THE INFERENCE THEORY IN MULTIVARIATE RANDOM EFFECT MODEL (I)

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1. Introduction.

In this paper we shall be concerned with the estimation of the parameters in the multivariate random effect model. The multivariate random effect model is understood to be a model where the observations are given by the multi-dimensional vectors and consequently the treatment effects, which are the normal variables, are also the multi-dimensional vectors. We shall discuss this problem under certain restrictions for the covariance matrices, while it seems to be more desirable to prove the problem under more general assumptions free from these restrictions.

Concerning the theory of estimation in the similar model as ours, the author should like to mention here in the first place the work of S. N. Roy and R. Gnanadesikan [5][1], in which their detailed discussions are concerned with the restricted model having the treatment effects whose covariance matrices are proportional to each other, and secondly the work of F. Grybill and R. A. Hultquist [9] which is concerned with the case of the univariate model.

The main results of this paper are Theorem 4.1 and Theorem 6.2. The former theorem gives the necessary and sufficient condition for the covariance matrices to be estimable, under our resticted model, while the latter gives the theorem concerning the completeness of the family of the distributions of the sufficient statistics in our concern. Section 3 is devoted to the derivation of the covariance matrix of all observations, and Section 5 to the discussion of some properties concerning the characteristic roots of covariance matrix, which seems to be crucial for the estimation theory under our model.

2. Preliminaries.

Let $\mathbf{Y}(N\times p)$ be a set of N observable stochastic p-dimensional vectors whose model equation is given by the following.

$$\mathbf{Y}(N\times p)=\mathbf{X}_0\mathbf{B}_0(1\times p)+\sum_{i=1}^{k}\mathbf{X}_i(N\times m_i)\mathbf{B}_i(m_i\times p)+\mathbf{X}_{k+1}(N\times N)\mathbf{B}_{k+1}(N\times p), \tag{2. 1}$$

where we assume

(i) $\mathbf{B}_0(1\times p)=[\mu_1, \mu_2, \cdots, \mu_p]$ is a p-dimensional vector with the fixed but unknown constants μ_i's$(i=1, 2, \cdots, p)$;

(ii) $\mathbf{B}_i(m_i\times p)$ is a random sample of size m_i from the p-variate normal population $\mathrm{N}[\mathbf{O}(1\times p), \Sigma_i(p\times p)]$ for $i=1, 2, \cdots, k+1$;

1) Numbers in brackets refer to the references of the end of the paper.

(iii) $\mathbf{B}_{k+1}(N\times p)$, which denotes the error term, is a random sample of size N from the p-variate normal population $N[\mathbf{O}(1\times p), \Sigma_{k+1}(p\times p)]$;

(iv) $\mathbf{B}_i$'s $(i=1, 2, \cdots, k)$ and $\mathbf{B}_{k+1}(N\times p)$ are mutually independent;

(v) $\mathbf{X}_0(N\times 1)=\mathbf{1}(N\times 1)$ is a N-dimensional vector of 1's, $\mathbf{X}_i$'s $(i=1, 2, \cdots, k)$ are the matrices of known constants, and $\mathbf{X}_{k+1}(N\times N)=\mathbf{I}(N\times N)$ is the identity matrix.

In what follows for the sake of simplicity we write sometimes $\mathbf{B}_0, \mathbf{B}_i, \mathbf{B}_{k+1}$ and X_i etc. instead of $\mathbf{B}_0(1\times p), \mathbf{B}_i(m_i\times p), \mathbf{B}_{k+1}(N\times p)$ and $X_i(N\times m_i)$ etc..

Throughout this paper we shall write $n\times n$ identity matrix as $\mathbf{I}(n\times n)$, $\mathbf{E}(n\times n)$ denotes the $n\times n$ matrix with the elements all equal to 1.

Let $\mathbf{H}(n\times n)$ be the $n\times n$ matrix with the elements all equal to zero except for the (1, 1)-element equal to 1. $\mathbf{A}_i(N\times N)$ denotes $\mathbf{X}_i\mathbf{X}_i'$ for $i=1, 2, \cdots, k+1$ and $\mathbf{A}_{k+1}$ is equal to $\mathbf{I}(N\times N)$.

Further let $\mathbf{P}(N\times N)$ be defined as any orthogonal matrix whose elements in the first row are all equal to $\frac{1}{\sqrt{N}}$, and also let $\mathbf{Q}(p\times p)$ be any orthogonal matrix whose elements in the first row are all equal to $\frac{1}{\sqrt{p}}$.

Then we have easily

$$\mathbf{P}(N\times N)\mathbf{E}(N\times N)\mathbf{P}'(N\times N)=N\mathbf{H}(N\times N) \tag{2. 2}$$

and

$$\mathbf{Q}(p\times p)\mathbf{E}(p\times p)\mathbf{Q}'(p\times p)=p\mathbf{H}(p\times p). \tag{2. 3}$$

In this paper, the Kronecker product of two matrices are defined in the way reverse to the usual ones for the sake of convenience. Thus for $\mathbf{C}=(C_{ij})$ and $\mathbf{D}=(d_{ij})$, the Kronecker product denoted by $\mathbf{C}\otimes\mathbf{D}$ is defined as the matrix $(\mathbf{C}d_{ij})$. The Kronecker product of any number of matrices is defined as the natural generalization of two matrices. And we shall make use of the well-known relations concerning the Kronecker product of two matrices such as $(\mathbf{C}\otimes\mathbf{D})(\mathbf{L}\otimes\mathbf{M})=\mathbf{CL}\otimes\mathbf{DM}$, $(\mathbf{C}\otimes\mathbf{D})^{-1}=\mathbf{C}^{-1}\otimes\mathbf{D}^{-1}$, $(\mathbf{C}\otimes\mathbf{D})'=\mathbf{C}'\otimes\mathbf{D}'$, and their generalization to the products of any number of matrices without mentioning explicitly.

3. Covariance matrix.

At first we observe

THEOREM 3. 1. *Let*

$$\mathbf{Y}(N\times p)=[\mathrm{Y}_1, \mathrm{Y}_2, \cdots, \mathrm{Y}_p]\}N, \tag{3. 1}$$

and

$$\mathbf{y}(Np\times 1)=[\mathrm{Y}_1', \mathrm{Y}_2', \cdots, \mathrm{Y}_p']', \tag{3. 2}$$

then, under the model (2. 1), $\mathbf{y}(Np\times 1)$ *is distributed in normal distribution* $N[\xi(p\times 1), \mathbf{V}(Np\times Np)]$, *where*

$$\xi(p\times 1)=[\mu_1\mathbf{1}'(1\times N),\ \mu_2\mathbf{1}'(1\times N),\ \cdots,\ \mu_p\mathbf{1}'(1\times N)]', \tag{3. 3}$$

and

$$\mathbf{V}(Np\times Np)=\sum_{i=1}^{k+1}\mathbf{A}(N\times N)\otimes\Sigma_i(p\times p). \tag{3. 4}$$

PROOF. It is easily seen that the expectation of $\mathbf{y}(Np\times 1)$ is given by (3. 1). Now, in virtue of (2. 1), the model equation of $\mathbf{Y}_l(N\times 1)$ is given by

$$\mathbf{Y}_l(N\times 1)=\mu_l\mathbf{1}(N\times 1)+\sum_{i=1}^{k}\mathbf{X}_i(N\times m_i)\beta_{il}(m_i\times 1)+\mathbf{X}_{k+1}(N\times N)\beta_{k+1,l}(N\times 1), \quad l=1,2,\cdots p, \tag{3. 5}$$

where β_{il} is the 1-th column vector of $\mathbf{B}_i$ for $i=1,2,\cdots,\ k+1$, and we have

$$\begin{aligned} E[\mathbf{Y}_l(N\times 1)\mathbf{Y}_s'(1\times N)]&=E[(\mu_l\mathbf{1}+\sum_{i=1}^{k+1}\mathbf{X}_i\beta_{il})(\mu_s\mathbf{1}+\sum_{i=1}^{k+1}\mathbf{X}_i\beta_{is})'] \\ &=\mu_l\mu_s\mathbf{E}(N\times N)+E(\sum_{i=1}^{k+1}\mathbf{X}_i\beta_{il}\beta_{is}'\mathbf{X}_i) \\ &=\mu_l\mu_s\mathbf{E}(N\times N)+\sum_{i=1}^{k+1}\mathbf{X}_i(\sigma_{ls}^{(i)}\mathbf{I}(N\times N))\mathbf{X}_i' \\ &=\mu_l\mu_s\mathbf{E}(N\times N)+\sum_{i=1}^{k+1}\sigma_{ls}^{(i)}\mathbf{A}_i(N\times N), \end{aligned} \tag{3. 6}$$

where $\sigma_{ls}^{(i)}$ is the (l,s)-element of Σ_i.

On the other hand, considering that

$$\begin{aligned} E[\mathbf{y}(Np\times 1)]E[\mathbf{y}'(Np\times 1)]&=\mathbf{E}(N\times N)\otimes\begin{pmatrix} \mu_1^2 & \mu_1\mu_2 & \mu_1\mu_3\cdots\cdots & \mu_1\mu_p \\ \mu_2\mu_1 & \mu_2^2 & \mu_2\mu_3\cdots\cdots & \mu_2\mu_p \\ \vdots & & & \vdots \\ \mu_p\mu_1 & \mu_p\mu_2 & \mu_p\mu_3\cdots\cdots & \mu_p^2 \end{pmatrix} \\ &=\mathbf{E}(N\times N)\otimes(\mathbf{B}_0'\mathbf{B}_0), \end{aligned} \tag{3. 7}$$

we obtain (3. 2).

Now we shall set up some combinations of the following assumptions in certain sections of this paper.

ASSUMPTION (I) *$\mathbf{A}_i$'s $(i=0,1,2,\cdots,k+1)$ commute in pairs.*

ASSUMPTION (II) *The elements of $\mathbf{A}_i$ are equal to 0 or 1 and it holds that $\mathbf{1}'(1\times N)\mathbf{A}_i=r_i\mathbf{1}'(1\times N)$.*

ASSUMPTION (III) *$\mathbf{A}_i$'s $(i=0,1,\cdots,k+1)$ are linearly independent.*

ASSUMPTION (IV) *The diagonal elements of Σ_i are equal to each other, while other elements of Σ_i are also equal among themselves for $i=1,2,\cdots,\ k+1$, hence Σ_i has the form*

$$\Sigma_i=\begin{pmatrix} \sigma_i^2 & & & \\ & \sigma_i^2 & & \tau_i \\ & & \ddots & \\ & \tau_i & & \sigma_i^2 \end{pmatrix}, \quad (i=1,2,\cdots,k+1). \tag{3. 8}$$

The Assumptions (I), (II) and (III) are concerned with the layout of experiments. The models of the experimental designs with equal numbers in the subclasses, which include the r-way layout models with or without interaction, the r-fold nested classfication models, the split-plot models, etc., satisfy these assumptions. For in these experimental designs all $\mathbf{A}_i$ are expressed in the form of the Kronecker product of several numbers of $\mathbf{E}$ and $\mathbf{I}$ such that the dimensions of the corresponding component matrices $\mathbf{E}$ or $\mathbf{I}$ are equal among themselves, and all $\mathbf{A}_i$ are different from each other.

The models of the experimental designs with unequal numbers in the subclass do not satisfy these assumptions. For example, in the 2-way layout without interaction such that the treatment combinations are given by (12), (21), (22), (33), (34), (43) and (44), $\mathbf{A}_1$, $\mathbf{A}_2$ and $\mathbf{A}_3$ can be written as

(3. 9) $$\mathbf{A}_1=\begin{pmatrix} 1&0&0&0&0&0&0\\ 0&1&1&0&0&0&0\\ 0&1&1&0&0&0&0\\ 0&0&0&1&1&0&0\\ 0&0&0&1&1&0&0\\ 0&0&0&0&0&1&1\\ 0&0&0&0&0&1&1 \end{pmatrix}, \qquad \mathbf{A}_2=\begin{pmatrix} 1&0&1&0&0&0&0\\ 0&1&0&0&0&0&0\\ 1&0&1&0&0&0&0\\ 0&0&0&1&0&1&0\\ 0&0&0&0&1&0&1\\ 0&0&0&1&0&1&0\\ 0&0&0&0&1&0&1 \end{pmatrix}, \qquad \mathbf{A}_3=\mathbf{I}(7\times 7),$$

which implies that this is the case. The models of B.I.B. designs satisfy the Assumption (II), but not the Assumption (I).

THEOREM 3. 2. *Under the Assumptions* (I), (II), *and* (IV), *there exists an orthogonal transformation which transforms* $\mathbf{V}$ *given by (3. 4) into a diagonal matrix.*

PROOF. In virtue of the Assumptoin (I) and the symmetricity of $\mathbf{A}_i$, there exists an orthogonal matrix which diagonalizes $\mathbf{A}_0, \mathbf{A}_1, \cdots\cdots, \mathbf{A}_{k+1}$. And under the Assumption (II) this can be realized by an orthogonal matrix $\mathbf{P}$, which is defined in Section 2. Therefore let us consider such a particular matrix $\mathbf{P}$. Let $\mathbf{Q}$ be any orthogonal matrix defined in Section 2.

Then, since Σ_i is expressed as follows

(3. 10) $$\Sigma_i=\tau_i\mathbf{E}(p\times p)+(\sigma_i^2-\tau_i)\,\mathbf{I}(p\times p),$$

we have

(3. 11) $$(\mathbf{P}(N\times N)\otimes\mathbf{Q}(p\times p))\mathbf{V}(\mathbf{P}(N\times N)\otimes\mathbf{Q}(p\times p))'$$
$$=\sum_{i=1}^{k+1}\{p\tau_i(\mathbf{PA}_i\mathbf{P}')\otimes\mathbf{H}(p\times p)+(\sigma_i^2-\tau_i)(\mathbf{PA}_i\mathbf{P}')\otimes\mathbf{I}(p\times p)\}$$
$$=\sum_{i=1}^{k+1}\left\{\begin{pmatrix} p\tau_i\Lambda_i(N\times N) & \mathbf{0} & & \\ & \mathbf{0} & & \\ & & \mathbf{0} & \\ \mathbf{0} & & & \ddots\ \mathbf{0} \end{pmatrix}+\begin{pmatrix} (\sigma_i^2-\tau_i)\Lambda_i(N\times N) & & \\ & (\sigma_i^2-\tau_i)\Lambda_i(N\times N) & \mathbf{0} \\ \mathbf{0} & & \ddots \\ & & (\sigma_i^2-\tau_i)\Lambda_i(N\times N) \end{pmatrix}\right\}$$

$$
=\begin{pmatrix}
\sum_{i=1}^{k+1}\{\sigma_i^2+(p-1)\tau_i\}\Lambda_i(N\times N) & & & & \mathbf{0} \\
& \sum_{i=1}^{k+1}(\sigma_i^2-\tau_i)\Lambda_i(N\times N) & & & \\
& & \sum_{i=1}^{k+1}(\sigma_i^2-\tau_i)\Lambda_i(N\times N) & & \\
& & & \ddots & \\
\mathbf{0} & & & & \sum_{i=1}^{k+1}(\sigma_i^2-\tau_i)\Lambda_i(N\times N)
\end{pmatrix},
$$

where each $\Lambda_i(N\times N)=\mathbf{PA}_i\mathbf{P}'$ is a diagonal matrix for $i=1, 2, \cdots, k+1$, which completes the proof. q. e. d..

The direct consequence of Theorem 3. 2. is given by

COROLLARY 3. 1. *Let* $\mathbf{P}$ *and* $\mathbf{Q}$ *be be the orthogonal matrices defined in Section 2. Then we have*

$$
(3.\ 12)\qquad ((\mathbf{P}\otimes\mathbf{Q})\mathbf{V}(\mathbf{P}\otimes\mathbf{Q})')^{-1}=\begin{pmatrix}
\mathbf{D}_1(N\times N) & & & & \mathbf{0} \\
& \mathbf{D}_2(N\times N) & & & \\
& & \mathbf{D}_2(N\times N) & & \\
& & & \ddots & \\
\mathbf{0} & & & & \mathbf{D}_2(N\times N)
\end{pmatrix},
$$

where

$$
\mathbf{D}_1=\begin{pmatrix}
\frac{1}{g_1} & & & 0 \\
& \times & & \\
& & \times & \\
0 & & & \ddots \\
& & & & \times
\end{pmatrix}\quad \textit{and}\quad \mathbf{D}_2=\begin{pmatrix}
\frac{1}{g_2} & & & 0 \\
& \times & & \\
& & \times & \\
0 & & & \ddots \\
& & & & \times
\end{pmatrix},
$$

and where g_1 *and* g_2 *are the distinct elements of the first row in the first column on the matrices* $\sum_{i=1}^{k+1}\{\sigma_i^2+(p-1)\tau_i\}\Lambda_i$ *and* $\sum_{i=1}^{k+1}(\sigma_i^2-\tau_i)\Lambda_i$, *respectively.*

4. Estimability Theorem.

We shall define the notion of estimability for Σ_i in our concern and then refer to the necessary and the sufficient conditions for Σ_i to be estimable.

Definition. Under the Assumption (II), the parameter matrix Σ_i is said to be estimable if and only if the quadratic forms $\mathbf{y}'\mathbf{G}_i\mathbf{y}$ and $\mathbf{y}'\mathbf{M}_i\mathbf{y}$ exist such that $E[\mathbf{y}'\mathbf{G}_i\mathbf{y}]=\sigma_i^2$ and $E[\mathbf{y}'\mathbf{M}_i\mathbf{y}]=\tau_i$.

Now we observe

THEOREM 4. 1. *Under the Assumption* (IV) *a necessary condition for* Σ_i's $(i=1,2,\cdots,k+1)$ *to be estimable is that* $\mathbf{A}_1,\cdots\cdots,\mathbf{A}_{k+1}$ *are linearly independent.*

PROOF. Let

$$\mathbf{B}_i=[\beta_{i1},\beta_{i2},\cdots\cdots,\beta_{ip}],$$

and

$$\tilde{\mathbf{B}}_i=[\beta'_{i1},\beta'_{i2}\cdots\cdots,\beta'_{ip}]', \qquad (i=1,2,\cdots,k+1).$$

Then it holds that

$$\mathbf{y}(Np\times 1)=\mathbf{1}(N\times N)\otimes\mathbf{B}_0+\sum_{i=1}^{k+1}(\mathbf{X}_i\otimes\mathbf{I}(p\times p))\tilde{\mathbf{B}}_i. \tag{4. 1}$$

Let Σ_i's $(i=1,2,\cdots,k+1)$ be estimable. Then there exist $\mathbf{G}_u$'s and $\mathbf{M}_u$'s such that

$$E[\mathbf{y}'\mathbf{G}_u\mathbf{y}]=\sigma^2_u, \qquad (u=1,2,\cdots,k+1), \tag{4. 2}$$

and

$$E[\mathbf{y}'\mathbf{M}_u\mathbf{y}]=\tau_u, \qquad (u=1,2,\cdots,k+1). \tag{4. 3}$$

The left hand side of (3. 11) is expressed as follows.

$$\begin{aligned}
&E[\{\mathbf{1}\otimes\mathbf{B}_0+\sum_{i=1}^{k+1}(\mathbf{X}_i\otimes\mathbf{I})\tilde{\mathbf{B}}_i\}'\mathbf{G}_u\{\mathbf{1}\otimes\mathbf{B}_0+\sum_{i=1}^{k+1}(\mathbf{X}_i\otimes\mathbf{I})\tilde{\mathbf{B}}_i\}]\\
&=E[\{\sum_{i=1}^{k+1}\tilde{\mathbf{B}}'_i(\mathbf{X}_i\otimes\mathbf{I}')\}\mathbf{G}_u\{\sum_{i=1}^{k+1}(\mathbf{X}_i\otimes\mathbf{I})\tilde{\mathbf{B}}_i\}]+(\mathbf{1}'\otimes\mathbf{B}'_0)\mathbf{G}_u(\mathbf{1}\otimes\mathbf{B}_0)\\
&=E\sum_{i=1}^{k+1}\mathrm{tr}[(\mathbf{X}_i\otimes\mathbf{I})'\mathbf{G}_u(\mathbf{X}_i\otimes\mathbf{I})\tilde{\mathbf{B}}_i\tilde{\mathbf{B}}'_i]+\mathrm{tr}[\mathbf{G}_u(\mathbf{1}\mathbf{1}'\otimes\mathbf{B}_0\mathbf{B}'_0)]\\
&=\sum_{i=1}^{k+1}\mathrm{tr}[(\mathbf{X}_i\otimes\mathbf{I})'\mathbf{G}_u(\mathbf{X}_i\otimes\mathbf{I})\{\tau_i\mathbf{I}\otimes\mathbf{E}+(\sigma_i^2-\tau_i)\mathbf{I}\otimes\mathbf{I}\}]\\
&\qquad+\mathrm{tr}[\mathbf{G}_u(\mathbf{E}\otimes\mathbf{B}_0\mathbf{B}'_0)]\\
&=\sum_{i=1}^{k+1}\tau_i\mathrm{tr}[\{(\mathbf{X}_i\mathbf{X}'_i)\otimes\mathbf{E}\}\mathbf{G}_u]+\sum_{i=1}^{k+1}(\sigma_i^2-\tau_i)\mathrm{tr}[\{(\mathbf{X}_i\mathbf{X}'_i)\otimes\mathbf{I}\}\mathbf{G}_u]\\
&\quad+\mathrm{tr}[\mathbf{G}_u(\mathbf{E}\otimes\mathbf{B}_0\mathbf{B}'_0)].
\end{aligned} \tag{4. 4}$$

Therefore it follows that, in virtue of (4. 11),

$$\sum_{i=1}^{k+1}\sigma_i^2\mathrm{tr}[(\mathbf{A}_i\otimes\mathbf{I})\mathbf{G}_u]+\sum_{i=1}^{k+1}\tau_i\mathrm{tr}[\{\mathbf{A}_i\otimes(\mathbf{E}-\mathbf{I})\}\mathbf{G}_u]+\mathrm{tr}[\mathbf{G}_u(\mathbf{E}\otimes\mathbf{B}_0\mathbf{B}'_0)]=\sigma^2_u, \qquad (u=1,2,\cdots,k+1). \tag{4. 5}$$

Similarly, it holds that

$$\sum_{i=1}^{k+1}\sigma_i^2\mathrm{tr}[(\mathbf{A}_i\otimes\mathbf{I})\mathbf{M}_u]+\sum_{i=1}^{k+1}\tau_i\mathrm{tr}[\{\mathbf{A}_i\otimes(\mathbf{E}-\mathbf{I})\}\mathbf{M}_u]+\mathrm{tr}[\mathbf{M}_u(\mathbf{E}\otimes\mathbf{B}_0\mathbf{B}'_0)] \tag{4. 6}$$

$$=\tau_u, \qquad (u=1,2,\cdots,k+1).$$

Since the equation (4. 5) holds true for all non-negative values of all σ_i^2, we obtain that

$$\mathrm{tr}[(\mathbf{A}_i\otimes\mathbf{I})\mathbf{G}_u]=\begin{cases}0 & \text{when } i\neq u,\\ 1 & \text{when } i=u,\end{cases} \qquad (u=1,2,\cdots,k+1).$$

In order to show the linear independency of $\mathbf{A}_i$'s $(i=1,\cdots,k+1)$, let $c_1,\cdots,c_{k+1}$ be any set of constants such that

$$\sum_{i=1}^{k+1} c_i(\mathbf{A}_i\otimes\mathbf{I})=\mathbf{O}.$$

Then, since it holds that

$$\sum_{i=1}^{k+1} c_i\,\mathrm{tr}[(\mathbf{A}_i\otimes\mathbf{I})\mathbf{G}_u]=c_u, \qquad (u=1,2,\cdots,k+1),$$

we have that

$$\mathrm{tr}\,[\mathbf{G}_u\{\sum_{i=1}^{k+1} c_i(\mathbf{A}_i\otimes\mathbf{I})\}]=\sum_{i=1}^{k+1} c_i\,\mathrm{tr}\,[(\mathbf{A}_i\otimes\mathbf{I})\mathbf{G}_u]=c_u,$$

which implies that $c_i=0$ for $i=1,2,\cdots,k+1$.

Therefore $\mathbf{A}_1\otimes\mathbf{I},\cdots,\mathbf{A}_{k+1}\otimes\mathbf{I}$ are linearly independent, which implies that $\mathbf{A}_1,\cdots,\mathbf{A}_{k+1}$ are linearly independent.

The same result can be obtained from (4. 6) in the similar way.

THEOREM 4. 2. *Under the Assumption* (IV) *a sufficient condition for* Σ_i's $(i=1,2,\cdots,k+1)$ *to be estimable is that* $\mathbf{A}_0,\mathbf{A}_1,\cdots,\mathbf{A}_{k+1}$ *are linearly independent.*

PROOF. Let us assume that $\mathbf{A}_0,\mathbf{A}_1,\cdots,\mathbf{A}_{k+1}$ are linearly independent. Then we have

$$E[Y_lY_s']=\mu_l\mu_s\mathbf{A}_0+\sum_{i=1}^{k+1}\tau_i\mathbf{A}_i, \text{ for } l\neq s, \tag{4. 7}$$

and

$$E[Y_lY_l']=\mu_l^2\mathbf{A}_0+\sum_{i=1}^{k+1}\sigma_i^2\mathbf{A}_i. \tag{4. 8}$$

Now let $\omega_{\alpha\beta}=y_\alpha^{(l)}y_\beta^{(l)}$ where $y_\alpha^{(l)}$ is the element in the α-th row of Y_l and let the vector $W(\frac{n(n+1)}{2}\times 1)$ be defined by

$$W(\tfrac{n(n+1)}{2}\times 1)=[\omega_{11},\omega_{12},\cdots,\omega_{1N},\omega_{22},\cdots,\omega_{2N},\cdots,\omega_{pp}]'.$$

And let the (α,β)-th element of $\mathbf{A}_i$ be $a_{\alpha\beta}^{(i)}$ and let the vector $\mathfrak{A}_i$ be defined by

$$\mathfrak{A}_i(\tfrac{n(n+1)}{2}\times 1)=[a_{11}^{(i)},a_{12}^{(i)},\cdots,a_{1N}^{(i)},a_{22}^{(i)},\cdots,a_{2N}^{(i)},\cdots,a_{NN}^{(i)}]'.$$

Then it holds that

(4. 9) $$E(\mathrm{W})=\mu_i^2\mathfrak{A}_0+\sum_{i=1}^{k+1}\sigma_i^2\mathfrak{A}_i.$$

Since $\mathbf{A}_0, \mathbf{A}_1, \cdots, \mathbf{A}_{k+1}$ are linearly independent, $\mathfrak{A}_0, \mathfrak{A}_1, \cdots, \mathfrak{A}_{k+1}$ are also linearly independent.

Putting $\mathfrak{A}(\frac{n(n+1)}{2}\times(k+2))=[\mathfrak{A}_0, \mathfrak{A}_1, \cdots, \mathfrak{A}_{k+1}]$ and $\mathfrak{C}((k+2)\times 1)=[\mu_i^2, \sigma_1^2, \cdots, \sigma_{k+1}^2]'$, (4.9) can be written as $E(\mathrm{W})=\mathfrak{A}\mathfrak{C}$. Since $\mathfrak{A}$ has rank $k+2$, $\mathfrak{C}$ is given in the form $\mathfrak{C}=\mathbf{L}E(\mathrm{W}^*)$, where W^* is a subvector of W, in virtue of (4. 8). Thus all σ_i^2's are estimable.

Further, from (4. 7) it is showed that all τ_i's are estimable in the similar way.

q. e. d.

5. Characteristic roots of the variance matrix.

In this section we shall discuss some of the properties of the characteristic roots, which play an important role in the following sections.

THEOREM 5. 1. *Under the Assumptions* (I), (II), (III) *and* (IV), *the number of the distinct characteristic roots of the matrix* $\mathbf{V}$ *is not less than* $2k+2$.

PROOF. In the proof of Theorem 3. 2, we showed that $\mathbf{V}$ was transformed to the diagonal matrix. And all $\{\sigma_i^2+(p-1)\tau_i\}$ and all $(\sigma_i^2-\tau_i)$ are functionally independent. Therefore it can be shown that the number of the distinct elements of

$$\begin{pmatrix} \sum_{i=1}^{k+1}\{\sigma_i^2+(p-1)\tau_i\}\Lambda_i & \mathbf{0} \\ \mathbf{0} & \sum_{i=1}^{k+1}(\sigma_i^2-\tau_i)\Lambda_i \end{pmatrix}$$

is not less than $2k+2$, along the same line as that of Theorem 3 in [9], which establishes the proof. q. e. d.

Secondly, we shall show

THEOREM 5. 2. *Under the Assumptions* (I), (II), (III) *and* (IV), *the* $2(k+1)$ *of the distinct characteristic roots of* $\mathbf{V}$ *are functionally independent.*

PROOF. Consider the last form in (3. 8). Let Λ^* and Λ_i^* be defined as the vectors of the diagonal elements of the diagonal matrices $(\mathbf{P}\otimes\mathbf{Q})\mathbf{V}(\mathbf{P}\otimes\mathbf{Q})'$ and Λ_i respectively. Then it holds that

(5. 1) $$\Lambda^*(Np\times 1)=\begin{pmatrix} \sum_{i=1}^{k+1}\{\sigma_i^2+(p-1)\tau_i\}\Lambda_i^*(N\times 1) \\ \sum_{i=1}^{k+1}(\sigma_i^2-\tau_i)\Lambda_i^*(N\times 1) \\ \sum_{i=1}^{k+1}(\sigma_i^2-\tau_i)\Lambda_i^*(N\times 1) \\ \vdots \\ \sum_{i=1}^{k+1}(\sigma_i^2-\tau_i)\Lambda_i^*(N\times 1) \end{pmatrix}.$$

Now we have

$$\sum_{i=1}^{k+1}\{\sigma_i^2+(p-1)\tau_i\}\Lambda_i^*(N\times 1)=[\Lambda_1^*,\Lambda_2^*,\cdots,\Lambda_{k+1}^*]\begin{pmatrix}\sigma_1^2+(p-1)\tau_1\\ \sigma_2^2+(p-1)\tau_2\\ \vdots\\ \sigma_{k+1}^2+(p-1)\tau_{k+1}\end{pmatrix}.$$

Since $\mathbf{A}_1, \mathbf{A}_2, \cdots, \mathbf{A}_{k+1}$ are linearly independent, $\Lambda_1^*, \Lambda_2^*, \cdots, \Lambda_{k+1}^*$ are linearly independent, which implies that the rank of the matrix $[\Lambda_1^*, \Lambda_2^*, \cdots, \Lambda_{k+1}^*]$ is equal to $k+1$. But $\{\sigma_i^2+(p-1)\tau_i\}$'s are functionally independent. Therefore $\sum_{i=1}^{k+1}\{\sigma_i^2+(p-1)\tau_i\}\Lambda_i^*$ has $k+1$ functionally independent elements.

Similarly, it holds that $\sum_{i=1}^{k+1}(\sigma_i^2-\tau_i)\Lambda_i^*$ has also $k+1$ functionally independent elements.

Since all $\{\sigma_i^2+(p-1)\tau_i\}$ and all $(\sigma_i^2-\tau_i)$ are functionally independent, the vector Λ^* has $2(k+1)$ functionally independent elements, which establishes the theorem. q. e. d.

6. Complete sufficient set.

In this section we shall derive the sufficient statistics in the model defined in Section 2 under the assumptions (I), (II), (III) and (IV) and then we shall discuss their distributions.

Now let us consider the quadratic form

$$Z=(\mathbf{y}-(\mathbf{1}\otimes\mathbf{B}_0))'\mathbf{V}^{-1}(\mathbf{y}-(\mathbf{1}\otimes\mathbf{B}_0)), \tag{6. 1}$$

and let introduce an orthogonal transformation $\mathbf{P}\otimes\mathbf{Q}$, where $\mathbf{P}$ and $\mathbf{Q}$ are defined in Section 2. Then, in virtue of Corollary 3. 1, Z is given by

$$\begin{aligned}(6.\ 2)\quad Z=&[(\mathbf{P}\otimes\mathbf{Q})\mathbf{y}-(\mathbf{P}\otimes\mathbf{Q})(\mathbf{1}\otimes\mathbf{B}_0)]'[(\mathbf{P}\otimes\mathbf{Q})\mathbf{V}(\mathbf{P}\otimes\mathbf{Q})']^{-1}\\ &[(\mathbf{P}\otimes\mathbf{Q})\mathbf{y}-(\mathbf{P}\otimes\mathbf{Q})(\mathbf{1}\otimes\mathbf{B}_0)]\\ =&\frac{1}{g_1}\{(P_1(1\times N)\otimes Q_1(1\times p))\mathbf{y}(Np\times 1)-\sqrt{N}\sum_{j=1}^{p}q_{1j}\mu_j\}^2\\ &+\frac{1}{g_2}[\sum_{h=2}^{p}\{(P_1(1\times N)\otimes Q_h(1\times p))\mathbf{y}(Np\times 1)-\sqrt{N}\sum_{j=1}^{p}q_{hj}\mu_j\}^2]\\ &+\sum_{u=3}^{s}\frac{1}{g_u}\mathbf{y}'(1\times Np)\mathbf{R}_u'(Np\times m_u)\mathbf{R}_u(m_u\times Np)\mathbf{y}(Np\times 1),\end{aligned}$$

where $P_i(1\times N)$ is the i-th row vector of $\mathbf{P}, \mathbf{Q}_j$ is the j-th row vector of $\mathbf{Q}, g_u$'s ($u=1, 2, \cdots, s$) are the distinct characteristic roots of $\mathbf{V}$, each row vector of all $\mathbf{R}_u$ is equal to one of $p(N-1)$ row vectors $P_i\otimes Q_j$'s ($i=2, \cdots, N$; $j=1, \cdots, p$), all row vectors of all $\mathbf{R}_u$ are distinct from each other and n_u, the row dimension of $\mathbf{R}_u$, is equal to the multiplicity of the characteristic root g_u. From the last form of (6. 2) it is easily seen that a set of $p+s-2$ statistics $(P_1\otimes Q_j)\mathbf{y}$'s ($j=1, 2, \cdots, p$) and $\mathbf{y}'\mathbf{R}_u'\mathbf{R}_u\mathbf{y}$'s ($u$

$=3, 4, \cdots, s$) are the sufficient statistics for the family of distribution of all observations under our model.

Now we shall derive the distributions of these statistics. $(P_1 \otimes Q_1)\,\mathbf{y}$ is distributed as a univariate normal, whose mean is given by $E[(P_1 \otimes Q_1)\mathbf{y}] = (P_1 \otimes Q_1)(\mathbf{1} \otimes \mathbf{B}_0) = \sqrt{N}\sum_{j=1}^{p} q_{1j}\mu_j$ and the variance is given by

$$(6.\ 3)\qquad \begin{aligned} &E[\{(P_1 \otimes Q_1)\mathbf{y} - \sqrt{N}\sum_{j=1}^{p} q_{1j}\mu_j\}\{(P_1 \otimes Q_1)\mathbf{y} - \sqrt{N}\sum_{j=1}^{p} q_{1j}\mu_j\}'] \\ &= E[(P_1 \otimes Q_1)\mathbf{y}\mathbf{y}'(P_1 \otimes Q_1)'] - N\Big(\sum_{j=1}^{p} q_{1j}\mu_j\Big)^2 \\ &= (P_1 \otimes Q_1)\mathbf{V}(P_1 \otimes Q_1)' \\ &= g_1. \end{aligned}$$

Similarly, it is seen that $(P_1 \otimes Q_j)\,\mathbf{y}$ is distributed as a univariate normal with mean $(P_1 \otimes Q_j)(\mathbf{1} \otimes \mathbf{B}_0) = \sqrt{N}\sum_{h=1}^{p} q_{jh}\mu_h$ and variance $(P_1 \otimes Q_j)\mathbf{V}(P_1 \otimes Q_j)' = g_2$ for $j = 2, \cdots, p$.

On the other hand we obtain the following;

(A) $\mathbf{R}_u'\mathbf{R}_u\mathbf{V}/g_u$ is idempotent since it holds that

$$\mathbf{R}_u'(\mathbf{R}_u\mathbf{V}\mathbf{R}_u')\mathbf{R}_u\mathbf{V}/g_u^2 = \mathbf{R}_u' g_u \mathbf{I}(n_u \times n_u)\mathbf{R}_u\mathbf{V}/g_u^2 = \mathbf{R}_u'\mathbf{R}_u/g_u.$$

(B) $E[\mathbf{R}_u\mathbf{y}] = (P_l \otimes Q_m)(\mathbf{1} \otimes \mathbf{B}_0) = \mathbf{0}$ for $l \neq 1$, and $\mathrm{Var}[\mathbf{R}_u\mathbf{y}] = g_u$.

(C) rank $(\mathbf{R}_u'\mathbf{R}_u\mathbf{V}) = n_u$.

(D) $\mathbf{R}_u\mathbf{V}\mathbf{R}_v' = \mathbf{0}$ for $u \neq v$, since $(P_i \otimes Q_j)\mathbf{V}(P_k \otimes Q_l)'$

is not zero if and only if $(i, j) = (k, l)$.

The above results show that all $\mathbf{y}\mathbf{R}_u'\mathbf{R}_u\mathbf{y}$ are distributed independently to each other in central chi-square distributions with n_u degrees of freedom and also independently to all $(P_1 \otimes Q_j)\,\mathbf{y}$.

This consideration will be summarized in the following.

THEOREM 6. 1. *In addition to the Assumptions* (I), (II), (III) *and* (IV), *let us assume that* $\mathbf{V}$ *has s distinct characteristic roots. Then the sufficient statistics for the family of the distribution of all observations are given by* $(P_1 \otimes Q_j)\,\mathbf{y}$'s $(j = 1, 2, \cdots, p)$ *and* $\mathbf{y}'\mathbf{R}_u'\mathbf{R}_u\mathbf{y}$'s $(u = 3, 4, \cdots, s)$.

Lastly in this section, we shall add the following theorem which seems to be much useful in seeking for the minimum variance unbiased estimate of Σ_i.

THEOREM 6. 2. *In addition to the Assumptions* (I), (II), (III) *and* (IV), *let us add that* $\mathbf{V}$ *has* $2k+4$ *distinct characteristic roots. Then the* $2(k+1)+p$ *statistics* $(P_1 \otimes Q_j)\,\mathbf{y}$'s $(j = 1, 2, \cdots, p)$ *and* $\mathbf{y}'\mathbf{R}_u'\mathbf{R}_u\mathbf{y}$'s $(u = 3, 4, \cdots, 2k+4)$ *form a complete sufficient set for the family of the distribution of all observations in our concern.*

PROOF. If $\mathbf{V}$ has $2k+4$ distinct characteristic roots, then the quadratic form in exponent (6. 2) is equal to

$$Z=\frac{1}{g_1}\{(P_1(1\times N)\otimes Q_1(1\times p))\mathrm{y}(Np\times 1)\}^2+\frac{1}{g_2}\sum_{h=2}^{p}\{(P_1(1\times N)\otimes Q_h(1\times p))\mathrm{y}(Np\times 1)\}^2+\sum_{u=3}^{2k+4}\frac{1}{g_u}\mathrm{y}'\mathbf{R}_u\mathbf{R}_u\mathrm{y}$$
$$-\frac{2\sqrt{N}\left(\sum_{j=1}^{p}g_{1j}\mu_j\right)}{g_1}(P_1(1\times N)\otimes Q_1(1\times p))\,\mathrm{y}(Np\times 1)$$
$$-\frac{2\sqrt{N}}{g_2}\sum_{h=2}^{p}\left(\sum_{j=1}^{p}g_{hj}\mu_j\right)(P_1(1\times N)\otimes Q_h(1\times p))\,\mathrm{y}(Np\times 1)$$
$$+\varphi(\mu_1,\mu_2,\cdots,\mu_p), \tag{6. 4}$$

where $\varphi(\mu_1, \mu_2, \cdots, \mu_p)$ is the function of $\mu_1, \mu_2, \cdots, \mu_p$.

Now we shall consider the transformations of the original parameters and the sufficient statistics such that

$$\theta_1=-\frac{2\sqrt{N}\left(\sum_{j=1}^{p}q_{1j}\mu_j\right)}{g_1}, \tag{6. 5}$$

$$\theta_h=-\frac{2\sqrt{N}\left(\sum_{j=1}^{p}q_{hj}\mu_j\right)}{g_2}, \quad (h=2,3,\cdots,p), \tag{6. 6}$$

$$\theta_k=\frac{1}{g_{k-(p-2)}}, \quad (k=p+1,p+2,\cdots,2k+2+p), \tag{6. 7}$$

$$U_1=(P_1(1\times N)\otimes Q_1(1\times p))\,\mathrm{y}(Np\times 1), \tag{6. 8}$$

$$U_h=(P_1(1\times h)\otimes Q_h(1\times p))\,\mathrm{y}(Np\times 1), \quad (h=2,3,\cdots,p), \tag{6. 9}$$

$$U_k=\mathrm{y}'\mathbf{R}_{k-(p-2)}\mathbf{R}_{k-(p-2)}\mathrm{y}, \quad (k=p+1,p+2,\cdots,2k+2+p). \tag{6. 10}$$

Then we should notice that the transformations (6. 5), $\cdots$, (6. 10) from $\tau=(\mu_1, \mu_2, \cdots, \mu_p, \sigma_1^2, \sigma_2^2, \cdots, \sigma_{k+1}^2, \tau_1, \tau_2, \cdots, \tau_{k+1})$ to $\theta=(\theta_1, \theta_2, \cdots, \theta_{2k+2+p})$ is one-to-one, because of the orthogonality of $\mathbf{Q}$ and the functional independency among g_u's ($u=3, 4, \cdots, 2k+4$), which is proved in Theorem 5. 2. Consequently it can be seen also that g_1 and g_2 are the functions of the new parameters θ_k's ($k=p+1, p+2, \cdots, 2k+2+p$).

Thus, under the new parameters, the quadratic form in exponent is given by

$$\sum_{i=1}^{2k+2+p}\theta_i U_i+g_1(\theta_{p+1},\cdots,\theta_{2k+2+p})U_1^2+g_2(\theta_{p+1},\cdots,\theta_{2k+2+p})\sum_{j=2}^{p}U_j^2+\varphi(\theta_1,\cdots,\theta_p), \tag{6. 11}$$

where $g_1(\theta_{p+1}, \cdots, \theta_{2k+2+p})$ and $g_2(\theta_{p+1}, \cdots, \theta_{2k+2+p})$ are the function of θ_k's $(k=p+1, p+2, \cdots, 2k+2+p)$.

The theorem is completed by applying the result of Lemma 4. 8 in our previous paper [4] to (6. 11). q. e. d.

7. Examples.

EXAMPLE 1. Concider the p-variate complete 2-way layout model without interaction in which the levels of all treatments are equal to three, and Assumption (IV) is satisfied. Then the design matrix is given by

$$(7.1)\qquad \left(\begin{array}{ccccccc:c} 1&1&0&0&1&0&0& \\ 1&1&0&0&0&1&0& \\ 1&1&0&0&0&0&1& \\ 1&0&1&0&1&0&0& \\ 1&0&1&0&0&1&0& \mathbf{I}(9\times 9) \\ 1&0&1&0&0&0&1& \\ 1&0&0&1&1&0&0& \\ 1&0&0&1&0&1&0& \\ 1&0&0&1&0&0&1& \end{array}\right),$$

and $\mathbf{A}_i$'s can be written as follows: $\mathbf{A}_1=\mathbf{E}(3\times 3)\otimes\mathbf{I}(3\times 3)$, $\mathbf{A}_2=\mathbf{I}(3\times 3)\otimes\mathbf{E}(3\times 3)$, $\mathbf{A}_3=\mathbf{I}(3\times 3)\otimes\mathbf{I}(3\times 3)$. Obviously Asumptions (I), (II) and (III) are satisfied in this case.

Moreover, hence $\mathbf{A}_1, \mathbf{A}_2$ and $\mathbf{A}_3$ are transformed to $\Lambda_1=3\mathbf{H}(3\times 3)\otimes\mathbf{I}(3\times 3)$, $\Lambda_2=3\mathbf{I}(3\times 3)\otimes\mathbf{H}(3\times 3)$ and $\Lambda_3=\mathbf{I}(3\times 3)\otimes\mathbf{I}(3\times 3)$ respectively, it is showed that for any c_1, c_2 and c_3.

$$(7.2)\qquad \sum_{i=1}^{3} c_i\Lambda=\begin{pmatrix} 3c_1+3c_2+c_3 &&&&&&&& \\ & 3c_2+c_3 &&&&&& 0 & \\ && 3c_2+c_3 &&&&&& \\ &&& 3c_1+c_3 &&&&& \\ &&&& c_3 &&&& \\ &&&&& c_3 &&& \\ &&&&&& 3c_1+c_3 && \\ & 0 &&&&&& c_3 & \\ &&&&&&&& c_3 \end{pmatrix}.$$

Therefore, in virtue of the last form of (3. 8), $\mathbf{V}$ has eight distinct characteristic roots $3\alpha_1+3\alpha_2+\alpha_3$, $3\alpha_1+\alpha_3$, $3\alpha_2+\alpha_3$, α_3, $3\beta_1+3\beta_2+\beta_3$, $3\beta_1+\beta_3$, $3\beta_2+\beta_3$, β_3, where $\alpha_i=\sigma_i^2+(p-1)\tau_i$ and $\beta_i=\sigma_i^2-\tau_i$.

Thus, from Theorem 6. 2, it can be seen that there exist the unique minimum variance unbiased estimates of Σ_i's $(i=1, 2, 3)$.

EXAMPLE 2. Consider the incomplete 2-way layout model without interaction in which the treatment combinations are given by (11), (12), (21), (22), (33), (34), (43) and (44) and Assumption (IV) is satisfied.
Then $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3$ can be written as follows: $\mathbf{A}_1=\mathbf{E}(2\times 2)\otimes\mathbf{I}(2\times 2)\otimes\mathbf{I}(2\times 2)$, $\mathbf{A}_2=\mathbf{I}(2\times 2)\otimes\mathbf{E}(2\times 2)\otimes\mathbf{I}(2\times 2)$, $\mathbf{A}_3=\mathbf{I}(8\times 8)$. And these satisfy the Assumptions (I), (II) and (III).

Hence $\mathbf{A}_1, \mathbf{A}_2$ and $\mathbf{A}_3$ are transformed to $\Lambda_1=2\mathbf{H}(2\times2)\otimes\mathbf{I}(2\times2)\otimes\mathbf{I}(2\times2)$, $\Lambda_2=2\mathbf{I}(2\times2)\otimes\mathbf{H}(2\times2)\otimes\mathbf{I}(2\times2)$ and $\Lambda_3=\mathbf{I}(8\times8)$, eight distinct characteristic roots of $\mathbf{V}$ are given by $2\alpha_1+2\alpha_2+\alpha_3$, $2\alpha_1+\alpha_3$, $2\alpha_2+\alpha_3$, α_3, $2\beta_1+2\beta_2+\beta_3$, $2\beta_1+\beta_3$, $2\beta_2+\beta_3$, β_3 where α_i and β_i are defined in Example 1.

Therefore there exist the unique munimun variance unbiased estimates of Σ_i's $(i=1,2,3)$.

8. Remark.

After treating the multivariate random effect model under the Assumptions (I), (II), (III) and (IV) in this paper, there naturally arises the corresponding problem for more general situation without the Assumption (IV), which is important for general application of random models. The similar problems for the case of mixed model are worthwhile to be discussed in detail. The author should like to have another ocasion to discuss some of these problems.

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