171

ON REPRESENTATIONS OF JORDAN TRIPLE SYSTEMS

Kiyosi YAMAGUTI

(Received Nov. 25, 1961)

The purpose of this paper is to abstract the notion of a Jordan triple system which has been introduced by Jacobson in [7][1] and to study its representation. Duffin [4] and Kemmer [11] first considered such system, β-matrices, for describing the meson and these matrices were studied by many authors. On the other hand, a Jordan triple system is an example of the so called affine structure which appeared in the study of 0-connection on the group space by E. Cartan [2], therefore it seems that it is appropriate to study this system from the Lie triple system-like stand point. Garnir [6] and Jacobson [7] have already used this fact and the latter obtained also many general properties of Jordan triple systems. But it seems that contrary to the cases of Jordan algebras and Lie triple systems, the abstract studies of Jordan triple systems are relatively few.

We give the definition of Jordan triple system of type I and of type II on the notion of triple derivations. Following Eilenberg [5], we define the generalized representations for these systems. Next, for the Jordan triple system J_{I} of type I we consider a cohomology group $H^n(J_{\text{I}}, V)$ which is associated with the representation by an analogous way to the method of Chevalley and Eilenberg [3]. Let J_{I}^* be an associated Lie triple system of J_{I}, then $H^n(J_{\text{I}}, V)$ is mapped homomorphically into the cohomology group $H^n(J_{\text{I}}^*, V)$ of J_{I}^*. Also, for the Jordan triple system J_{II} of type II we define a cohomology group $H^n(J_{\text{II}}, V)$ which is associated with the representation. Let J_{II} be an associated Jordan triple system of type II of J_{I}, then $H^n(J_{\text{II}}, V)$ is isomorphic to $H^n(J_{\text{I}}, V)$. Throughout this paper, we assume that the base field is of characteristic 0 and the dimension of Jordan triple system is finite.

1. Basic definitions. We begin with the abstraction of the notion of subspaces of associative algebras which are closed relative to the Jordan triple product $\{a\{bc\}\}$, where $\{ab\}=ab+ba$, in two ways.

DEFINITION 1.1. A *Jordan triple system* (J. t. s.) J_{I} *of type* I is a vector space over a field Φ with a trilinear multiplication $\{abc\}$ and satisfying

$$\{abc\}=\{acb\}, \tag{1.1}$$

$$\begin{aligned}&\{ab\{cde\}\}+\{\{bac\}de\}+\{ce\{bad\}\}+\{cd\{bae\}\}\\&=\{ba\{cde\}\}+\{\{abc\}de\}+\{ce\{abd\}\}+\{cd\{abe\}\}.\end{aligned} \tag{1.2}$$

DEFINITION 1.2. A *J. t. s.* J_{II} *of type* II is a vector space over a field Φ with a

1) Numbers in brackets refer to the references at the end of the paper.

trilinear multiplication $\langle abc\rangle$[2] and satisfying

$$\langle abc\rangle = \langle cba\rangle, \tag{1.3}$$

$$\begin{aligned} &\langle ab\langle cde\rangle\rangle + \langle\langle bac\rangle de\rangle + \langle c\langle bad\rangle e\rangle + \langle cd\langle bae\rangle\rangle \\ &= \langle ba\langle cde\rangle\rangle + \langle\langle abc\rangle de\rangle + \langle c\langle abd\rangle e\rangle + \langle cd\langle abe\rangle\rangle. \end{aligned} \tag{1.4}$$

EXAMPLES. A vector space over Φ spanned by the Dirac matrices γ_i, $i=1,2,3,4,5$, is a J.t.s. of type I relative to a ternary composition $\{\gamma_i\{\gamma_j\gamma_k\}\}$, since they satisfy the relation $\{\gamma_i\gamma_j\}=2\delta_{ij}I$, I being a unit matrix.

Consider a wave equation $\partial_\mu\beta_\mu\Psi+\kappa\Psi=0$ for the meson, then the operators β_i, $i=1,2,3,4$, satisfy, by definition, the following relation

$$\beta_i\beta_j\beta_k+\beta_k\beta_j\beta_i=\delta_{ij}\beta_k+\delta_{kj}\beta_i\text{[3]}.$$

Therefore, a vector space spanned by these β-matrices is a J.t.s. of type II relative to a ternary composition $\langle\beta_i\beta_j\beta_k\rangle=\beta_i\beta_j\beta_k+\beta_k\beta_j\beta_i$. A J.t.s. of this type is called a meson triple system [7].

Let J be a subspace of an associative algebra A, which is closed with respect to a ternary composition $\{a\{bc\}\}$, then J is a J.t.s. of type I relative to $\{abc\}=\{a\{bc\}\}$ and it is a J.t.s. of type II relative to $\langle abc\rangle=abc+cba$.

DEFINITION 1.3. A linear transformation D of J.t.s. J_{I} of type I is called a *derivation* of J_{I} if

$$D\{xyz\}=\{(Dx)yz\}+\{x(Dy)z\}+\{xy(Dz)\}$$

for all x, y, z in J_{I}. A derivation of a J.t.s. of type II is defined similarly.

In a J.t.s. J_{I} of type I, if we put

$$[abc]=\{abc\}-\{bac\},$$

then (1.2) can be rewritten as

$$[ab\{cde\}]=\{[abc]de\}+\{c[abd]e\}+\{cd[abe]\}, \tag{1.5}$$

hence a linear mapping $\sum_i D_{(a_i,b_i)}: x\to\sum_i[a_ib_ix]$ is a derivation of J_{I}, which is called an inner derivation of J_{I}.

Similarly, for a J.t.s. J_{II} of type II if we put

$$[abc]=\langle abc\rangle-\langle bac\rangle,$$

then (1.4) shows that a linear mapping $\sum_i D_{(a_i,b_i)}: x\to\sum_i[a_ib_ix]$ is an (inner) derivation of J_{II}, in each case the inner derivations form an ideal of a Lie algebra $\mathfrak{D}$ generated by derivations of J_{I} (or J_{II}), since $[D, D_{(a,b)}]=D_{(Da,b)}+D_{(a,Db)}$ for every $D\in\mathfrak{D}$.

2) In [7], Jacobson denoted a ternary product $abc+cba$ in associative algebras by (abc). Later he generalized this product to the case of abstract Jordan algebras and denoted by $\{abc\}$ [10]. Here we use the notation $\langle abc\rangle$ instead of $\{abc\}$. See Lemma 1.1.

3) Duffin [4] and Kemmer [11].

Now, we consider the relations of J. t. s. with the other algebras. A (nonassociative) commutative algebra J over a field Φ is called a Jordan algebra if $(a^2b)a=a^2(ba)$ for all $a, b\in J$.

LEMMA 1.1. *Any Jordan algebra J is a J.t.s. of type I relative to* $\{abc\}=a(bc)$ *and a J.t.s. of type II relative to* $\langle abc\rangle=\frac{1}{2}a(bc)-\frac{1}{2}b(ca)+\frac{1}{2}c(ab)$.

Proof. In a Jordan algebra J, it holds that $a(b(cd))+c(b(ad))+d(b(ca))=a(d(bc))+b(d(ca))+c(d(ab))$ for all $a,b,c,d\in J$ [1, (7)]. Hence a linear mapping $x\to[abx]$ is a derivation of a Jordan algebra J, where $[abc]=a(bc)-b(ac)$. From this fact we can easily prove this lemma.

LEMMA 1.2. *A J.t.s. of type I is a J.t.s. of type II relative to* $\langle abc\rangle=\frac{1}{2}\{abc\}-\frac{1}{2}\{bca\}+\frac{1}{2}\{cab\}$. *Conversely a J.t.s. of type II is a J.t.s. of type I relative to* $\{abc\}=\langle abc\rangle+\langle acb\rangle$. In this case J_{II} (or J_I) is called an *associated J.t.s.* of J_I (or J_{II}).

A vector space T over Φ with a trilinear composition $[abc]$ is called a Lie triple system if

$$[aab]=0,$$
$$[abc]+[bca]+[cab]=0,$$
$$[ab[cde]]=[[abc]de]+[c[abd]e]+[cd[abe]].$$

The following lemma will be usefull in the sequel.

LEMMA 1.3. *A J.t.s. J_I of type I is a Lie triple system relative to* $[abc]=\{abc\}-\{bac\}$ *and a J.t.s. J_{II} of type II is a Lie triple system relative to* $[abc]=\langle abc\rangle-\langle bac\rangle$.

To prove this lemma we use that a linear mapping $x\to[abx]$ is a derivation of J_I (or J_{II}). A corollary of this lemma is that any Jordan algebra is a Lie triple system relative to $[abc]=a(bc)-b(ac)$ [8]. We call a Lie triple system derived from J_I (or J_{II}) by above lemma the *associated Lie triple system* of J_I (or J_{II}).

We prove now an identity in Jordan algebras:

$$(1.6)\qquad [\langle aba\rangle, b, ab]=\frac{1}{2}a((ab)(b(ab)))-\frac{1}{2}b((ab)(a(ba)))-\frac{1}{4}[a^2, b^2, ab].$$

Proof. In a Jordan algebra the following identity holds [1, (6)]:

$$(1.7)\qquad a((bc)d)+b((ca)d)+c((ab)d)=(ab)(cd)+(bc)(ad)+(ca)(bd).$$

From this, we have (α) $2(a(ab))(b(ab))=a((ab)(b(ab)))+b((ab)(a(ba)))$. If we apply (1.7) with $a=b$, $b=a^2$, $c=b$, $d=ab$, we obtain $2((a^2b)(b(ab))-b((a^2b)(ab)))=a^2(b^2(ab))-b^2(a^2(ab))$, i.e. ($\beta$) $[a^2b, b, ab]=\frac{1}{2}[a^2, b^2, ab]$. Using ($\alpha$) and ($\beta$) we obtain (1.6). If we use (1.6) and [10, (19)], we see that the expression $[a, b, \langle aba\rangle b]-[\langle aba\rangle, b, ab]$ is skew-symmetric in a and b.

REMARK 1.1. We consider the geometrical meanning of the axiom (1.2) or (1.5) for a J.t.s. J_{I} of type I. Let L_n be a space with a symmetric affine connection and suppose that the curvature tensor $R_{ijk}{}^{l}$ in L_n decomposes as $R_{ijk}{}^{l}=2K_{[ij]k}{}^{l}$, where the tensor $K_{ijk}{}^{l}$ satisfies $K_{i[jk]}{}^{l}=0$ and $\nabla_m K_{ijk}{}^{l}=0$, ∇ denoting a covariant derivation. If we apply the operator $\nabla_{[e}\nabla_{f]}$ to $K_{ijk}{}^{l}$ we have

$$R_{efm}{}^{l}K_{ijk}{}^{m}-R_{efi}{}^{m}K_{mjk}{}^{l}-R_{efj}{}^{m}K_{imk}{}^{l}-R_{efk}{}^{m}K_{ijm}{}^{l}=0.$$

This relation is an expression of (1.5) in the form of structure constants of J_{I} and the curvature tensor $R_{ijk}{}^{l}$ satisfies three conditions in coefficient forms for the associated Lie triple system of J_{I}. For the geometrical meanning of the axiom (1.4) for a J.t.s. of type II we can consider a space L_n whose curvature tensor satisfies a special identity $R_{ijk}{}^{l}=2K_{[ij]k}{}^{l}$, where $K_{[i|j|k]}{}^{l}=0$ and $\nabla_m K_{ijk}{}^{l}=0$.

REMARK 1.2. Let J be a J.t.s. of type I. Since a Lie triple system T can be 1-to-1 imbedded in a Lie algebra L in such a way that the given composition $[abc]$ in T is a product $[[ab]c]$ in L [8 § 5, 14 Theorem 2.1], we can consider J as a subspace of a Lie algebra such that $\{abc\}-\{bac\}=[[ab]c]$ by Lemma 1.3.

We shall now state some concepts for J.t.s. of type I[4] which will be necessary for the later use. Let J and J' be J.t.s. of type I over Φ, a homomorphism of J into J' is a linear mapping f of J into J' satisfying $f(\{abc\})=\{f(a)f(b)f(c)\}$ for all $a,b,c \in J$. A subspace K of J is called a subsystem of J if $a,b,c\in K$ implies $\{abc\}\in K$. A subspace K of J is a subsystem if and only if $[abb]\in K$ and $\{aaa\}\in K$ for all $a,b\in K$. In fact, suppose that a subspace K satisfies these conditions, then $[abc]+[acb]\in K$ and $[bac]+[bca]\in K$, hence $3[abc]\in K$ and K is a subsystem of the associated Lie triple system of J. Also, using the identity $2(\{abc\}+\{bca\}+\{cab\})=\{a+b+c, a+b+c, a+b+c\}-\{a+b, a+b, a+b\}-\{b+c, b+c, b+c\}-\{c+a, c+a, c+a\}+\{aaa\}+\{bbb\}+\{ccc\}$ we have $\{abc\}+\{bca\}+\{cab\}\in K$ hence $3\{abc\}=[abc]+[acb]+\{abc\}+\{bca\}+\{cab\}\in K$. An ideal of J is a subspace K satisfying $\{JJK\}\subseteqq K$ and $\{KJJ\}\subseteqq K$. Let K be an ideal of J, then the factor space J/K becomes a J.t.s. with a trilinear product $\{a+K\ b+K\ c+K\}=\{abc\}+K$ and a natural mapping $a\rightarrow a+K$ is a homomorphism of J onto J/K. Conversely, let f be a homomorphism of a J.t.s. J onto a J.t.s. J' with a kernel K, then K is an ideal of J and the factor system J/K is isomorphic to J'. An ideal K of a J.t.s. J is a Lie triple system ideal[5] of an associated Lie triple system of J, because $[kab]=\{kab\}-\{abk\}$, $a,b\in J$, $k\in K$ and this relation shows that an ideal K of an associated Lie triple system of a J.t.s. J is an ideal of J if and only if either $\{JJK\}\subseteqq K$ or $\{KJJ\}\subseteqq K$.

2. Representations of Jordan triple systems. In this section we define a general representation of J.t.s. and consider the relations among the representations of Jordan algebras, of J.t.s. and of Lie triple systems. We begin with the natural definition of

4) In [13] Ôno stated these concepts in the form $\{a\{bc\}\}$.

5) [12, Definition 1.3].

representations for J. t. s.

DEFINITION 2.1. A linear mapping ρ of a J. t. s. J_{I} of type I into the algebra of linear transformations on a vector space V over Φ is called a *special representation* of J_{I} if $\rho(\{abc\})=\{\rho(a)\{\rho(b)\rho(c)\}\}$.

But, for our purpose it is necessary to define a more generalized representation than the special representation.

DEFINITION 2.2. Let J_{I} be a J. t. s. of type I. A pair (L, R) of bilinear mappings of J_{I} into the algebra of linear transformations on a vector space V over Φ is called a *(bi-)representation* of J_{I} if

$$R(a, b)=R(b, a), \tag{2.1}$$

$$\begin{aligned} R(a, \{bcd\})-L(a, \{bcd\}) &= R(c, d)(R(a, b)-L(a, b))+L(b, d)(R(a, c)-L(a, c)) \\ &\quad +L(b, c)(R(a, d)-L(a, d)), \end{aligned} \tag{2.2}$$

$$[L(a, b)-L(b, a),\ R(c, d)]=R([abc], d)+R(c, [abd]), \tag{2.3}$$

$$[L(a, b)-L(b, a),\ L(c, d)]=L([abc], d)+L(c, [abd]), \tag{2.4}$$

where $[L(a, b), R(c, d)]$ denotes, as usual, $L(a, b)\,R(c, d)-R(c, d)\,L(a, b)$.

From (2.4) we have $[L(a, b)-L(b, a), L(c, d)-L(d, c)]=L([abc], d)-L(d, [abc])+L(c, [abd])-L([abd], c)$, hence $\sum_i (L(a_i, b_i)-L(b_i, a_i))$ generate a subalgebra $\mathfrak{L}$ of a Lie algebra $\mathfrak{gl}(V)$. Let K be an ideal of J_{I} and let $(\bar{L}, \bar{R})$ be a restriction of a representation (L, R) of J_{I} to K, then a Lie algebra generated by $\sum_i (\bar{L}(a_i, b_i)-\bar{L}(b_i, a_i))$, $a_i, b_i \in K$, is an ideal of $\mathfrak{L}$.

For a, b in J_{I}, if we denote the mapping $x \to \{abx\}$ by $L(a, b)$ and the mapping $x \to \{xab\}$ by $R(a, b)$, then L and R satisfy (2.1), $\cdots$, (2.4). As usual we call this representation (L, R) the *regular representation.* We note that if (L, R) is a regular representation, then $L(a, b)-L(b, a)$ is an inner derivation of J_{I}. If K is a subsystem of J_{I} and A is an ideal of J_{I}, then the regular representation (L, R) induces a representation of K in A. If k is in a kernel of the regular representation, then $[kab]=0$ for every a, b in J_{I}. The inner derivation $\sum_i (L(a_i, b_i)-L(b_i, a_i))$ becomes a trivial mapping on the set of all elements with this property.

Let ρ be a special representation of J_{I}, then we have $\rho([abc])=[[\rho(a)\rho(b)]\rho(c)]$. Hence, if we put $L(a, b)=\rho(a)\rho(b)$ and $R(a, b)=\rho(a)\rho(b)+\rho(b)\rho(a)$, then it follows that (L, R) is a representation of J_{I}.

DEFINITION 2.3. Let J_{II} be a J. t. s. of type II. A pair (λ, τ) of bilinear mappings of J_{II} into the algebra of linear transformations on a vector space V over Φ is called a *(bi-)representation* of J_{II} if

$$\tau(a, b)=\tau(b, a), \tag{2.5}$$

(2.6) $$\tau(\langle abc\rangle, d)-\lambda(\langle abc\rangle, d) =\tau(a, c)(\tau(b, d)-\lambda(b, d))+\lambda(a, b)(\tau(c, d)-\lambda(c, d)) +\lambda(c, b)(\tau(a, d)-\lambda(a, d)),$$

(2.7) $$[\lambda(a, b)-\lambda(b, a), \tau(c, d)]=\tau([abc], d)+\tau(c, [abd]),$$

(2.8) $$[\lambda(a, b)-\lambda(b, a), \lambda(c, d)]=\lambda([abc], d)+\lambda(c, [abd]).$$

For a, b in a J.t.s. J_{II} of type II, two linear mappings $\lambda(a, b)$: $x\rightarrow\langle abx\rangle$ and $\tau(a, b)$: $x\rightarrow\langle axb\rangle$ satisfy the conditions (2.5), $\cdots$, (2.8), hence (λ, τ) is a representation of J_{II} which we call a regular representation. In this case $\lambda(a, b)-\lambda(b, a)$ is an inner derivation of J_{II}.

LEMMA 2.1. *Let $a\rightarrow\rho(a)$ be a representation*[6] of a *Jordan algebra J. Then ρ induces a representation (L, R) for an associated J.t.s. of type I of J.*

In fact, if we put $L(a, b)=\rho(a)\rho(b)$ and $R(a, b)=\rho(ab)$, then we easily see that L and R satisfy the conditions (2.1), $\cdots$, (2.4). Also, from Lemma 1.2 we have the following lemma by a direct verification.

LEMMA 2.2. *Let (L, R) be a representation for a J.t.s. J_{I} of type I. If we put $\lambda(a, b)=\frac{1}{2}L(a, b)-\frac{1}{2}L(b, a)+\frac{1}{2}R(a, b)$ and $\tau(a, b)=\frac{1}{2}L(a, b)+\frac{1}{2}L(b, a)-\frac{1}{2}R(a, b)$, then (λ, τ) is a representation for an associated J.t.s. of type II of J_{I}. Conversely, let (λ, τ) be a representation for a J.t.s. J_{II} of type II. If we put $L(a, b)=\lambda(a, b)+\tau(a, b)$ and $R(a, b)=\lambda(a, b)+\lambda(b, a)$, then (L, R) is a representation for an associated J.t.s. of type I of J_{II}.*

LEMMA 2.3. *Let (L, R) be a representation for a J.t.s. of type I. If we put $\theta(a, b)=R(a, b)-L(a, b)$, then we have*

$$\theta(c, d)\theta(a, b)-\theta(b, d)\theta(a, c)-\theta(a, [bcd])+D(b, c)\theta(a, d)=0,$$
$$[D(a, b), \theta(c, d)]=\theta([abc], d)+\theta(c, [abd]),$$

where $D(a, b)=\theta(b, a)-\theta(a, b)$.

From Lemma 1.3 and Lemma 2.3, it follows that a representation (L, R) of a J.t.s. J_{I} of type I induces a representation θ[7] of an associated Lie triple system of J_{I}. Similarly, a representation (λ, τ) of a J.t.s. J_{II} of type II induces a representation θ of an associated Lie triple system of J_{II} by putting $\theta(a, b)=\lambda(b, a)-\tau(b, a)$. Therefore, by [15] from a representation of a J.t.s. J we can define a cohomology group of an associated Lie triple system of J.

For a Jordan algebra J, there is a concept of Jordan bimodule equivalent to that of representation for J. In the case of J.t.s. of type I we define a Jordan triple bimodule

6) [8, Definition 2.1].

7) See [15, Definition 2], there we called a representation space V a $\mathfrak{T}$-module instead of a representation θ.

as follows.

Let J_{I} be a J. t. s. of type I. A *Jordan triple bimodule* for J_{I} is a vector space M with trilinear compositions $\{mab\}$, $\{amb\}$ and $\{abm\}$ for $a, b \in J_{\mathrm{I}}$, $m \in M$ such that these compositions are contained in M and satisfy

$$\begin{aligned}
&\{mab\} = \{mba\},\\
&\{amb\} = \{abm\},\\
&\{ma\{bcd\}\} + \{\{abm\}cd\} + \{bd\{acm\}\} + \{bc\{adm\}\}\\
&\quad = \{am\{bcd\}\} + \{\{mab\}cd\} + \{bd\{mac\}\} + \{bc\{mad\}\},\\
&\{ab\{mcd\}\} + \{\{bam\}cd\} + \{md\{bac\}\} + \{mc\{bad\}\}\\
&\quad = \{ba\{mcd\}\} + \{\{abm\}cd\} + \{md\{abc\}\} + \{mc\{abd\}\},\\
&\{ab\{cdm\}\} + \{cd\{bam\}\} + \{\{bac\}dm\} + \{cm\{bad\}\}\\
&\quad = \{ba\{cdm\}\} + \{cd\{abm\}\} + \{\{abc\}dm\} + \{cm\{abd\}\}.
\end{aligned}$$

Let (L, R) be a representation of J_{I} acting in the vector space M. Putting $\{abm\} = \{amb\} = L(a, b)m$ and $\{mab\} = R(a, b)m$ for $m \in M$, M is a Jordan triple bimodule for J_{I}. Conversely, let M be a Jordan triple bimodule for J_{I}. If we define the linear mappings $L(a, b)$ and $R(a, b)$ of M by $L(a, b)m = \{abm\}$ and $R(a, b)m = \{mab\}$ respectively, then (L, R) is a representation of J_{I} with representation space M. Hence, the concept of Jordan triple bimodule for J_{I} is equivalent to that of representation of J_{I}. As in [9, § 2] from a given Jordan triple bimodule M we can construct a semi-direct sum of J_{I} and the bimodule M or a split null extension of J_{I} by M, and we can discuss this problem in a more general situation.

3. Cohomology group of Jordan triple systems of type I. Let (L, R) be a representation of a J. t. s. J_{I} of type I acting in a vector space V over Φ, and let f be an n-linear mapping of $J_{\mathrm{I}} \times \cdots \times J_{\mathrm{I}}$ (n times) into V satisfying

$$f(x_1, \cdots, x_{n-2}, x, y) = f(x_1, \cdots, x_{n-2}, y, x) \qquad \text{for } n \geqq 3.$$

We call such a mapping f an *n-cochain* and denote a vector space spanned by n-cochains by $C^n(J_{\mathrm{I}}, V)$, $n = 0, 1, 2, \cdots$, where we identify $C^0(J_{\mathrm{I}}, V)$ with V.

We define a linear mapping δ of $C^n(J_{\mathrm{I}}, V)$ into $C^{n+2}(J_{\mathrm{I}}, V)$ as follows:

(3.1) $$(\delta f)(x_1, x_2) = (L(x_1, x_2) - R(x_1, x_2))f \qquad \text{for } f \in C^0(J_{\mathrm{I}}, V),$$

(3.2) $$\begin{aligned}(\delta f)(x_1, x_2, x_3) = {} & L(x_1, x_2)f(x_3) + L(x_1, x_3)f(x_2)\\ & + R(x_2, x_3)f(x_1) - f(\{x_1x_2x_3\})\end{aligned} \qquad \text{for } f \in C^1(J_{\mathrm{I}}, V),$$

(3.3) $$\begin{aligned}&(\delta f)(x_1, x_2, x_3, x_4)\\ &\quad = L(x_2, x_3)f(x_1, x_4) + L(x_2, x_4)f(x_1, x_3)\\ &\qquad + R(x_3, x_4)f(x_1, x_2) - f(x_1, \{x_2x_3x_4\})\end{aligned} \qquad \text{for } f \in C^2(J_{\mathrm{I}}, V),$$

$$
\begin{aligned}
&(\delta f)(x_1, x_2, \cdots, x_{2n+1}) \\
&=(-1)^{n+1}[L(x_{2n-1}, x_{2n})f(x_1, x_2, \cdots, x_{2n-2}, x_{2n+1}) \\
&\qquad -L(x_{2n-1}, x_{2n})f(x_1, x_2, \cdots, x_{2n-4}, x_{2n-2}, x_{2n-3}, x_{2n+1}) \\
&\quad +L(x_{2n-1}, x_{2n+1})f(x_1, x_2, \cdots, x_{2n-2}, x_{2n}) \\
&\qquad -L(x_{2n-1}, x_{2n+1})f(x_1, x_2, \cdots, x_{2n-4}, x_{2n-2}, x_{2n-3}, x_{2n}) \qquad (3.4)\\
&\quad +R(x_{2n}, x_{2n+1})f(x_1, x_2, \cdots, x_{2n-1})-R(x_{2n}, x_{2n+1})f(x_1, x_2, \cdots, x_{2n-4}, x_{2n-2}, x_{2n-3}, x_{2n-1}) \\
&\quad -f(x_1, x_2, \cdots, x_{2n-2}, \{x_{2n-1}x_{2n}x_{2n+1}\})+f(x_1, x_2, \cdots, x_{2n-4}, x_{2n-2}, x_{2n-3}, \{x_{2n-1}x_{2n}x_{2n+1}\})] \\
&+\sum_{k=1}^{n-1}(-1)^{k+1}(L(x_{2k-1}, x_{2k})-L(x_{2k}, x_{2k-1}))f(x_1, x_2, \cdots, \hat{x}_{2k-1}, \hat{x}_{2k}, \cdots, x_{2n+1}) \\
&+\sum_{k=1}^{n-1}\sum_{j=2k+1}^{2n+1}(-1)^k f(x_1, x_2, \cdots, \hat{x}_{2k-1}, \hat{x}_{2k}, \cdots, [x_{2k-1}x_{2k}x_j], \cdots, x_{2n+1}) \\
&\qquad\qquad\qquad \text{for} \quad f\in C^{2n-1}(J_{\mathrm{I}}, V), n=2, 3, \cdots,
\end{aligned}
$$

$$
\begin{aligned}
&(\delta f)(y, x_1, x_2, \cdots, x_{2n+1}) \\
&=(-1)^{n+1}[L(x_{2n-1}, x_{2n})f(y, x_1, x_2, \cdots, x_{2n-2}, x_{2n+1}) \\
&\qquad -L(x_{2n-1}, x_{2n})f(y, x_1, x_2, \cdots, x_{2n-4}, x_{2n-2}, x_{2n-3}, x_{2n+1}) \\
&\quad +L(x_{2n-1}, x_{2n+1})f(y, x_1, x_2, \cdots, x_{2n-2}, x_{2n}) \\
&\qquad -L(x_{2n-1}, x_{2n+1})f(y, x_1, x_2, \cdots, x_{2n-4}, x_{2n-2}, x_{2n-3}, x_{2n}) \\
&\quad +R(x_{2n}, x_{2n+1})f(y, x_1, x_2, \cdots, x_{2n-1}) \qquad (3.5)\\
&\qquad -R(x_{2n}, x_{2n+1})f(y, x_1, x_2, \cdots, x_{2n-4}, x_{2n-2}, x_{2n-3}, x_{2n-1}) \\
&\quad -f(y, x_1, x_2, \cdots, x_{2n-2}, \{x_{2n-1}x_{2n}x_{2n+1}\}) \\
&\qquad +f(y, x_1, x_2, \cdots, x_{2n-4}, x_{2n-2}, x_{2n-3}, \{x_{2n-1}x_{2n}x_{2n+1}\})] \\
&+\sum_{k=1}^{n-1}(-1)^{k+1}(L(x_{2k-1}, x_{2k})-L(x_{2k}, x_{2k-1}))f(y, x_1, x_2, \cdots, \hat{x}_{2k-1}, \hat{x}_{2k}, \cdots, x_{2n+1}) \\
&+\sum_{k=1}^{n-1}\sum_{j=2k+1}^{2n+1}(-1)^k f(y, x_1, x_2, \cdots, \hat{x}_{2k-1}, \hat{x}_{2k}, \cdots, [x_{2k-1}x_{2k}x_j], \cdots, x_{2n+1}) \\
&\qquad\qquad\qquad \text{for} \quad f\in C^{2n}(J_{\mathrm{I}}, V), n=2, 3, \cdots,
\end{aligned}
$$

where the sign ^ over a letter indicates that this letter is to be omitted.

Then, we obtain the following

THEOREM 3.1. *For the operator δ defined above, we have $\delta\delta f=0$ for every cochain f.*

Proof. If $f\in C^0(J_{\mathrm{I}}, V)$, then

$$
\begin{aligned}
&(\delta\delta f)(x_1, x_2, x_3, x_4) \\
&=(R(x_1, \{x_2x_3x_4\})-L(x_1, \{x_2x_3x_4\})+R(x_3, x_4)(L(x_1, x_2)-R(x_1, x_2)) \\
&\qquad +L(x_2, x_4)(L(x_1, x_3)-R(x_1, x_3))+L(x_2, x_3)(L(x_1, x_4)-R(x_1, x_4)))f=0
\end{aligned}
$$

by (2.2). Similarly, a direct verification shows that $\delta\delta f=0$ for 1- and 3-cochain f.

To prove the general case, it is useful to consider two linear mappings. For a pair a, b in J_{I} we define a linear mapping $\kappa(a, b)$ of $C^{2n-1}(J_{\mathrm{I}}, V)$ into $C^{2n-1}(J_{\mathrm{I}}, V)$ and a linear mapping $\iota(a, b)$ of $C^{2n-1}(J_{\mathrm{I}}, V)$ into $C^{2n-3}(J_{\mathrm{I}}, V)$ by the following formulas:

(3.6) $$(\kappa(a,b)f)(x_1,\cdots,x_{2n-1})=(L(a,b)-L(b,a))f(x_1,\cdots,x_{2n-1}) - \sum_{j=1}^{2n-1} f(x_1,\cdots,[abx_j],\cdots,x_{2n-1}), \qquad n=2,3,4,\cdots,$$

(3.7) $$(\iota(a,b)f)(x_1,\cdots,x_{2n-3})=f(a,b,x_1,\cdots,x_{2n-3}), \qquad n=3,4,5,\cdots.$$

Then, we have by a direct calculation the following two formulas for $f\in C^{2n-1}(J_{\rm I},V), n\geqq 3$,

(3.8) $$\iota(a,b)\delta f+\delta\iota(a,b)f=\kappa(a,b)f,$$

(3.9) $$[\kappa(a,b),\iota(c,d)]f=\iota([abc],d)f+\iota(c,[abd])f.$$

Next we have

(3.10) $$[\kappa(a,b),\kappa(c,d)]f=\kappa([abc],d)f+\kappa(c,[abd])f \qquad \text{for } f\in C^{2n-1}(J_{\rm I},V), n\geqq 3.$$

For if $f\in C^5(J_{\rm I},V)$, then we obtain (3.10) directly. Using Lemma 1.3 and (3.9) we can prove the general case by an inductive method. Moreover, we have

(3.11) $$\kappa(a,b)\delta f=\delta\kappa(a,b)f \qquad \text{for } f\in C^{2n-1}(J_{\rm I},V), n\geqq 3.$$

For if $f\in C^5(J_{\rm I},V)$, then we obtain (3.11) by a direct computation. The general case follows by using the induction and (3.8). We have next for all $f\in C^n(J_{\rm I},V)$

(3.12) $$\delta\delta f=0.$$

Since (3.12) holds in case of $n=0,1,3$, we assume that (3.12) has been proved for all $f\in C^{2n-3}(J_{\rm I},V)$ and suppose $f\in C^{2n-1}(J_{\rm I},V), n\geqq 3$. Then for every pair $a,b\in J_{\rm I}$, from (3.8) and (3.11)

$$\begin{aligned}\iota(a,b)\delta\delta f&=\kappa(a,b)\delta f-\delta\iota(a,b)\delta f\\&=\delta\delta\iota(a,b)f\\&=0.\end{aligned}$$

Hence (3.12) holds for all cochains f with odd dimension and from this by (3.3) and (3.5), (3.12) holds for all cochains f with even dimension. Theorem 3.1 is therefore proved.

An n-cochain f is called an *n-cocycle* if $\delta f=0$. Denote $Z^n(J_{\rm I},V)$ a subspace of $C^n(J_{\rm I},V)$ spanned by n-cocycles. An n-cochain f of the form δg, where $g\in C^{n-2}(J_{\rm I},V)$, is called an *n-coboundary*. We denote $B^n(J_{\rm I},V)$ a subspace of $C^n(J_{\rm I},V)$ spanned by n-coboundaries, where $B^0(J_{\rm I},V)=B^1(J_{\rm I},V)=0$ by definition. Then, by Theorem 3.1 $B^n(J_{\rm I},V)$ is a subspace of $Z^n(J_{\rm I},V)$, hence we can define the quotient space $H^n(J_{\rm I},V)=Z^n(J_{\rm I},V)/B^n(J_{\rm I},V)$ which is called the *nth cohomology group* of $J_{\rm I}$.

From Lemma 2.3 a representation (L,R) of $J_{\rm I}$ with representation space V induces a representation θ of an associated Lie triple system of $J_{\rm I}$. Hence

$H^0(J_{\mathrm{I}}, V)$ is the subspace of the invariant elements for the induced representation θ of V.

A linear mapping f of J_{I} into V is called a derivation of J_{I} into V if $f(\{x_1x_2x_3\}) = R(x_2, x_3)f(x_1) + L(x_1, x_3)f(x_2) + L(x_1, x_2)f(x_3)$. Then,

$H^1(J_{\mathrm{I}}, V)$ is the vector space spanned by derivations of J_{I} into V.

We shall next consider the relation between a cohomology group $H^n(J_{\mathrm{I}}, V)$ of a J.t.s. J_{I} of type I and a cohomology group $H^n(J_{\mathrm{I}}^*, V)$ of an associated Lie triple system J_{I}^* of J_{I}. For this purpose, we modify slightly the coboundary operator δ introduced for Lie triple system in [15, (10), (11), (12)] as follows. Thus, for a Lie triple system J_{I}^*, let $C^n(J_{\mathrm{I}}^*, V)$ be a vector space spanned by n-cochains. We define a linear mapping $f \to \delta^* f$ of $C^n(J_{\mathrm{I}}^*, V)$ into $C^{n+2}(J_{\mathrm{I}}^*, V)$ by the following formulas:

$$(\delta^* f)(x_1, x_2) = -(\delta f)(x_1, x_2) \qquad \text{for } f \in C^0(J_{\mathrm{I}}^*, V),$$

$$(\delta^* f)(x_1, x_2, \cdots, x_{2n+1}) = (-1)^{n+1}(\delta f)(x_1, x_2, \cdots, x_{2n+1}) \qquad \text{for } f \in C^{2n-1}(J_{\mathrm{I}}^*, V),\ n \geqq 1,$$

$$(\delta^* f)(y, x_1, x_2, \cdots, x_{2n+1}) = (-1)^{n+1}(\delta f)(y, x_1, x_2, \cdots, x_{2n+1}) \qquad \text{for } f \in C^{2n}(J_{\mathrm{I}}^*, V),\ n \geqq 1.$$

Then $\delta^*\delta^* f = 0$ for all $f \in C^n(J_{\mathrm{I}}^*, V)$ by [15, Theorem 1], and we define the nth cohomology group $H^n(J_{\mathrm{I}}^*, V)$ of J_{I}^* as the quotient space $Z^n(J_{\mathrm{I}}^*, V)/B^n(J_{\mathrm{I}}^*, V)$.

Let $f \in C^n(J_{\mathrm{I}}, V)$. Define an n-linear mapping g of $J_{\mathrm{I}}^* \times \cdots \times J_{\mathrm{I}}^*$ (n times) into V as

$$g = f \qquad \text{for } f \in C^n(J_{\mathrm{I}}, V),\ n = 0, 1, 2,$$

$$g(x_1, \cdots, x_n) = f(x_1, \cdots, x_n) - f(x_1, \cdots, x_{n-3}, x_{n-1}, x_{n-2}, x_n) \qquad \text{for } f \in C^n(J_{\mathrm{I}}, V),\ n \geqq 3,$$

then the mapping $\phi: f \to g$ is a linear mapping of $C^n(J_{\mathrm{I}}, V)$ into $C^n(J_{\mathrm{I}}^*, V)$. Denote δ the coboundary operator for the elements of $C^n(J_{\mathrm{I}}, V)$ and δ^* the coboundary operator for the elements of $C^n(J_{\mathrm{I}}^*, V)$, then we have the following relations.

$$(\delta^* g)(x_1, x_2) = (\delta f)(x_1, x_2) \qquad \text{for } f \in C^0(J_{\mathrm{I}}, V),$$

$$(\delta^* g)(x_1, \cdots, x_{n+2}) = (\delta f)(x_1, \cdots, x_{n+2}) - (\delta f)(x_1, \cdots, x_{n-1}, x_{n+1}, x_n, x_{n+2}) \qquad \text{for } f \in C^n(J_{\mathrm{I}}, V),\ n \geqq 1.$$

Hence, $\phi(Z^n(J_{\mathrm{I}}, V)) \subseteqq Z^n(J_{\mathrm{I}}^*, V)$ and $\phi(B^n(J_{\mathrm{I}}, V)) \subseteqq B^n(J_{\mathrm{I}}^*, V)$ and from this ϕ induces a homomorphism ϕ^* of $H^n(J_{\mathrm{I}}, V)$ into $H^n(J_{\mathrm{I}}^*, V)$. Thus we obtain the following theorem.

THEOREM 3.2. *Let $H^n(J_{\mathrm{I}}, V)$ be the nth cohomology group of a J.t.s. J_{I} of type I and let $H^n(J_{\mathrm{I}}^*, V)$ be the nth cohomology group of an associated Lie triple system J_{I}^* of J_{I}. Then, there exists a homomorphism of $H^n(J_{\mathrm{I}}, V)$ into $H^n(J_{\mathrm{I}}^*, V)$.*

4. Cohomology group of Jordan triple systems of type II. Let J_{II} be a J.t.s. of type II and let (λ, τ) be a representation of J_{II} acting in a vector space V over Φ. An

n-cochain is an n-linear mapping f of $J_{\text{II}}\times\cdots\times J_{\text{II}}$ (n times) into V such that

$$f(x_1, \cdots, x_{n-3}, x, y, z)=f(x_1, \cdots, x_{n-3}, z, y, x) \qquad \text{for } n\geqq 3.$$

Denote $C^n(J_{\text{II}}, V)$ a vector space spanned by n-cochains, where we define $C^0(J_{\text{II}}, V)=V$. The coboundary operator is a linear mapping δ of $C^n(J_{\text{II}}, V)$ into $C^{n+2}(J_{\text{II}}, V)$ defined by the formulas:

(4.1) $$(\delta f)(x_1, x_2)=(\tau(x_2, x_1)-\lambda(x_2, x_1))f \qquad \text{for } f\in C^0(J_{\text{II}}, V),$$

(4.2) $$(\delta f)(x_1, x_2, x_3)=\lambda(x_3, x_2)f(x_1)+\tau(x_1, x_3)f(x_2) +\lambda(x_1, x_2)f(x_3)-f(\langle x_1x_2x_3\rangle) \qquad \text{for } f\in C^1(J_{\text{II}}, V),$$

(4.3) $$(\delta f)(x_1, x_2, x_3, x_4) =\lambda(x_4, x_3)f(x_1, x_2)+\tau(x_2, x_4)f(x_1, x_3) +\lambda(x_2, x_3)f(x_1, x_4)-f(x_1, \langle x_2x_3x_4\rangle) \qquad \text{for } f\in C^2(J_{\text{II}}, V),$$

(4.4)
$$\begin{aligned}
&(\delta f)(x_1, x_2, \cdots, x_{2n+1})\\
&=(-1)^{n+1}[\lambda(x_{2n-1}, x_{2n})f(x_1, x_2, \cdots, x_{2n-2}, x_{2n+1})\\
&\qquad -\lambda(x_{2n-1}, x_{2n})f(x_1, x_2, \cdots, x_{2n-4}, x_{2n-2}, x_{2n-3}, x_{2n+1})\\
&\quad +\tau(x_{2n-1}, x_{2n+1})f(x_1, x_2, \cdots, x_{2n-2}, x_{2n})\\
&\qquad -\tau(x_{2n-1}, x_{2n+1})f(x_1, x_2, \cdots, x_{2n-4}, x_{2n-2}, x_{2n-3}, x_{2n})\\
&\quad +\lambda(x_{2n+1}, x_{2n})f(x_1, x_2, \cdots, x_{2n-1})\\
&\qquad -\lambda(x_{2n+1}, x_{2n})f(x_1, x_2, \cdots, x_{2n-4}, x_{2n-2}, x_{2n-3}, x_{2n-1})\\
&\quad -f(x_1, x_2, \cdots, x_{2n-2}, \langle x_{2n-1}x_{2n}x_{2n+1}\rangle)\\
&\qquad +f(x_1, x_2, \cdots, x_{2n-4}, x_{2n-2}, x_{2n-3}, \langle x_{2n-1}x_{2n}x_{2n+1}\rangle)]\\
&\quad +\sum_{k=1}^{n-1}(-1)^{k+1}(\lambda(x_{2k-1}, x_{2k})-\lambda(x_{2k}, x_{2k-1}))f(x_1, x_2, \cdots, \hat{x}_{2k-1}, \hat{x}_{2k}, \cdots, x_{2n+1})\\
&\quad +\sum_{k=1}^{n-1}\sum_{j=2k+1}^{2n+1}(-1)^k f(x_1, x_2, \cdots, \hat{x}_{2k-1}, \hat{x}_{2k}, \cdots, [x_{2k-1}x_{2k}x_j], \cdots, x_{2n+1})
\end{aligned}$$
$$\text{for } f\in C^{2n-1}(J_{\text{II}}, V), n=2, 3, \cdots,$$

(4.5)
$$\begin{aligned}
&(\delta f)(y, x_1, x_2, \cdots, x_{2n+1})\\
&=(-1)^{n+1}[\lambda(x_{2n-1}, x_{2n})f(y, x_1, x_2, \cdots, x_{2n-2}, x_{2n+1})\\
&\qquad -\lambda(x_{2n-1}, x_{2n})f(y, x_1, x_2, \cdots, x_{2n-4}, x_{2n-2}, x_{2n-3}, x_{2n+1})\\
&\quad +\tau(x_{2n-1}, x_{2n+1})f(y, x_1, x_2, \cdots, x_{2n-2}, x_{2n})\\
&\qquad -\tau(x_{2n-1}, x_{2n+1})f(y, x_1, x_2, \cdots, x_{2n-4}, x_{2n-2}, x_{2n-3}, x_{2n})\\
&\quad +\lambda(x_{2n+1}, x_{2n})f(y, x_1, x_2, \cdots, x_{2n-1})\\
&\qquad -\lambda(x_{2n+1}, x_{2n})f(y, x_1, x_2, \cdots, x_{2n-4}, x_{2n-2}, x_{2n-3}, x_{2n-1})
\end{aligned}$$

$$-f(y, x_1, x_2, \cdots, x_{2n-2}, \langle x_{2n-1}x_{2n}x_{2n+1}\rangle)$$
$$+f(y, x_1, x_2, \cdots, x_{2n-4}, x_{2n-2}, x_{2n-3}, \langle x_{2n-1}x_{2n}x_{2n+1}\rangle)]$$
$$+\sum_{k=1}^{n-1}(-1)^{k+1}(\lambda(x_{2k-1}, x_{2k})-\lambda(x_{2k}, x_{2k-1}))f(y, x_1, x_2, \cdots, \hat{x}_{2k-1}, \hat{x}_{2k}, \cdots, x_{2n+1})$$
$$+\sum_{k=1}^{n-1}\sum_{j=2k+1}^{2n+1}(-1)^k f(y, x_1, x_2, \cdots, \hat{x}_{2k-1}, \hat{x}_{2k}, \cdots, [x_{2k-1}x_{2k}x_j], \cdots, x_{2n+1})$$
$$\text{for } f\in C^{2n}(J_{\mathrm{II}}, V), n=2, 3, \cdots,$$

where the sign ^ over a letter indicates that this letter is to be omitted.

Then, using the same method as in § 3 we obtain

THEOREM 4.1. *For the operator δ defined above, we have $\delta\delta f=0$ for every cochain f.*

An n-cochain f with $\delta f=0$ is called an n-cocycle and a subspace of $C^n(J_{\mathrm{II}}, V)$ spanned by n-cocycles is denoted by $Z^n(J_{\mathrm{II}}, V)$. A cochain of the form δf, where $f\in C^{n-2}(J_{\mathrm{II}}, V)$ is called an n-coboundary and a subspace of $C^n(J_{\mathrm{II}}, V)$ spanned by n-coboundaries is denoted by $B^n(J_{\mathrm{II}}, V)$. By Theorem 4.1, $B^n(J_{\mathrm{II}}, V)$ is a subspace of $Z^n(J_{\mathrm{II}}, V)$. The quotient space $H^n(J_{\mathrm{II}}, V)=Z^n(J_{\mathrm{II}}, V)/B^n(J_{\mathrm{II}}, V)$ is called the nth *cohomology group* of J_{II}.

Let (L, R) be a representation of a J. t. s. J_{I} of type I with representation space V and let $H^n(J_{\mathrm{I}}, V)$ be the nth cohomology group of J_{I}. Then a representation (λ, τ) for an associated J. t. s. J_{II} of type II of J_{I} is induced from (L, R) and we have an n-th cohomology group $H^n(J_{\mathrm{II}}, V)$. Define a linear mapping $f\to\phi f$ of $C^n(J_{\mathrm{I}}, V)$ into $C^n(J_{\mathrm{II}}, V)$ by

$$\phi f=f \qquad \text{for } f\in C^n(J_{\mathrm{I}}, V), n=0, 1, 2,$$

$$(\phi f)(x_1, \cdots, x_n)=\frac{1}{2}f(x_1, \cdots, x_n)-\frac{1}{2}f(x_1, \cdots, x_{n-3}, x_{n-1}, x_n, x_{n-2})$$
$$+\frac{1}{2}f(x_1, \cdots, x_{n-3}, x_n, x_{n-2}, x_{n-1})$$
$$\text{for } f\in C^n(J_{\mathrm{I}}, V), n\geqq 3,$$

and define a linear mapping $g\to\varphi g$ of $C^n(J_{\mathrm{II}}, V)$ into $C^n(J_{\mathrm{I}}, V)$ by

$$\varphi g=g \qquad \text{for } g\in C^n(J_{\mathrm{II}}, V), n=0, 1, 2,$$

$$(\varphi g)(x_1, \cdots, x_n)=g(x_1, \cdots, x_n)+g(x_1, \cdots, x_{n-2}, x_n, x_{n-1})$$
$$\text{for } g\in C^n(J_{\mathrm{II}}, V), n\geqq 3.$$

Then, ϕ and φ are inverse isomorphisms since both $\varphi\phi$ and $\phi\varphi$ are identity mappings and $C^n(J_{\mathrm{I}}, V)\approx C^n(J_{\mathrm{II}}, V)$.

Denote $\delta_{\mathrm{I}} f$ a coboundary of $f\in C^n(J_{\mathrm{I}}, V)$ and denote $\delta_{\mathrm{II}} g$ a coboundary of $g\in C^n(J_{\mathrm{II}}, V)$, then we have by a direct calculation the following relations:

$$(\delta_{\mathrm{II}}\phi f)(x_1, x_2)=(\delta_{\mathrm{I}} f)(x_1, x_2) \qquad \text{for } f\in C^0(J_{\mathrm{I}}, V),$$

$$(\delta_{II}\phi f)(x_1, \cdots, x_{n+2})$$
$$=\frac{1}{2}(\delta_I f)(x_1, \cdots, x_{n+2})-\frac{1}{2}(\delta_I f)(x_1, \cdots, x_{n-1}, x_{n+1}, x_{n+2}, x_n)$$
$$+\frac{1}{2}(\delta_I f)(x_1, \cdots, x_{n-1}, x_{n+2}, x_n, x_{n+1})$$
$$\text{for } f\in C^n(J_I, V), n\geqq 1.$$

Conversely,

$$(\delta_I\varphi g)(x_1, x_2)=(\delta_{II}g)(x_1, x_2) \qquad \text{for } g\in C^0(J_{II}, V),$$
$$(\delta_I\varphi g)(x_1, \cdots, x_{n+2})=(\delta_{II}g)(x_1, \cdots, x_{n+2})+(\delta_{II}g)(x_1, \cdots, x_n, x_{n+2}, x_{n+1})$$
$$\text{for } g\in C^n(J_{II}, V), n\geqq 1,$$

From these relations ϕ maps $Z^n(J_I, V)$ onto $Z^n(J_{II}, V)$ and $Z^n(J_I, V)\approx Z^n(J_{II}, V)$. Assume that $f\in B^n(J_I, V)$, $n\geqq 2$, then f is the form $\delta_I f'$ with $f'\in C^{n-2}(J_I, V)$. We have $\phi f=\delta_{II}\phi f'$ and ϕ maps $B^n(J_I, V)$ onto $B^n(J_{II}, V)$, therefore $B^n(J_I, V)\approx B^n(J_{II}, V)$, $n\geqq 0$.

Thus we have the following theorem.

THEOREM 4.2. *Let $H^n(J_I, V)$ be the nth cohomology group of a J.t.s. J_I of type I and let $H^n(J_{II}, V)$ be the nth cohomology group of an associated J.t.s. J_{II} of type II of J_I. Then $H^n(J_{II}, V)$ is isomorphic to $H^n(J_I, V)$.*

Department of Mathematics,
Faculty of Science,
Kumamoto University

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