tau)')$.

For any ordinal number α we define $\tau^\alpha=\tau^{\beta,1}(=\beta(E, E(\tau^\beta)')$ if $\alpha=\beta+1$, and $=$ the limit projective topology $\bigcup_{\beta<\alpha}\tau^\beta$ if α is a limit number.

For any τ there exists an ordinal number α such that $\tau^\alpha=\tau^{\alpha+1}$. Evidently such a topology τ^α is identical with the tonnelé topology $\bar{\tau}$ defined by τ.

THEOREM 2.1. *Let E be a linear space with a locally convex topology τ. Then the τ-completion of E contains the $\bar{\tau}$-completion of it, where $\bar{\tau}$ is the tonnelé topology defined by τ.*

PROOF. We shall prove using transfinite induction. We assume that the τ^γ-completion of E contains the τ^β-completion of it for any γ and any β such that $\gamma<\beta<\alpha$. When $\alpha=\beta+1$ for some β, the τ^α-completion of E is contained in the τ^β-completion of it, by virtue of Grothendieck's theorem. When α is a limit number, then $E(\tau^\alpha)'=\bigcup_{\beta<\alpha}E(\tau^\beta)'$ and any equi-continuous set in $E(\tau^\alpha)'$ is contained and equi-continuous in some $E(\tau^\beta)'$.

On the other hand, the τ^γ-completion of E (= the set of all linear functionals weakly continuous on each τ^γ-equi-continuous set) contains the τ^β-completion of E (= the set of all linear functionals weakly continuous on each τ^β-equi-continuous set), by the assumptions of induction. Hence a linear functional which is weakly continuous on each equi-continuous set in some $E(\tau^\gamma)'$ is, if extensible, uniquely extended to a linear functional which is weakly continuous on each equi-continuous set of $E(\tau^\beta)'$ for $\gamma<\beta<\alpha$.

Let l be an arbitrary element of the τ^α-completion of E. Then l is weakly continuous on each τ^α-equi-continuous set in $E(\tau^\alpha)'$, hence it is weakly continuous on each τ^γ-equi-continuous set in $E(\tau^\gamma)'$ for any $\gamma<\alpha$. Since the restriction l^γ of l to $E(\tau^\gamma)'$ is uniquely extended to the restriction l^β of l to $E(\tau^\beta)'$ for $\gamma<\beta<\alpha$, we may identify $l^\gamma=l^\beta$. The relation $E(\tau^\alpha)'=\bigcup_{\beta<\alpha}E(\tau^\beta)'$ implies that the inverse of the mapping: $l\to l^\beta$ is unique, hence the canonical mapping from the τ^α-completion of E to the τ^β-comletion of E is one-to-one. q. e. d.

Slightly modifying, we can verify the following corollary, a generalization of Grothendieck's theorem.

COROLLARY. *Let σ and τ be two locally convex topologies on some linear space E. The σ-completion of E contains the τ-completion of E if the following conditions are satisfied:*
1) $\sigma<\tau$,
2) *a linear space F, such that $E'_\sigma\subset F\subset E'_\tau$ and the intersection of F with each weakly closed equi-continuous set in E'_τ is weakly closed, is necessarily identical with E'_τ.*

3. Closed graph theorem and minimal topology.

In this note we discuss the closed graph theorem in abstract form. The result is essentially contained in [2], [4], [5] and [8].

LEMMA 3.1. *Let E and F be locally convex spaces, and u be a linear operator from E to F. (The domain of u is a whole space E.) u is a closed operator if and only if u is continuous from E to $F_\tau=F$ with some weaker separated topology τ.*

PROOF. Suppose that such a topology τ exists. If a directed set $\{x_\varphi\}\subset E$ converges to x in E, then $u(x_\varphi)$ converges to y in F_τ. If moreover $u(x_\varphi)$ converges to y' in F, y coinsides with y' since the canonical mapping $F\to F_\tau$ is continuous. This means that u is a closed operator. q. e. d.

Conversely, we suppose that u is not continuous with respect to any weaker separated topology. Then the topology generated by a system of neighbourhoods $\{U+u(V)$: $U=$neighbourhood of F, $V=$neighbourhood of $E\}$ is not separated. Hence there exists some $x_0\in F, x_0\neq 0$, such that $x_0\in U+u(V)$ for any neighbourhood U of zero in E and any neighbourhood V of zero in F. Therefore there exists a directed set $\{x_{U,V}\}$ such that $x_{U,V}\in V$ and $u(x_{U,V})\in U+x_0$, that is, $x_{U,V}\to 0$, $u(x_{U,V})\to x_0\neq 0$. This means that u is not a closed operator. q. e. d.

We consider some property (α) of locally convex spaces. Property (α) is called *invariant under finite (limit) inductive operations* if the following condition is satisfied: (A) *Let $\{E_i\}$ be a set of finite (infinite) (α)-spaces, and u_i be a linear operator from E_i to same linear space E for any i, and some u_j be an operator onto E. Then the space E, given the strongest locally convex topology such that each u_i is continuous, is also an (α)-space.*

For example, the property of "*tonnelé*" is invariant under limit inductive operations.

PROPOSITION 3.1. *Let (α) be invariant under finite inductive operations. Any closed linear operator from any (α)-space to a fixed (α)-space E is continuous if any only if there is no separated (α)-topology on E weaker than the original topology.*

The proof follows immediately from the lemma 3.1.

COROLLARY. *Let E be a $\sigma(E,E')$-complete space, that is, E is isomorphic to a direct product space of finite dimensional spaces. Then any closed linear operator from a locally convex space to E is continuous.*

PROOF. On such a space E, there exists no weaker separated topology. Hence if we consider the property (α) as the property of locally convex topologies, our corollary is obtained immediately. q. e. d.

We call an (α)-topology on E the *(α)-minimal topology* if there is no separated weaker (α)-topology. Then we may say that *E has the property of the closed graph theorem with respect to (α) if and only if E is endowed with an (α)-minimal topology.*

A space E which is a limit inductive of (α)-spaces $\{E_\lambda\}$ is called *an $(\bar{\alpha})$-space.* Thus we have an extension $(\bar{\alpha})$ of the class of (α)-spaces. Evidently property $(\bar{\alpha})$ is invariant under limit inductive operations.

For any locally convex topology τ on E, there exists the unique $(\bar{\alpha})$-topology $\bar{\tau}$ which is the weakest among the $(\bar{\alpha})$-topologies stronger than τ. In fact, $\bar{\tau}$ is the limit inductive topology of all $(\bar{\alpha})$-topologies stronger than τ.

We call *$\bar{\tau}$ the $(\bar{\alpha})$-topology defined by τ.*

THEOREM 3.1. *Let E be a locally convex space with the topology τ. Any linear closed operator u from any $(\bar{\alpha})$-space F to E is necessarily continuous if and only if the $(\bar{\alpha})$-topology $\bar{\tau}$ defined by τ is identical with the $(\bar{\alpha})$-topology $\bar{\sigma}$ defined by any separated locally convex topology σ on E weaker than τ.*

PROOF. Let u be a closed operator from an $(\bar{\alpha})$-space F to E. Then by virtue of lemma 3.1 there exists some separated locally convex topology σ weaker than τ, and u is continuous from F to $E(\sigma)$. Since $(\bar{\alpha})$ is invariant under finite inductive operations, u is continuous from F to $E(\bar{\sigma})$. Therefore u is continuous from F to $E(\tau)$ if $\bar{\sigma}>\tau$. Since $\sigma<\tau$, the condition $\bar{\sigma}>\tau$ means $\bar{\sigma}=\bar{\tau}$. Conversely, if there exists some separated locally convex topology σ weaker than τ such that $\bar{\sigma}\neq\bar{\tau}$, then the identity mapping u: $E(\bar{\sigma})\rightarrow E(\tau)$ is not continuous. Since u is a closed operator from $E(\sigma)$ to $E(\tau)$, it is also a closed operator from $E(\bar{\sigma})$ to $E(\tau)$. q. e. d.

We explain the case of tonnelé spaces. The property of tonnelé is invariant under limit inductive operations, that is, a limit inductive space of tonnelé spaces is also tonnelé. Therefore the extension of the class of tonnelé spaces by limit inductive operations is also the class of tonnelé spaces.

By virtue of the proposition 3.1, a tonnelé space E has a tonnelé-minimal topology τ if and only if any closed operator from any tonnelé space to E is continuous. Evidently a locally convex topology τ is tonnele-minimal if and only if the tonnelé topology $\bar{\sigma}$, defined by any locally convex topology σ weaker than τ, is identical with τ. In other words, E is a tonnelé-minimal space if and only if the following condition is satisfied: "for a $\sigma(E',E)$-dense subspace F of E', if the intersection of F with each $\sigma(E',E)$-closed equi-continuous set in E' is necessarily $\sigma(E',E)$-closed, then we have $F=E'$". Since a subspace with co-dimension 1 is dense or closed, for a tonnelé-minimal space E a subspace F of E' with co-dimension 1 is $\sigma(E',E)$-closed when the intersection of F with each $\sigma(E',E)$-closed equi-continuous set is closed. Therefore such a space E is complete.

In addition, we give the following proposition.

PROPOSITION 3.2. *Let E be a locally convex space and E_0 be its dense subspace. If a closed linear operator from E to a locally convex complete space F is continuous on E_0, then it is continuous on E.*

The proof is almost obvious. It is to be noted that *for a non-complete space F the* *above proposition is not true in general.* In fact, let H be a dense subspace of E with co-dimension 1. (Such a subspace exists if and only if there exists some discontinuous linear functional on E. For instance, an infinite dimensional normed space has always such a subspace.) If F is the direct sum $H \oplus R^1$ of H and 1-dimensional space R^1, then the one-to-one linear operator u: $F \to E$, which is the identity map on H, is continuous. Therefore the inverse operator u^{-1}: $E \to F$ is a closed operator which is continuous on a dense subspace H, but it is not continuous.

4. A class of boundedly closed system.

A. Grothendieck showed in [3], lemma 9, that the following three conditions for a locally convex space E are equivalent to each other.

a) *All continuous linear operator from E to any Banach space is weakly compact.*

b) *For any convex circular neighbourhood V of the origin in E, there exists a convex circular neighbourhood $U \subset V$ such that the canonical mapping from the Banach space $\hat{E}_U$ (= the completion of the normed space E/N_U, $N_U = \{x \in E;\ \lambda x \in U$, for any scalar $\lambda\}$, with the unit ball U) to the Banach space $\hat{E}_V$, is weakly compact.*

c) *For any convex circular equi-continuous weakly closed subset A of E', there exists a convex circular equi-continuous weakly closed subset B of E' such that A is a weakly compact set in the Banach space E'_B (= the normed space generated by B with unit ball B).*

Spaces E satisfying the above property are a generalization of Schwartz spaces. In fact, if we replace the term "weakly compact" by "compact", then we have the definition of Schwartz spaces.

We shall prove that *the above conditions are equivalent to the following condition.*

d) *E is naturally embedded in a direct product ΠE_α of Banach spaces E_α (see § 1) such that each E_α is the dual space F'_α of some Banach space F_α and $E' = \Sigma F_\alpha / E^\perp$.*

PROOF. Let $\{V_\alpha\}$ be a fundamental system of neighbourhoods of the origin such that each V_α is convex circular and closed. Assume the condition b). Then there exists another fundamental system $\{U_\alpha\}$ of neighbourhoods such that for any α, $U_\alpha \subset V_\alpha$ and the canonical image of U_α in $\hat{E}_{V_\alpha}$ is relatively weakly compact. Then the closure $\bar{U}_\alpha$ of the image of U_α is $\sigma(\hat{E}_{V_\alpha}, \hat{E}'_{V_\alpha})$-compact. Hence $(\hat{E}_{V_\alpha})_{\bar{U}_\alpha} =$ the dual of the normed space $(E_{V_\alpha})'_{U_\alpha^0}$ (= the space generated by U^0_α with the unit ball U^0_α, where $U^0_\alpha = \{x' \in (E_{V_\alpha})';\ \sup_{x \in \bar{U}_\alpha} |\langle x, x \rangle| \leq 1\}$).

We put $E_\alpha = (\hat{E}_{V_\alpha})_{\bar{U}_\alpha}$ and $F_\alpha =$ the completion of $(E_{V_\alpha})'_{U_\alpha^0}$. For any two indices α and β such that $U_\alpha \subset U_\beta$ and $V_\alpha \subset V_\beta$, there exists the canonical mapping $u_{\beta\alpha}$: $F_\beta \to F_\alpha$.

For any F_α, by virtue of the condition c) there exists an F_β such that the mapping $u_{\beta\alpha}$ is weakly compact. Then the bi-transposed operator ${}^{tt}u_{\beta\alpha}$: $E'_\beta = F''_\beta \to F''_\alpha = E'_\alpha$, maps $E'_\beta = F''_\beta$ to F_α. For any pair x, y such that ${}^{tt}u_{\beta\alpha}(x) = y$, the element $x - y \in \Sigma E'_\alpha$ is contained in $E^\perp$, and therefore, we conclude that $\Sigma F_\alpha / E^\perp = \Sigma F''_\alpha / E^\perp$.

The converse part of our assertion is almost obvious. q. e. d.

From the condition d), if such a space E is complete, we have $E=(\Sigma F_\alpha/E^\perp)'$. In this case, its dual space E' with the Mackey topology $\tau(E',E)$ is a (β)-space (= a limit inductive of Banach spaces), and E is the set of all linear functionals bounded on each equi-continuous set in E'. In particular, we have the following corollary.

COROLLARY. *A Schwartz space E is complete if and only if it is the set of all linear functionals bounded on each equi-continuous set in E'.*

5. Closedness and quasi-closedness.

Let E be a locally convex space and $\mathfrak{B}$ be a system of bounded sets. A subset A of E is said to be *quasi-closed* (or *quasi-complete*) *with respect to* $\mathfrak{B}$ if, for any $B\in\mathfrak{B}$, $A\cap B$ is closed (or complete) in B. When $\mathfrak{B}$ is the set of all closed bounded sets, we shall omit the term "with respect to $\mathfrak{B}$" for the sake of brevity. For instance, when $\mathfrak{B}$ is the set of all metrizable precompact sets, a subset A is quasi-closed with respect to $\mathfrak{B}$ if and only if A is sequentially closed.

The problem, to ask in what case the closedness (or completeness) follows from the quasi-closedness (or quasi-completeness), may be one of the most important but difficult problems in the theory of linear topological spaces. (For instance, this is closely connected with the closed graph theorems on tonnelé spaces. See § 2). It is well known that in case of (DF)-spaces, the completeness follows from the quasi-completeness, and in case of (F)-spaces, the closedness of subspaces follows from the quasi-closedness. We shall give generalizations of these cases.

PROPOSITION 5.1. *A locally convex space E is closed in the bidual E'' with respect to the natural topology (= topology of uniform convergence on each equi-continuous set in E') if and only if E is quasi-complete.*

PROOF. Let $\bar{E}$ be the closure of E in E''. For any $x\in E''$, there exists a convex circular bounded set $B\subset E$ such that $x\in\tilde{B}=\sigma(E'',E')$-closure of B. If $x\in\bar{E}$, then $x\in\tilde{B}\cap\bar{E}=\sigma(\bar{E},E')$-closure of B. Since the dual of $\bar{E}$ with respect to the natural topology is equal to E', $\tilde{B}\cap\bar{E}=$ the closure of B with respect to the natural topology, in which x is contained. The quasi-completeness of E implies $\tilde{B}\cap\bar{E}\subset E$, and therefore $\bar{E}\subset E$.

Conversely, if E is closed in E'' with respect to the natural topology, then E is quasi-complete since E'' is quasi-complete with respect to $\mathfrak{B}=\{\tilde{B}=\sigma(E'',E')$-closure of B; B is any bounded set in $E\}$. q. e. d.

By virtue of the above proposition, for a space E such that E'' is complete with respect to the natural topology, the completeness follows from the quasi-completeness. This is a usual method to prove the completeness of quasi-complete (DF)-spaces. (See Köthe [6].)

We say that *a sequence $\{x_k\}$ converges to x_∞ in the sense of Mackey* if, for some infinitely increasing sequence $\{\lambda_k\}$ of scalars, the set $\{\lambda_k(x_\infty-x_k)\}$ is bounded.

PROPOSITION 5.2. (Mackey) *Let E be a bornologic space and H be its subspace with co-dimension* 1. *If H is sequentially closed in the sense of Mackey, then it is closed.*

PROOF. Assume that H is not closed. We pick up an element x_0 of $E, \notin H$. Let B be an arbitrary bounded set of E. Then for some positive number λ_0 and bounded set $A \subset H$, the set B is contained in the set $\{x=y+\lambda x_0;\ y \in A, |\lambda| \leq \lambda_0\}$. In fact, if, for $x_k=y_k+\lambda_k x_0 \in B$, $|\lambda_k|$ is infinitely increasing, then $-y_k/\lambda_k = x_0 - x_k/\lambda_k \to x_0$ in the sense of Mackey, which contradicts to the sequential closedness of H. Hence there exists a scalar λ_0 and a subset A of H such that $B \subset \{x=y+\lambda x_0;\ y \in A, |\lambda| \leq \lambda_0\}$. We see easily that A can be chosen to be bounded.

From the above fact, the convex circular set $V=\{x=y+\lambda x_0;\ y \in H, |\lambda| \leq 1\}$ absorbs each bounded set in E. In other words, there exists some bounded linear functional u on E which is identically zero on H and $u(x_0) \neq 0$. Since we assume that H is not closed, that is, H is dense in E, E is not bornologic. This is a contradiction. q. e. d.

Let E be a locally convex space and H be its subspace which is sequentially closed in the sense of Mackey. If for an element $x_0 \in E, \notin H$, the space $H \dotplus x_0$ generated by H and x_0 is a bornologic subspace of E, then H is closed in $H \dotplus x_0$ by virtue of the above proposition, that is, the closure of H does not contain x_0. Hence, if $H \dotplus x$ is bornologic for any $x \in E, \notin H$, then H is closed in E. Thus we have the following corollary.

COROLLARY. *Let E be a locally convex space whose any subspace is bornologic. Then an arbitrary sequentially closed subspace in the sense of Mackey is always closed.*

This gives another (and more complicated) proof of the fact that an arbitrary sequentially closed subspace of a locally convex metrizable space is closed.

It is well known that a locally convex space which possesses a dense tonnelé subspace is tonnelé too. This is not true for bornologic case. In fact, we know many example of bornologic and sequentially complete, but non-complete spaces. Such spaces are densely contained in (tonnelé and) non-bornologic spaces. In the following, we shall show *the existence of tonnelé (DF) and non-bornologic spaces*, which gives the negative answer to the problem 3) in [3].

G. Köthe gave an example of a non-complete limit inductive space E of countable Banach spaces $\{E_n\}$ (See [6]). Let x_0 be an element of the completion $\hat{E}$ of E, such that $x_0 \notin E$. Since E'', which is the dual space of an (F)-space, possesses a fundamental sequence $\{B_n\}$ of bounded sets (that is, an arbitrary bounded set $B \subset E''$ is contained in some (B_n), the subspace $\tilde{E}=E \dotplus x_0$ of E generated by E and x_0 possesses a fundamental sequence $\{B_n \cap \tilde{E}\}$ of bounded sets.

Since $\tilde{E}$ contains a dense tonnelé subspace $E, \tilde{E}$ itself is tonnelé. Therefore $\tilde{E}$ is a (DF)-space. If E is sequentially complete in the sense of Mackey, $\tilde{E}$ is not bornologic by virtue of the proposition 5.2. Hence it suffices to prove the following lemma.

LEMMA 5.1. *The limit inductive space E of a sequence $\{E_n\}$ of Banach spaces is sequentially complete in the sense of Mackey.*

PROOF. From the assumption, we have $E=(\Sigma E_n)/N$, where N is a closed subspace of ΣE_n. Hence $E'=N^{\perp}\subset\Pi E'_n$ and $E''=(\Sigma E''_n)/N^{\perp\perp}$, since the topology of E' is the topology induced by $\Pi E'_n$. Since E'' is complete, the completion $\hat{E}$ of E is contained in E''. If E is complete, our assertion is obvious. We consider the case $\hat{E}\neq E$. Let x_0 be an element of $\hat{E}$ such that $x_0\notin E$. If a sequence $\{x_k\}\subset E$ converges to x_0 in the sense of Mackey, for some infinitely increasing sequence λ_k and a bounded set $B\subset\hat{E}$, we have $\{\lambda_k(x_0-x_k)\}\subset B$. Since B and $\{x_k\}$ are bounded sets of E'', we may assume, without losing the generality, that B and $\{x_k\}$ are contained in the unit ball of some E''_n, that is, $\|x_0-x_k\|_n\leq 1/\lambda_k$, where $\|\ \|_n$ denotes the norm of E''_n. Therefore x_k converges to x_0 in E_n. Since E_n is complete, we have $x_0\in E_n$, which contradicts to the assumption that $x_0\notin E$. q. e. d.

We shall give *an example of semi-reflexive spaces which are not complete with respect to the Mackey topology.* (See [6] p. 311.) A locally convex space is semi-reflexive if and only if it is quasi-complete with respect to the weak topology (in other words, its any convex closed bounded set is weakly compact.)

EXAMPLE. $\aleph_0$, $\aleph_1$, or $\aleph_2$ denotes the first, second or third infinite cardinal number respectively. Let A be a set of power $\aleph_2$. We denote by $\omega(A)$ the direct product $\prod_{\alpha\in A} R_\alpha$ of each one dimensional space R_α, and denote by $\omega_0(A)$ the set $\{(x_\alpha)\in\omega(A);\ x_\alpha=0$ except countable indices $\alpha\}$. Then $\omega_0(A)$ is a tonnelé subspace of $\omega(A)$.

Let $E=\omega_0(A)\dotplus 1_A$ (that is, the space generated by $\omega_0(A)$ and constant function $1_A=$ the elements of $\omega(A)$ whose coordinates are identically 1). Then the space E, with the topology induced by $\omega(A)$, is tonnelé, and therefore the dual $E'(=\sum_{\alpha\in A} R_\alpha)$ is semi-reflexive with respect to the Mackey topology $\tau(E',E)$. We shall prove that E' is not complete with respect to the Mackey topology $\tau(E',E)$ We need the following lemma.

LEMMA 5.2. *Let B be a bounded set in $\omega_0(A)$. If the $\sigma(\omega(A),\sum_{\alpha\in A} R_\alpha)$-closure $\bar{B}$ of B contains 1_A, then it necessarily contains another element of $\omega(A)$ which is not contained in $\omega_0(A)$.*

By virtue of the above lemma, for any convex circular $\sigma(E',E)$-compact set B of E, the intersection $B\cap\omega_0(A)$ is closed in E. Hence the linear functional u on E, such that $u(1_A)=1$ and $u(x)=0$ for any $x\in\omega_0(A)$, is $\sigma(E',E)$-continuous on each convex circular $\sigma(E',E)$ compact set in E, that is, u is contained in the $\tau(E',E)$ completion of E'. However, u is not contained in E' since $\omega_0(A)$ is $\sigma(E',E)$-dense in E.

PROOF OF THE LEMMA. Let A_1 be a subset of A with the power $\aleph_1$. Then for any $\alpha\in A_1$, there exists a countable subset $B_1^\alpha=\{(x_\beta)_n(=(x_\beta^{(n)}));\ n=1,2,\cdots\}$ of B such that $|x_\alpha^{(n)}-1|<1/n$. We put $B_1=\bigcap_{\alpha\in A_1} B_1^\alpha$ and $A_2=\{\beta\in A$; the β-th coordinate x_β of some element $(x_\beta^{(n)})$ of B_1 is not zero$\}$. Then A_2 is of power $\aleph_1$, since the coordinates of an element $(x_\beta^{(n)})$ of $\omega_0(A)$ are zero except countable indices.

In the similar way as above we can construct the sequence B_n of subsets of B

and the sequence A_n of sets of indices with power $\aleph_1$, satisfying

1) for any $\beta \in A_n$ and any $\varepsilon > 0$, there exists some $(x_\alpha) \in B_n$ such that $|x_\beta - 1| < \varepsilon$,

2) for any index $\beta \notin A_{n+1}$, the β-th coordinate x_β of any element (x_α) of B_n is zero.

Therefore for the sets $A_\infty = \bigcup_{n=1}^{\infty} A_n$ and $B_\infty = \bigcup_{n=1}^{\infty} B_n \subset B$, we have

1)′ for any $\beta \in A_\infty$ and any $\varepsilon > 0$, there exists some $(x_\alpha) \in B_\infty$ such that $|x_\beta - 1| < \varepsilon$,

2)′ for any index $\beta \notin A_\infty$, the β-th coordinate x_β of any element (x_α) of B_∞ is zero.

The conditions 1)′ and 2)′ imply that the $\sigma(\omega(A), \sum_{\alpha \in A} R_\alpha)$-closure of B_∞ contains the element 1_{A_∞} of $\omega(A)$, whose α-th coordinate is equal to 1 for $\alpha \in A_\infty$, is equal to 0 for $\alpha \notin A_\infty$. Since the power of A_∞ is $\aleph_1$, the element 1_{A_∞} is a required element.

q. e. d.

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