ON THE THEORY OF MALCEV ALGEBRAS

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Introduction. A *Malcev algebra* or a *Moufang–Lie algebra* is a non-associative algebra satisfying $x^2=0$ and $(xy)(zx)+(xy\cdot z)x+(yz\cdot x)x+(zx\cdot x)y=0$. Malcev first discussed this algebra in the study of Moufang–Lie loops [5][1]. Any associative algebra is a Lie algebra relative to a multiplication $ab-ba$. Any alternative algebra is a Malcev algebra relative to a multiplication $ab-ba$ and the notion of Malcev algebra is more general than that of Lie algebra. It was shown in a previous note [12] that a Malcev algebra is a class of so-called general Lie triple systems.

The purpose of the present paper is to study certain properties of Malcev algebras and their representations. The methods employed here are to use a trilinear multiplication $[xyz]=x(yz)-y(xz)+(xy)z$ with the original multiplication xy and to construct a Lie algebra $M\oplus\mathfrak{D}(M)$ from the given Malcev algebra M, where $\mathfrak{D}(M)$ is a Lie algebra generated by the derivations of the form $\sum_i[L_{x_i}, L_{y_i}]+L_{x_iy_i}$, L_x being the left multiplication by x in M.

In § 1, it is shown that a Malcev algebra is characterized as a subspace satisfying some conditions of Lie algebra. In § 2, the concepts of solvability, radical, and semi-simplicity are introduced and it is shown that the solvability and semi-simplicity of M induce the same properties for the Lie algebra $M\oplus\mathfrak{D}(M)$.

We recall that a representation of a Lie algebra $\mathfrak{L}$ into a vector space V is a homomorphism of $\mathfrak{L}$ into the Lie algebra of linear transformations of V. For every x in $\mathfrak{L}$ the mapping $x\to\mathrm{ad}\,x$ is a representation of $\mathfrak{L}$ with $\mathfrak{L}$ as a representation space. But for a Malcev algebra the mapping of this type is not necessary a representation in the sense stated above. Thus, in §§ 3 and 7 we generalize the concept of representation in three ways for a Malcev algebra, i.e. a generalized representation, a weak representation, and a representation which includes the representation in the above sense as a special case. A representation is a weak representation and a weak representation induces a generalized representation. The representation has been introduced by Eilenberg in [2]. Let ρ be a weak representation of a Malcev algebra M and P be a generalized representation of M. If M is solvable then the Lie algebras generated by $\rho(M)$ and $P(M, M)$ are solvable. § 4 is concerned with the cohomology ring of Malcev algebra and in §§ 5, 6, and 7 we consider the cohomology groups which are associated with three representations by an analogous way to the method of Chevalley and Eilenberg [1]. Sagle showed in [6] that a seven-dimensional simple Malcev algebra C^* which is not a Lie algebra is obtained from the Cayley-Dickson algebra. In § 8, the multiplication in a Lie algebra $C^*\oplus\mathfrak{D}(C^*)$ constructed from C^* is calculated as an application of § 1. Throughout this paper, we assume that the base field is of characteristic zero and the algebras or vector spaces are to be finite-

1) Numbers in brackets refer to the references at the end of the paper.

dimensional. The author is indebted for Sagle's work [6], without it this paper would not appear.

1. A characterization of Malcev algebras. Let $\mathfrak{L}$ be a Lie algebra over a field Φ with multiplication $[x, y]$. Assume that $\mathfrak{L}$ is a vector space direct sum of a subspace T and a subalgebra $\mathfrak{D}$ such that $[T, \mathfrak{D}]\subseteqq T$. Then, any element x of $\mathfrak{L}$ is uniquely expressed as $x_T+x_{\mathfrak{D}}$, where x_T and $x_{\mathfrak{D}}$ denote the T- and $\mathfrak{D}$-component of x. If we put

$$xy = [x, y]_T,$$

$$[xyz] = [[x, y]_{\mathfrak{D}}, z]$$

for x, y, z in T, then we have the following relations:

(1.1) $$x^2 = 0,$$

(1.2) $$[xxy] = 0,$$

(1.3) $$[xyz]+[yzx]+[zxy]+(xy)z+(yz)x+(zx)y=0,$$

(1.4) $$[xy, z, w]+[yz, x, w]+[zx, y, w]=0,$$

(1.5) $$[x, y, zw]=[xyz]w+z[xyw],$$

(1.6) $$[xy[zvw]]=[[xyz]vw]+[z[xyv]w]+[zv[xyw]].$$

A vector space T over Φ with bilinear composition xy and trilinear composition $[xyz]$ is called a *general Lie triple system* (*general L.t.s.*) if these compositions satisfy (1.1), (1.2),$\cdots$, (1.6) [10]. Any Lie algebra is a general L.t.s. relative to $xy=[x, y]$ and $[xyz]=[[x, y]z]$. If $[xyz]=0$ for all x, y, z in T, the axioms stated above reduce to that of Lie algebras and if $xy=0$ for all x, y in T, the axioms reduce to that of Lie triple systems. In this sense, the general L.t.s. is a more general concept than those of the Lie algebras and Lie triple systems.

A linear mapping D of a general L.t.s. T is called a derivation of T if $D(xy)=(Dx)y+x(Dy)$ and $D([xyz])=[(Dx)yz]+[x(Dy)z]+[xy(Dz)]$ for all x, y, z in T. (1.5) and (1.6) imply that $\sum_i D(x_i, y_i)$: $z\to\sum_i [x_iy_iz]$ is a derivation of T. The set of all derivations of T forms a Lie algebra $\mathfrak{D}$ under the multiplication $[D_1, D_2]=D_1D_2-D_2D_1$ and the set of all derivations of the form $\sum_i D(x_i, y_i)$ forms an ideal $\mathfrak{D}(T)$ of $\mathfrak{D}$ since $[D, D(x, y)]=D(Dx, y)+D(x, Dy)$ for every derivation D. In particular, we have

(1.7) $$[D(x, y), D(z, w)]=D([xyz], w)+D(z, [xyw]).$$

Let T be a general L.t.s. and let $\mathfrak{L}$ be a vector space direct sum $T\oplus\mathfrak{D}(T)$. If we define a multiplication in $\mathfrak{L}$ by

$$[x+\sum_i D(y_i, z_i),\ u+\sum_i D(v_i, w_i)]$$
$$= xu+\sum_i [y_iz_iu]-\sum_i [v_iw_ix]+D(x, u)+\sum_{i,j} [D(y_i, z_i),\ D(v_j, w_j)]$$

for x, u, y_i, z_i, v_i, w_i in T, then by (1.1), (1.2),$\cdots$,(1.6) $\mathfrak{L}$ becomes a Lie algebra relative to this multiplication and satisfies $[\mathfrak{D}(T), \mathfrak{D}(T)]\subseteqq\mathfrak{D}(T)$ and $[T, \mathfrak{D}(T)]\subseteqq T$. Therefore, *a structure of a subspace T of the Lie algebra $\mathfrak{L}$ such that $\mathfrak{L}=T\oplus\mathfrak{D}$ (vector space direct sum) and satisfying $[\mathfrak{D}, \mathfrak{D}]\subseteqq\mathfrak{D}$, $[T, \mathfrak{D}]\subseteqq T$ is characterized by* (1.1),$\cdots$, (1.6).

REMARK 1.1. In the Lie algebra $\mathfrak{L}=T\oplus\mathfrak{D}(T)$ constructed out of the given general L.t.s. T, $\sum_i D(x_i, y_i)=0$ is defined by $\sum_i [x_i y_i z]=0$ for all $z\in T$. This means that $\mathfrak{D}(T)$ does not contain a non-zero ideal of $\mathfrak{L}$. Thus, assume a Lie algebra $\mathfrak{L}$ is a vector space direct sum $\mathfrak{M}\oplus\mathfrak{N}$ such that $[\mathfrak{M}, \mathfrak{N}]\subseteq\mathfrak{M}$ and $[\mathfrak{N}, \mathfrak{N}]\subseteq\mathfrak{N}$, then that $\mathfrak{N}$ does not contain a non-zero ideal of $\mathfrak{L}$ is equivalent to say that $[n, \mathfrak{M}]=(0)$ for $n\in\mathfrak{N}$ implies $n=0$.

From (1.5), by using [12, Theorem 1.1] we have the following

THEOREM 1.1. *For all elements x, y, z in a general L.t.s. assume a relation*

$$[xyz]=x(yz)-y(xz)+(xy)z. \tag{1.8}$$

Then, we have the identity

$$(xy)(zx)+(xy\cdot z)x+(yz\cdot x)x+(zx\cdot x)y=0. \tag{1.9}$$

A non-associative algebra M over Φ with a binary composition xy is called a *Malcev algebra* or a *Moufang-Lie algebra* if it satisfies (1.1) and (1.9). Then, from Theorem 1.1 a general L.t.s. satisfying (1.8) has a structure of Malcev algebra relative to xy. Conversely, in a Malcev algebra M with a composition xy if we define a trilinear multiplication $[xyz]$ by (1.8), then M is a general L.t.s. relative to xy and $[xyz]$ [12, Theorem 2.1]. We call a general L.t.s. derived from M by using (1.8) the general L.t.s. associated with M. A linear transformation D of a Malcev algebra M is a derivation if $D(xy)=(Dx)y+x(Dy)$ for all x, y in M. Then (1.5) says $D(x, y)$: $z\to[xyz]=x(yz)-y(xz)+(xy)z$ is a derivation of M and the Lie algebra $\mathfrak{D}(M)$ generated by all derivations of the form $\sum_i D(x_i, y_i)$ is an ideal of the derivation algebra of M. Any derivation of M is a derivation of the general L.t.s. associated with M.

Summarizing above results, we have the following characterization for Malcev algebras.

THEOREM 1.2.[2] *For any Malcev algebra M over Φ, there exists a Lie algebra $\mathfrak{L}$ over Φ such that $\mathfrak{L}$ is a vector space direct sum of M and a subalgebra $\mathfrak{D}$ satisfying $[M, \mathfrak{D}]\subseteq M$. The product xy in M is an M-component $[x, y]_M$ of a product $[x, y]$ in $\mathfrak{L}$ and it holds:*

$$[[x, y]_{\mathfrak{D}}, z]=[x, [y, z]_M]_M-[y, [x, z]_M]_M+[[x, y]_M, z]_M. \tag{1.10}$$

for all x, y, z in M. Conversely, suppose that M is a subspace of a Lie algebra $\mathfrak{L}$ such that $\mathfrak{L}$ is expressed as a vector space direct sum of M and a subalgebra $\mathfrak{D}$ satisfying $[M, \mathfrak{D}]\subseteq M$ and (1.10). Define a product xy in M by $[x, y]_M$, then M is a Malcev algebra with respect to this product.

We note that, by the Jacobi identity, (1.10) is equivalent to:

$$2[[x, y]_M, z]_M=[[x, z]_{\mathfrak{D}}, y]+[x, [y, z]_{\mathfrak{D}}] \tag{1.11}$$

for all x, y, z in M.

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2) This theorem holds if the characteristic of Φ is not 2 or 3.

REMARK 1.2. The above construction of a Lie algebra from the given general L.t.s. T can be slightly generalized. Let $\mathfrak{D}$ be a Lie algebra formed from all derivations of T and let $\mathfrak{L}$ be a vector space direct sum $T\oplus\mathfrak{D}$. A multiplication in $\mathfrak{L}$ is defined by

$$(1.12) \qquad [x_1+D_1, x_2+D_2]=x_1x_2+D_1(x_2)-D_2(x_1)+D(x_1, x_2)+[D_1, D_2]$$

for $x_i\in T$, $D_i\in\mathfrak{D}$, $i=1,2$, then $\mathfrak{L}$ is a Lie algebra such that $[\mathfrak{D}, \mathfrak{D}]\subseteq\mathfrak{D}$, $[T, \mathfrak{D}]\subseteq T$ and $xy=[x, y]_T$.

We recall some definitions and results on the derivations of Malcev algebra M. Let L_x be the left multiplication by x in M. If we put $\Delta(x, y)=[L_x, L_y]-L_{xy}$, then $\Delta(x, y)z+(xy)z+(yz)x+(zx)y=0$. A J-nucleus of M is defined as the set $N=\{z\in M: \Delta(x, y)z=0$ for all $x, y\in M\}$, then N is a characteristic ideal of M. Let $L(M)$ be the vector space spanned by the set of all left multiplications of M. A subalgebra $\mathfrak{L}(M)$ of $\mathfrak{gl}(M)$, the Lie algebra of linear transformations of M, generated by $L(M)$ is called the Lie transformation algebra of M. From the relation $[[L_x, L_y] L_z]+[L_{xy}, L_z]=L_{[xyz]}$ it follows $\mathfrak{L}(M)=L(M)+[L(M), L(M)]$. A derivation D of M is said to be inner provided $D\in\mathfrak{L}(M)$. In [6] Sagle proved that a derivation D of M is inner if and only if D is of the form $L_n+\sum_i D(x_i, y_i)$, where $D(x, y)=[L_x, L_y]+L_{xy}$, $n\in N$, J-nucleus, x_i, $y_i\in M$. The set of all inner derivations forms an ideal of the derivation algebra of M since the J-nucleus is a characteristic ideal of M.

Theorem 2.1 in [12] holds also for the inner derivation algebra. Let a Malcev algebra M be given. Let $\mathfrak{D}(M)$ be the Lie algebra generated by all derivations of the form $\sum_i D(x_i, y_i)$ and let $\bar{\mathfrak{D}}(M)$ and $\mathfrak{D}$ be the inner derivation algebra and derivation algebra respectively. If we put $\mathfrak{L}=M\oplus\mathfrak{D}(M)$, $\bar{\mathfrak{L}}=M\oplus\bar{\mathfrak{D}}(M)$, and $\tilde{\mathfrak{L}}=M\oplus\mathfrak{D}$, then these vector spaces form the Lie algebras under multiplication (1.12). We see easily that $\mathfrak{L}$ and $\bar{\mathfrak{L}}$ are ideals of $\tilde{\mathfrak{L}}$. Thus we have

THEOREM 1.3. *For a Malcev algebra M let $\bar{\mathfrak{D}}(M)$ be the inner derivation algebra of M. Put $\bar{\mathfrak{L}}=M\oplus\bar{\mathfrak{D}}(M)$ (vector space direct sum) and define a product in $\bar{\mathfrak{L}}$ by*

$$[x+L_l+\sum_i D(y_i, z_i), u+L_m+\sum_i D(v_i, w_i)]$$
$$=xu+lu+xm+\sum_i [y_iz_iu]-\sum_i [v_iw_ix]+L_n+D(x, u)+\sum_{i,j} [D(y_i, z_i), D(v_j, w_j)],$$

where l, $m\in N$ and n denotes $lm+2\sum_i (y_iz_i)m-2\sum_i (v_iw_i)l$, then $\bar{\mathfrak{L}}$ is a Lie algebra relative to this product. The original product xy in M is the M-component of product $[x, y]$ in $\bar{\mathfrak{L}}$ and $[M, \bar{\mathfrak{D}}]\subseteq M$, $[\bar{\mathfrak{D}}, \bar{\mathfrak{D}}]\subseteq\bar{\mathfrak{D}}$. Let $\mathfrak{D}(M)$ and $\mathfrak{D}$ be the Lie algebra generated by all derivations of the form $\sum_i D(x_i, y_i)$ and derivation algebra respectively, then the Lie algebras $M\oplus\mathfrak{D}(M)$ and $\bar{\mathfrak{L}}$ are the ideals of Lie algebra $M\oplus\mathfrak{D}$.

The above construction of a Lie algebra $\mathfrak{L}=M\oplus\mathfrak{D}(M)$ from the given Malcev algebra M is called the *standard construction.* Its meaning will be explained by the following examples.

The standard construction of a Lie algebra from the given Malcev algebra M is not necessary the most general among the Lie algebras $\mathfrak{L}$ such that $\mathfrak{L}$ is a direct sum

$M\oplus\mathfrak{D}$ and satisfying (*): $[\mathfrak{D},\mathfrak{D}]\subseteq\mathfrak{D}$, $[M,\mathfrak{D}]\subseteq M$ and $xy=[x,y]_M$ for $x,y\in M$. For example, let M be a 2-dimensional abelian Malcev algebra, then $\mathfrak{L}=M\oplus\mathfrak{D}(M)=M$ is a 2-dimensional abelian Lie algebra. Let $\mathfrak{L}'$ be a Lie algebra with base X_1, X_2, X_3, in which a multiplication is defined by $[X_1,X_2]=X_3$, $[X_1,X_3]=X_1$, $[X_3,X_2]=X_2$. Denote M' a subspace of $\mathfrak{L}'$ spanned by X_1 and X_2, then $\mathfrak{L}=M'+\Phi X_3$. Define a new multiplication in M' by $X_iX_j=[X_i,X_j]_{M'}$, $i,j=1,2$, then M' is isomorphic to M and $\mathfrak{L}'$ is a 3-dimensional simple Lie algebra satisfying the property (*).

Next, if M is a simple Malcev algebra with base X_1, X_2, X_3 such that $X_1X_2=X_2$, $X_1X_3=-X_3$, $X_2X_3=X_1$, then $\mathfrak{L}=M\oplus\mathfrak{D}(M)$ is a 6-dimensional semi-simple Lie algebra. Let $\mathfrak{L}'$ be a Lie algebra with base X_1', X_2', X_3', X_4', in which a multiplication is defined by $[X_1',X_2']=X_2'$, $[X_1',X_3']=-X_3'$, $[X_1',X_4']=0$, $[X_2',X_3']=X_1'-X_4'$, $[X_2',X_4']=X_2'$, $[X_3',X_4']=-X_3'$, (e.g. $X_1'=x\frac{\partial}{\partial x}$, $X_2'=x\frac{\partial}{\partial y}$, $X_3'=y\frac{\partial}{\partial x}$, $X_4'=y\frac{\partial}{\partial y}$). If we put $M'=\Phi X_1'+\Phi X_2'+\Phi X_3'$ and define a product $X_i'X_j'$ in M' by $[X_i',X_j']_{M'}$, then M' is a Malcev algebra isomorphic to M. The Lie algebra $\mathfrak{L}'$ satisfies the property (*) and is not semi-simple.

By using [12, Theorem 1.1] we obtain another proof of Malcev's result concerning a commutator algebra of the alternative algebra, which is a non-associative algebra defined by $x(yy)=(xy)y$ and $(xx)y=x(xy)$.

THEOREM 1.4. (Malcev) *Any alternative algebra A with multiplication xy is a Malcev algebra relative to $[x,y]=xy-yx$.*

PROOF. Let L_x and R_x denote the mappings $z\to xz$ and $z\to zx$, respectively, for all z in A. From [8, (12)], $2[L_x,L_y]+2[L_x,R_y]+2[R_x,R_y]$ is a derivation of A, hence of a commutator algebra of A. If we put $\lambda_x=L_x-R_x$, then $[\lambda_x,\lambda_y]+\lambda_{xy}=2[L_x,L_y]+2[L_x,R_y]+2[R_x,R_y]$, hence $[xy[z,w]]=[[xyz],w]+[z,[xyw]]$, where $[xyz]=[x[y,z]]-[y[x,z]]+[[x,y]z]$. Therefore, the theorem is proved from [12, Theorem 1.1]. A Malcev algebra derived from an alternative algebra A by Theorem 1.4 is called to be associated with A.

The following diagram shows the relations among certain classes of non-associative algebras[3)], where

3) (i) An alternative algebra is a (special) Jordan algebra relative to $ab+ba$ [8].

(ii) An associative (alternative) algebra is a Lie (Malcev) algebra relative to $ab-ba$.

(iii) A Jordan algebra is a Jordan triple system of type I (of type II) relative to $x(yz)$ $(\frac{1}{2}x(yz)-\frac{1}{2}y(zx)+\frac{1}{2}z(xy))$ [3; 11].

(iv) A Jordan triple system of type I with multiplication $\{xyz\}$ is that of type II relative to $\frac{1}{2}\{xyz\}-\frac{1}{2}\{yzx\}+\frac{1}{2}\{zxy\}$ [3; 11].

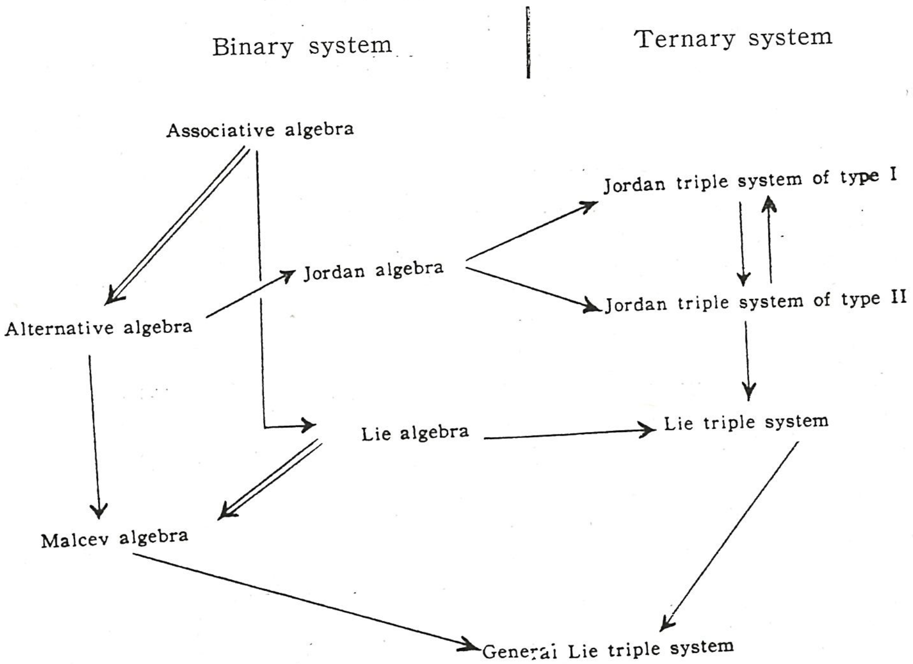
(iv)′ A Jordan triple system of type II with multiplication $\langle xyz\rangle$ is that of type I relative to $\langle xyz\rangle+\langle xzy\rangle$ [3; 11].

(v) A Jordan triple system of type II with multiplication $\langle xyz\rangle$ is a Lie triple system relative to $\langle xyz\rangle-\langle yxz\rangle$ [3; 11].

(vi) A Lie algebra with multiplication $[x,y]$ is a Lie triple system relative to $[[x,y]z]$ [3].

(vii) A Lie triple system T with multiplication $[xyz]$ is a general Lie triple system by defining $xy=0$ for all x,y in T.

$A \Rightarrow B$ means B is a more generalized concept than A,
$A \rightarrow B$ means A has the structure of B.



2. Solvability.[4] Let A, B and C be the subspaces of a Malcev algebra M. $A+B$ denotes the subspace spanned by A and B. AB and $[A\ B\ C]$ means, respectively, the subspaces of M spanned by all elements of the forms xy and $[xyz]=x(yz)-y(xz)+(xy)z$, where $x \in A$, $y \in B$, $z \in C$. If A and B are ideals of M, then $A \cap B$ and $A+B$ are ideals of M, but AB is not necessary so [6, Example 3.4]. We have the following lemma, which is fundamental in the sequel.

LEMMA 2.1. *Let A and B be the ideals of a Malcev algebra M, then $AB+[M\ A\ B]+[M\ B\ A]$ is an ideal of M.*

PROOF. An identity $2(xy)z+[zxy]-[zyx]=0$ in M implies $M(AB) \subseteqq [M\ A\ B]+[M\ B\ A]$. Since $z \rightarrow [xyz]$ is a derivation of M, we have $x[yzw]=[yz(xw)]-[yzx]w$ for every $x, y \in M$, $z \in A$, $w \in B$, hence $M[M\ A\ B] \subseteqq [M\ A\ B]+AB$. Similarly $M[M\ B\ A] \subseteqq [M\ B\ A]+AB$ and the lemma is proved.

Let A be an ideal of M. For an integer $k \geqq 0$ we define inductively $A^{(k)}$ by $A^{(0)}=A$, $A^{(1)}=AA+[M\ A\ A]$, $A^{(k)}=A^{(k-1)}A^{(k-1)}+[M\ A^{(k-1)}A^{(k-1)}]$. Since $A^{(k)}=(A^{(k-1)})^{(1)}$, by Lemma 2.1, $A^{(k)}$ is an ideal of M. A is called to be *solvable in* M provided $A^{(n)}=(0)$ for some integer $n \geqq 0$. Any ideal of abelian Malcev algebra M is solvable in M.

4) For the proofs of the existence of radical, the relations between the properties of M and Lie algebra $M \oplus \mathfrak{D}(M)$ in this section, and the relations between the properties of M and a weak (or generalized) representation of M in the next section, we follow the method of Lister [4, II].

Let $\mathfrak{L}$ be a Lie algebra with multiplication $[x, y]$ and let A be a (Lie) solvable ideal of $\mathfrak{L}$. Denote the derived subalgebra of Lie algebra A by $A^{[k]}$, $k=0,1,2,\cdots$. Since $[x\,y\,z]=2[[x, y]\,z]$, $A^{(1)}=[A, A]+[\mathfrak{L}\,A\,A]\subseteqq[A, A]=A^{[1]}$. Assume $A^{(k)}\subseteqq A^{[k]}$, then $A^{(k+1)}\subseteqq [A^{[k]}, A^{[k]}]+[[\mathfrak{L}, A^{[k]}]A^{[k]}]\subseteqq[A^{[k]}, A^{[k]}]=A^{[k+1]}$, hence, A is (Malcev) solvable in the Malcev algebra $\mathfrak{L}$. Conversely, suppose A is (Malcev) solvable in $\mathfrak{L}$. Since $A^{[k]}\subseteqq A^{(k)}$ by definition, A is a (Lie) solvable ideal of $\mathfrak{L}$.

PROPOSITION 2.1. *Let $\mathfrak{L}$ be the Lie algebra $M\oplus\mathfrak{D}(M)$ (standard construction) and let $\mathfrak{R}$ be a solvable ideal of $\mathfrak{L}$, then $\mathfrak{R}_M$ is a solvable ideal in M.*

PROOF. $M\mathfrak{R}_M=[M, \mathfrak{R}_M]_M\subseteqq[M, \mathfrak{R}]_M\subseteqq\mathfrak{R}_M$, hence $\mathfrak{R}_M$ is an ideal of M. $\mathfrak{R}_M\mathfrak{R}_M=[\mathfrak{R}_M, \mathfrak{R}_M]_M\subseteqq[\mathfrak{R}, \mathfrak{R}]=\mathfrak{R}^{[1]}$, and $[M\mathfrak{R}_M\mathfrak{R}_M]=[[M, \mathfrak{R}_M]_{\mathfrak{D}(M)}, \mathfrak{R}_M]\subseteqq[[M, \mathfrak{R}]\mathfrak{R}]\subseteqq[\mathfrak{R}, \mathfrak{R}]=\mathfrak{R}^{[1]}$, hence $\mathfrak{R}_M{}^{(1)}\subseteqq\mathfrak{R}^{[1]}$. Assume $\mathfrak{R}_M{}^{(k)}\subseteqq\mathfrak{R}^{[k]}$, then similarly $\mathfrak{R}_M{}^{(k+1)}\subseteqq\mathfrak{R}^{[k+1]}$ therefore $\mathfrak{R}_M{}^{(k)}\subseteqq\mathfrak{R}^{[k]}$ for any integer $k\geqq 0$, from this $\mathfrak{R}_M{}^{(n)}=(0)$ for some integer n and the proposition is proved.

LEMMA 2.2. *Let A and B be solvable ideals in a Malcev algebra M, then $A+B$ is so.*

PROOF. $(A+B)^{(1)}=(A+B)(A+B)+[M\;\;A+B\;\;A+B]\subseteqq AA+BB+[M\,A\,A]+[M\,B\,B]+A\cap B\subseteqq A^{(1)}+B^{(1)}+A\cap B$. Assume that $(A+B)^{(k)}\subseteqq A^{(k)}+B^{(k)}+A\cap B$, then $(A+B)^{(k+1)}\subseteqq A^{(k)}A^{(k)}+B^{(k)}B^{(k)}+[M\,A^{(k)}\,A^{(k)}]+[M\,B^{(k)}B^{(k)}]+A\cap B\subseteqq A^{(k+1)}+B^{(k+1)}+A\cap B$. Hence $(A+B)^{(k)}\subseteqq A^{(k)}+B^{(k)}+A\cap B$ for each integer k, and there exists an integer n such that $(A+B)^{(n)}\subseteqq A\cap B$, from this $(A+B)^{(n+l)}=((A+B)^{(n)})^{(l)}\subseteqq(A\cap B)^{(l)}\subseteqq A^{(l)}=(0)$ for some integer l, therefore $A+B$ is solvable in M.

From Lemma 2.2 it follows that in a finite-dimensional Malcev algebra M there exists an unique maximal solvable ideal. We call this maximal solvable ideal the *radical* of M. A Malcev algebra M is called *semi-simple* if the radical of M is (0). If R is the radical of M, then M/R is semi-simple. Let S be an ideal of M such that M/S is semi-simple, then $S\supseteqq R$.

THEOREM 2.1. *Let $\mathfrak{L}$ be a Lie algebra $M\oplus\mathfrak{D}(M)$ (standard construction). Denote by $M^{(k)}$ and $\mathfrak{L}^{(k)}$ the derived subalgebras of order k of M and $\mathfrak{L}$ respectively, then*

$$\mathfrak{L}^{(2k-1)}\subseteqq M^{(k)}+\sum_{i=0}^{k-1}D(M^{(i)}, M^{(2k-2-i)}),\tag{2.1}$$

$$\mathfrak{L}^{(2k)}\subseteqq M^{(k)}+\sum_{i=0}^{k-1}D(M^{(i)}, M^{(2k-1-i)}).$$

Hence, if M is solvable, then $\mathfrak{L}$ is a solvable Lie algebra.

PROOF. $\mathfrak{L}^{(1)}=[\mathfrak{L}, \mathfrak{L}]\subseteqq MM+D(M, M)+[M\;M\;M]+[D(M, M),\;\;D(M, M)]\subseteqq M^{(1)}+D(M, M)$ by (1.7). Similarly $\mathfrak{L}^{(2)}\subseteqq M^{(1)}+D(M, M^{(1)})$. Suppose that the theorem has been proved in case $2k-1$. Then,

$$\mathfrak{L}^{(2k)}\subseteqq[M^{(k)}, M^{(k)}]+\sum_{i=0}^{k-1}[M^{(i)}M^{(2k-2-i)}M^{(k)}]+\sum_{i=0}^{k-1}\sum_{j=0}^{k-1}[D(M^{(i)}, M^{(2k-2-i)}), D(M^{(j)}, M^{(2k-2-j)})].$$

Now, $0\leqq i\leqq k-1$, so $2k-2-i\geqq k-1$, hence $[M^{(i)}\ M^{(2k-2-i)}\ M^{(k)}]\subseteqq[M\ M^{(k-1)}M^{(k-1)}]\subseteqq M^{(k)}$. By the skew-symmetry of the product $[D(x, y), D(u, v)]$ we can assume $i\leqq j$.

$$\begin{aligned}&[D(M^{(i)}, M^{(2k-2-i)}),\ D(M^{(j)}, M^{(2k-2-j)})]\\ &\subseteqq D([M^{(i)}\ M^{(2k-2-i)}\ M^{(j)}],\ M^{(2k-2-j)})+D(M^{(j)},\ [M^{(i)}\ M^{(2k-2-i)}M^{(2k-2-j)}])\\ &\subseteqq D([M\ M^{(j)}\ M^{(j)}],\ M^{(2k-2-j)})+D(M^{(j)},\ [M\ M^{(2k-2-j)}\ M^{(2k-2-j)}])\\ &\subseteqq D(M^{(j+1)},\ M^{(2k-2-j)})+D(M^{(j)},\ M^{(2k-1-j)}).\end{aligned}$$

Hence,

$$\begin{aligned}\mathfrak{L}^{(2k)}&\subseteqq M^{(k)}M^{(k)}+D(M^{(k)},\ M^{(k)})+M^{(k)}+\sum_{i=0}^{k-1} D(M^{(i)},\ M^{(2k-1-i)})\\ &\subseteqq M^{(k)}+\sum_{i=0}^{k-1} D(M^{(i)},\ M^{(2k-1-i)}).\end{aligned}$$

Assume the formula has been proved in case of $2k$, then in case $2k+1$ the formula is proved in a similar fashion and this theorem is proved.

COROLLARY. *Let M be a solvable Malcev algebra and let $\mathfrak{D}(M)$ be the subalgebra of $\mathfrak{gl}(M)$ generated by all derivations of the form $\sum_i [L_{x_i}, L_{y_i}]+L_{x_iy_i}$, where L_x is the left multiplication by x. Then $\mathfrak{D}(M)$ is a solvable Lie algebra.*

THEOREM 2.2. *Let M be a semi-simple Malcev algebra, then $\mathfrak{L}=M\oplus\mathfrak{D}(M)$ (standard construction) is the semi-simple Lie algebra.*

PROOF. Denote $\mathfrak{R}$ the radical of $\mathfrak{L}$, then $\mathfrak{R}=\mathfrak{R}_M\oplus\mathfrak{R}_{\mathfrak{D}(M)}$ (vector space direct sum). By Proposition 2.1 $\mathfrak{R}_M$ is a solvable ideal in M hence $\mathfrak{R}_M=(0)$ and $\mathfrak{R}=\mathfrak{R}_{\mathfrak{D}(M)}$. We show next $\mathfrak{R}_{\mathfrak{D}(M)}=(0)$. $[\mathfrak{R}_{\mathfrak{D}(M)}, M]=[\mathfrak{R}, M]\subseteqq\mathfrak{R}$ and the construction of $\mathfrak{L}$ implies $[\mathfrak{R}_{\mathfrak{D}(M)}, M]\subseteqq M$ hence $[\mathfrak{R}_{\mathfrak{D}(M)}, M]\subseteqq\mathfrak{R}\cap M=\mathfrak{R}_{\mathfrak{D}(M)}\cap M=(0)$, i.e. the elements of $\mathfrak{R}_{\mathfrak{D}(M)}$ trivialy operate on M, so $\mathfrak{R}_{\mathfrak{D}(M)}=(0)$ and $\mathfrak{L}$ is a semi-simple Lie algebra.

THEOREM 2.3.[5] *If a Malcev algebra M is semi-simple, then every derivation D of M is inner.*

PROOF. Put $\mathfrak{L}=M\oplus\mathfrak{D}(M)$ (standard construction). The element of $\mathfrak{L}$ is of the form $x+\sum_i D(y_i, z_i)$, $x, y_i, z_i\in M$. By using that $\mathfrak{D}(M)$ is an ideal of the derivation algebra of M, we define a linear mapping $\tilde{D}$ of $\mathfrak{L}$ as follows:

$$\tilde{D}(x+\sum_i D(y_i, z_i))=D(x)+\sum_i [D, D(y_i, z_i)].$$

Then $\tilde{D}(M)\subseteqq M$ and $\tilde{D}(\mathfrak{D}(M))\subseteqq\mathfrak{D}(M)$ since $[D, D(y, z)]=D(Dy, z)+D(y, Dz)$. It is easy to see that $\tilde{D}$ is an derivation of $\mathfrak{L}$. From Theorem 2.2 $\mathfrak{L}$ is a semi-simple Lie algebra, hence every derivation of $\mathfrak{L}$ is inner, so there exists l in $\mathfrak{L}$ such that $\tilde{D}=\operatorname{ad} l$. Put $l=a+\sum_i D(b_i, c_i)$. For arbitrary $x\in M$ $\tilde{D}(x)=ax+D(a, x)+\sum_i [b_i\, c_i\, x]$, hence $D(a, x)=0$ which implies $[a\, x\, y]=0$ for all $x, y\in M$ and $D(x)=(L_a+\sum_i D(b_i, c_i))(x)$. We show

5) In [7], Sagle proved this result. However, he called a Malcev algebra M is to be semi-simple if M is a direct sum of simple ideals.

that a is an element of J-nucleus of M. Since $\tilde{D}(D(x, y))=-[xya]+\sum_i [D(b_i, c_i), D(x, y)]$ we have $[x y a]=0$ for all $x, y \in M$. On the other hand $[x y z]+[z x y]-[z y x] = \varDelta(x, y)z$ for all $x, y, z \in M$, hence $\varDelta(x, y)a=0$ for all $x, y \in M$, i.e. a belongs to the J-nucleus of M. Thus this theorem is proved.

Since every representation of semi-simple Lie algebra is completely reducible we obtain the following

COROLLARY.[5] *The derivation algebra* $\mathfrak{D}$ *of any semi-simple Malcev algebra* M *is completely reducible in* M.

3. Weak and generalized representations. We first recall the representation of a Lie algebra $\mathfrak{L}$ over Φ. A representation of $\mathfrak{L}$ into a vector space V is a linear mapping $x \to \rho(x)$ of $\mathfrak{L}$ into the algebra $\mathfrak{E}(V)$ of linear transformations of V satisfying

$$\rho([x, y])=\rho(x)\rho(y)-\rho(y)\rho(x)$$

for all $x, y \in \mathfrak{L}$. This says that

(a) ρ is a homomorphism of $\mathfrak{L}$ into the Lie algebra $\mathfrak{gl}(V)$ of linear transformations of V,

and implies that

(b) the vector space spanned by all $\rho(x)$ forms a subalgebra of $\mathfrak{gl}(V)$.

As an example of the representation of $\mathfrak{L}$ we have

(c) an adjoint representation $l \to \operatorname{ad} l$, where $\operatorname{ad} l$ denotes a linear transformation $x \to [l, x]$ in $\mathfrak{L}$.

In Malcev algebras, a mapping of this type is not necessary a representation in the sense stated above. In fact, let L_x be the left multiplication by x, then that $x \to L_x$ is a homomorphism is equivalent to say that the Jacobi identity holds in the Malcev algebras. Also, the vector space spanned by all L_x does not necessary form a subalgebra of $\mathfrak{gl}(M)$.

Let ρ be a linear mapping $x \to \rho(x)$ of a Malcev algebra M into the algebra $\mathfrak{E}(V)$. Put

$$\varDelta(x, y)=[\rho(x), \rho(y)]-\rho(xy) \tag{3.1}$$

for $x, y \in M$. If $\varDelta(x, y)=0$ for all $x, y \in M$, i.e. ρ is a homomorphism, then we call ρ a *special representation* of M and we shall generalize the notion of representation for Malcev algebras in three ways which correspond to (a), (b), and (c). In the case of (c), this generalization was already done by Eilenberg [2].

Standing on a viewpoint of (a) we define a more general representation than the special representations for Malcev algebras as follows:

DEFINITION 3.1. A linear mapping ρ of a Malcev algebra M into the algebra of linear transformations of a vector space V over Φ is called a *weak representation* of M if

$$[D(x, y), \rho(z)]=\rho([x y z]) \tag{3.2}$$

for all $x, y, z \in M$, where

(3.3) $$D(x, y)=[\rho(x), \rho(y)]+\rho(xy).$$

If ρ is a special representation of M, then ρ is a representation in the sense defined above, particulary a representation of Lie algebra $\mathfrak{L}$ is a weak representation of Malcev algebra $\mathfrak{L}$.

For $x \in M$, let L_x be a left multiplication $z \rightarrow xz$ in M, then $x \rightarrow L_x$ is a weak representation of M which is called to be *regular* [6, Proposition 8.3]. For a regular representation, $D(x, y)$ is an inner derivation in M. We note that in an anti-commutative algebra, for the left multiplication L_x (3.2) characterizes a Malcev algebra [12, Theorem 1.1]. Let A be a subalgebra of M and let B be an ideal of M, then a regular representation of M induces a weak representation of A into B. Let K be a kernel of weak representation. If $k \in K$, then $[x\,y\,k] \in K$ for all $x, y \in M$, hence K is an invariant subspace of the inner derivation of the form $[L_x, L_y]+L_{xy}$. Let (ρ_i, V), $i=1,2,\cdots,n$, be the weak representations of M such that $\rho_i(x)$ commutes with $\rho_j(y)$ for all $x, y \in M$; $i \neq j$. If we put $\rho(x)=\sum_{i=1}^{n} \rho_i(x)$, then ρ is a weak representation of M into V. This fact implies the following two results. Let (ρ_i, V_i) be the n weak representations of M and V the tensor product $V_1 \otimes \cdots \otimes V_n$. For $x \in M$, define an n-linear mapping $\tilde{\rho}(x)$ of $V_1 \times \cdots \times V_n$ into V as $\tilde{\rho}(x)(v_1, \cdots, v_n)=\sum_{i=1}^{n} v_1 \otimes \cdots \otimes \rho_i(x) v_i \otimes \cdots \otimes v_n$, then there exists a linear transformation $\rho(x)$ of V satisfying $\tilde{\rho}(x)(v_1, \cdots, v_n)=\rho(x)(v_1 \otimes \cdots \otimes v_n)$. $x \rightarrow \rho(x)$ is a weak representation of M with V as a representation space. Next, let (ρ, V) and (σ, W) be the weak representations of M and denote $\mathfrak{L}(V, W)$ a vector space of all linear mappings of V into W. For $x \in M$, we define a linear transformation $\tau(x)$ of $\mathfrak{L}(V, W)$ by $(\tau(x)f)(v)=\sigma(x)(f(v))-f(\rho(x)v)$ for $f \in \mathfrak{L}(V, W)$, $v \in V$. Put $(\tau_1(x)f)(v)=\sigma(x) \cdot f(v)$, $(\tau_2(x)f)(v)=-f(\rho(x)v)$, then $(\tau_1(x)(\tau_2(y)f))(v)=(\tau_2(x)(\tau_1(x)f))(v)=-(\sigma(x)(f\rho(y)))(v)$, hence $x \rightarrow \tau(x)$ is a weak representation of M with $\mathfrak{L}(V, W)$ as the representation space.

A representation of an alternative algebra A into a vector space V is a pair (L, R) of linear mappings of A into $\mathfrak{E}(V)$ satisfying

$$[L_x, R_y]=[R_x, L_y]=L_y L_x-L_{yx}=R_y R_x-R_{xy}$$

for all x, y in A [8].

PROPOSITION 3.1. *Let (L, R) be a representation of an alternative algebra A. Then, $x \rightarrow L_x-R_x$ is a weak representation of a Malcev algebra associated with A.*

PROOF is similar to that of Theorem 1.4. We see that $[L_x, L_y]=L_{[x,y]}-2[L_x, R_y]$ and $[R_x, R_y]=R_{[y,x]}-2[L_x, R_y]$, and we can use these relations to obtain $\Delta(x, y)+6[L_x, R_y]=0$ which implies $D(x, y)=2[L_x, L_y]+2[L_x, R_y]+2[R_x, R_y]$. The proposition follows from these relations.

For a weak representation ρ of Malcev algebra, we have the following relations.

(3.4) $$D(xy, z)+D(yz, x)+D(zx, y)=0,$$

or

(3.5) $$\Delta(xy, z)+\Delta(yz, x)+\Delta(zx, y)+2\rho(J(x, y, z))=0,$$
where $J(x, y, z)=(xy)z+(yz)x+(zx)y$.

REMARK 3.1. (3.4) (or (3.5)) and a relation $[D(x, y), \rho(y)]=\rho([xyy])$ are equivalent to (3.2). In fact, a linearlization of $[D(x, y), \rho(y)]=\rho([xyy])$ implies $[D(x, y), \rho(z)]+[D(x, z), \rho(y)]=\rho([xyz])+\rho([xzy])$, then by interchange of x and y and subtract we have $3([D(x, y), \rho(z)]-\rho([xyz]))=D(xy, z)+D(yz, x)+D(zx, y)$, hence (3.2).

(3.6) $$[D(x, y), D_k(z, w)]=D_k([xyz], w)+D_k(z, [xyw]),$$
where $D_k(z, w)=[\rho(z), \rho(w)]+k\rho(zw)$ and k is an integer. In particular,

(3.6)′ $$[D(x, y), D(z, w)]=D([xyz], w)+D(z, [xyw]).$$

Hence, we have the following

LEMMA 3.1. *For a weak representation ρ of a Malcev algebra M into V, a vector space spanned by $\sum_i D(x_i, y_i)$ forms a subalgebra of the Lie algebra $\mathfrak{gl}(V)$.*

(3.7) $$2[\Delta(x, y), \rho(z)]+3\Delta(xy, z)=\Delta(yz, x)+\Delta(zx, y),$$
from which, by using (3.6) with $k=-1$

(3.8) $$[\Delta(x, y), \Delta(z, w)]=\Delta([xyz]+xy\cdot z, w)+\Delta(z, [xyw]+xy\cdot w)+3\Delta(xy, zw).$$

Therefore we have the following

LEMMA 3.2. *For a weak representation ρ of a Malcev algebra M into V, a vector space spanned by $\sum_i \Delta(x_i, y_i)$ forms a subalgebra of the Lie algebra $\mathfrak{gl}(V)$.*

THEOREM 3.1. *For a weak representation of a Malcev algebra M let $\Delta(M, M)$ be the Lie algebra generated by all $\sum_i \Delta(x_i, y_i)$, $x_i, y_i \in M$ and denote by $\Delta(M, M)^{(n)}$ the nth derived subalgebra of $\Delta(M, M)$. Then,*

(3.9)
$$\Delta(M, M)^{(2k-1)}\subseteq\sum_{i=0}^{k-1} \Delta(M^{(i)}, M^{(2k-1-i)}),$$
$$\Delta(M, M)^{(2k)}\subseteq\sum_{i=0}^{k} \Delta(M^{(i)}, M^{(2k-i)}).$$

Hence, if M is a solvable Malcev algebra, then $\Delta(M, M)$ is a solvable Lie algebra.

PROOF. The formula is easily proved in case $n=1, 2$ by using (3.8). Assume that the formula has been proved in case $n=2k-1$. Then,

$$\begin{aligned}\Delta(M, M)^{(2k)}&\subseteq\sum_{i=0}^{k-1}\sum_{j=0}^{k-1} [\Delta(M^{(i)}, M^{(2k-1-i)}), \Delta(M^{(j)}, M^{(2k-1-j)})]\\ &\subseteq\sum_{i=0}^{k-1}\sum_{j=0}^{k-1} \Delta([M^{(i)}M^{(2k-1-i)}M^{(j)}]+M^{(i)}M^{(2k-1-i)}\cdot M^{(j)}, M^{(2k-1-j)})\\ &\quad+\sum_{i=0}^{k-1}\sum_{j=0}^{k-1} \Delta(M^{(j)}, [M^{(i)}M^{(2k-1-i)}M^{(2k-1-j)}]+M^{(i)}M^{(2k-1-i)}\cdot M^{(2k-1-j)})\\ &\quad+\sum_{i=0}^{k-1}\sum_{j=0}^{k-1} \Delta(M^{(i)}M^{(2k-1-i)}, M^{(j)}M^{(2k-1-j)}).\end{aligned}$$

We can assume $i \leqq j$ by the skew-symmetry of the product $[\Delta(x, y), \Delta(z, w)]$, then $2k-1-i>j$. Using that $M^{(n)}$ is an ideal of M

$$\begin{aligned}\Delta(M, M)^{(2k)} &\subseteqq \sum_{j=0}^{k-1} \Delta(M^{(j+1)}, M^{(2k-1-j)}) + \Delta(M^{(j)}, M^{(2k-j)}) + \Delta(M^{(2k-1-j)}, M^{(2k-1-j)}) \\ &= \sum_{i=0}^{k} \Delta(M^{(i)}, M^{(2k-i)}).\end{aligned}$$

Next, we obtain $\Delta(M, M)^{(2k+1)} \subseteqq \sum_{i=0}^{k} \Delta(M^{(i)}, M^{(2k+1-i)})$ similarly and the theorem is proved.

COROLLARY. *If M is a solvable Malcev algebra, then the Lie algebra generated by all $\sum_i [L_{x_i}, L_{y_i}] - L_{x_i y_i}$ is solvable.*

Denote $D(A, B)$ $(\Delta(A, B))$ a vector space spanned by all $\sum_i D(x_i, y_i)$ $(\sum_i \Delta(x_i, y_i))$, $x_i \in A$, $y_i \in B$. A subalgebra $\mathfrak{A}$ of a Lie algebra $\mathfrak{L}$ is called to be subinvariant in $\mathfrak{L}$ if there exists a finite sequence of subalgebras $\mathfrak{L}=\mathfrak{A}_0, \mathfrak{A}_1, \cdots, \mathfrak{A}_r=\mathfrak{A}$ such that $\mathfrak{A}_i$ is an ideal of $\mathfrak{A}_{i-1}$, $i=1, 2, \cdots, r$.

PROPOSITION 3.2. *Let N be an ideal of a Malcev algebra M. Then the Lie algebra $D(N, N)$ is an ideal of $D(M, M)$ and the Lie algebra $\Delta(N, N)$ is subinvariant in $\Delta(M, M)$.*

PROOF. From (3.6)' $D(N, N)$ is an ideal of $D(M, M)$. Next, $\Delta(N, N)$ is an ideal of $\Delta(N, M)$ and $\Delta(N, M)$ is an ideal of $\Delta(M, M)$ by (3.8).

PROPOSITION 3.3. *Let ρ be a weak representation of Malcev algebra M. Then the eveloping Lie algebra of $\rho(M)$ is $\rho(M)+[\rho(M), \rho(M)]$. Let N be an ideal of M. Then a Lie algebra $\rho(N)+[\rho(N), \rho(N)]$ is subinvariant in $\rho(M)+[\rho(M), \rho(M)]$.*

PROOF. The first part is an immediate consequence of (3.2) and the second part follows from that $\rho(N)+[\rho(N), \rho(N)]$ is an ideal of $\rho(N)+[\rho(N), \rho(M)]$ and $\rho(N)+[\rho(N), \rho(M)]$ is an ideal of $\rho(M)+[\rho(M), \rho(M)]$.

THEOREM 3.2. *Let ρ be a weak representation of a Malcev algebra M. If M is solvable then the enveloping Lie algebra $\mathfrak{L}$ of $\rho(M)$ is solvable.*

PROOF. From Proposition 3.3 $\mathfrak{L}=\rho(M)+[\rho(M), \rho(M)]$ hence $\mathfrak{L}=\rho(M)+D(M, M)$. Denote by $M^{(k)}$ and $\mathfrak{L}^{(k)}$ the derived subalgebras of order k of M and $\mathfrak{L}$ respectively. By a similar manner as the proof of Theorem 3.1 we have:

$$\mathfrak{L}^{(2k-1)} \subseteqq \rho(M^{(k)}) + \sum_{i=0}^{k-1} D(M^{(i)}, M^{(2k-2-i)}),$$

$$\mathfrak{L}^{(2k)} \subseteqq \rho(M^{(k)}) + \sum_{i=0}^{k-1} D(M^{(i)}, M^{(2k-1-i)}).$$

Thus if M is solvable then there is an integer n such that $\mathfrak{L}^{(n)}=(0)$, which proves the theorem.

COROLLARY. *If M is a solvable Malcev algebra, then the inner derivation algebra of M is a solvable Lie algebra.*

PROPOSITION 3.4. *Let ρ be a weak representation of a Malcev algebra M with V as a representation space and let N be an ideal of M. Then, for any subspace W of V stable under ρ, $\Delta(M, N)W$ is a stable subspace of V.*

PROOF. This follows easily from (3.7). In this proposition, assume ρ is regular, then it follows that for ideals N_1, N_2 of M, $J(M, N_1, N_2)$ is an ideal of M [6, Theorem 3.5].

We shall next define a generalized representation for Malcev algebras from a viewpoint of (b).

DEFINITION 3.2. Let M be a Malcev algebra. Let P be a bilinear mapping of $M\times M$ into an algebra of linear transformations of a vector space V. P is called a *generalized representation* of M if

(3.10) $$P(x, y)+P(y, x)=0$$

and

(3.11) $$[P(x, y), P(z, w)]=P([xyz], w)+P(z, [xyw])$$

for all x, y, z, w in M.

From this definition, a vector space $P(M, M)$ spanned by all $\sum_i P(x_i, y_i)$, $x_i, y_i \in M$, forms a subalgebra of $\mathfrak{gl}(V)$. Let ρ be a weak representation of M into a vector space V, then (3.6)′ shows that $(x, y)\rightarrow D(x, y)=[\rho(x), \rho(y)]+\rho(xy)$ is a generalized representation of M with V as the representation space. Let A be a subalgebra of M and let B be an invariant subspace for all inner derivations of the form $D(x, y)=[L_x, L_y]+L_{xy}$, $x, y \in M$, then D induces a generalized representation of A into B. Let (P_i, V), $i=1, 2, \cdots, n$, be the generalized representations of M such that $P_i(M, M)$ commute with $P_j(M, M)$ for $i\neq j$. If we put $P(x, y)=\sum_{i=1}^{n} P_i(x, y)$, then P is a generalized representation of M into V.

Let $\mathfrak{L}$ be the Lie algebra $M\oplus\mathfrak{D}(M)$ constructed out of the Malcev algebra M. For arbitrary element x in M, define a linear transformation $\rho(x)$ in $\mathfrak{L}$ by $l\rightarrow[x, l]$, where $[l, m]$ denotes the product in $\mathfrak{L}$. If we put $P(x, y)=[\rho(x), \rho(y)]-\rho(xy)$, then $P(x, y)(l)=[D(x, y), l]$, $D(x, y)=[L_x, L_y]+L_{xy}$, from which we see that P is a generalized representation of M with $\mathfrak{L}$ as the representation space. In this case, $x\rightarrow\rho(x)$ is not necessary a weak representation of M. For example, let M be a vector space with base X_1, X_2, X_3, X_4 and define a multiplication in M by $X_1X_2=-X_2$, $X_1X_3=-X_3$, $X_1X_4=X_4$, $X_2X_3=2X_4$, $X_iX_j=-X_jX_i$, $i\neq j$, and the others are 0. Then, M is a Malcev algebra [6, Example 3.1] and the Lie algebra $\mathfrak{L}=M\oplus\mathfrak{D}(M)$ is a 7-dimensional Lie algebra with the following multiplication table, in which $Y_1=D(X_1, X_2)$, $Y_2=D(X_1, X_3)$, $Y_3=D(X_1, X_4)$

$(=-D(X_2, X_3))$.

	X_1	X_2	X_3	X_4	Y_1	Y_2	Y_3
X_1	0	$-X_2+Y_1$	$-X_3+Y_2$	X_4+Y_3	$2X_2$	$2X_3$	$2X_4$
X_2	X_2-Y_1	0	$2X_4-Y_3$	0	0	$2X_4$	0
X_3	X_3-Y_2	$-2X_4+Y_3$	0	0	$-2X_4$	0	0
X_4	$-X_4-Y_3$	0	0	0	0	0	0
Y_1	$-2X_2$	0	$2X_4$	0	0	$4Y_3$	0
Y_2	$-2X_3$	$-2X_4$	0	0	$-4Y_3$	0	0
Y_3	$-2X_4$	0	0	0	0	0	0

For $x=z=X_1$, $y=X_3$, $l=Y_1$, $[[\rho(x), \rho(y)]+\rho(xy), \rho(z)](l)=8(X_4-Y_4)$ and $\rho([xyz])(l)=4X_4$, hence ρ is not a weak representation of M.

THEOREM 3.3. *Let P be a generalized representation of a Malcev algebra M and let $P(M, M)$ be the Lie algebra generated by all $\sum_i P(x_i, y_i)$, $x_i, y_i \in M$. If $M^{(k)}$ and $P(M, M)^{(k)}$ are the derived subalgebras of order k of M and $P(M, M)$ respectively, then*

$$P(M, M)^{(2k-1)} \subseteq \sum_{i=0}^{k-1} P(M^{(i)}, M^{(2k-1-i)}),$$

(3.12)

$$P(M, M)^{(2k)} \subseteq \sum_{i=0}^{k} P(M^{(i)}, M^{(2k-i)}).$$

If M is a solvable Malcev algebra, then $P(M, M)$ is a solvable Lie algebra.

PROOF. By using (3.10) and (3.11) this is proved in a similar fashion as the proof of Theorem 3.1.

COROLLARY. *Let M be a solvable Malcev algebra. Let P be a generalized representation of M into a vector space V over an algebraically closed field. Then there exists a one-dimensional P-invariant subspace of V.*

PROOF. This follows from Theorem 3.3 and Lie's theorem.

A Malcev algebra M is said to be *nilpotent* if there is an integer s such that $L_{x_1}L_{x_2}\cdots L_{x_s}=0$ for every $x_i \in M$. A nilpotent Malcev algebra is solvable. If M is nilpotent then $D(x_1, y_1)D(x_2, y_2)\cdots D(x_t, y_t)=0$ for some integer t, where $D(x, y)=[L_x, L_y]+L_{xy}$.

THEOREM 3.4. *Let P be a generalized representation of a nilpotent Malcev algebra M into a vector space V. Then the Lie algebra $P(M, M)$ generated by all $\sum_i P(x_i, y_i)$ is a nilpotent subalgebra of $\mathfrak{gl}(V)$.*

PROOF. By the induction on n, it follows that $(\text{ad} \sum_i P(x_i, y_i))^n P(z, w)=\sum P(D(x_{j_1}, y_{j_1})\cdots D(x_{j_s}, y_{j_s})z, D(x_{k_1}, y_{k_1})\cdots D(x_{k_t}, y_{k_t})w)$, where $s+t=n$ and $D(x, y)z=x(yz)-y(xz)+(xy)z$. Since M is nilpotent there exists an integer l such that $D(x_{i_1}, y_{i_1})\cdots D(x_{i_l}, y_{i_l})=0$, so

$(\mathrm{ad}\ \sum_i P(x_i, y_i))^n=0$ for some integer n. Hence, by Engel's theorem, $P(M, M)$ is a nilpotent Lie algebra.

COROLLARY. *Let M be a nilpotent Malcev algebra and let $\mathfrak{D}(M)$ be the Lie algebra generated by all derivations of the form $\sum_i [L_{x_i}, L_{y_i}]+L_{x_i y_i}$. Then $\mathfrak{D}(M)$ is a nilpotent Lie algebra.*

4. The cohomology ring of a Malcev algebra. Let M be a Malcev algebra over a field Φ of characteristic 0. A $2p$-dimensional cochain in M, $p\geqq 1$, is a $2p$-linear function f of $M\times\cdots\times M$ ($2p$ times) into Φ such that if $x_{2k-1}=x_{2k}$, $k=1, 2,\cdots, p$,

$$f(x_1, x_2,\cdots, x_{2k-1}, x_{2k},\cdots, x_{2p})=0.$$

Denote by $C^{2p}(M)$, $p=0, 1, 2,\cdots$, the vector space over Φ spanned by $2p$-dimensional cochains of M, where we define $C^0(M)=\Phi$.

A coboundary operator δ is a linear mapping of $C^{2p}(M)$ into $C^{2p+2}(M)$ defined by the formula

$$(4.1)\quad \begin{aligned} &\delta f=0 && \text{for } f\in C^0(M),\\ &(\delta f)(x_1,\cdots, x_{2p+2})=\sum_{k=1}^{p}\sum_{j=2k+1}^{2p+2}(-1)^k f(x_1,\cdots, \hat{x}_{2k-1}, \hat{x}_{2k},\cdots, [x_{2k-1}x_{2k}x_j],\cdots, x_{2p+2}) \\ & && \text{for } f\in C^{2p}(M),\ p=1, 2, 3,\cdots, \end{aligned}$$

where the sign $\wedge$ over a letter indicates that this letter is to be omitted.

For $x, y\in M$, $\kappa(x, y)$ is a linear mapping of $C^{2p}(M)$ into $C^{2p}(M)$ defined by

$$(4.2)\quad \begin{aligned} &\kappa(x, y)f=0 && \text{for } f\in C^0(M),\\ &(\kappa(x, y)f)(x_1,\cdots, x_{2p})=-\sum_{j=1}^{2p} f(x_1,\cdots, [x\, y\, x_j],\cdots, x_{2p}) && \text{for } f\in C^{2p}(M),\ p=1, 2, 3,\cdots. \end{aligned}$$

For $x, y\in M$, $\iota(x, y)$ is a linear mapping of $C^{2p}(M)$ into $C^{2p-2}(M)$ defined by

$$(4.3)\quad \begin{aligned} &\iota(x, y)f=0 && \text{for } f\in C^0(M),\\ &(\iota(x, y)f)(x_1,\cdots, x_{2p-2})=f(x, y, x_1,\cdots, x_{2p-2}) && \text{for } f\in C^{2p}(M),\ p=1, 2, 3,\cdots. \end{aligned}$$

From above definitions we have immediately the following two relations:

$$(4.4)\quad \{\iota(x, y),\ \delta\}f=\kappa(x, y)f \qquad \text{for } f\in C^{2p}(M),\ p=0, 1, 2,\cdots,$$

[6]

and

$$(4.5)\quad [\kappa(x, y),\ \iota(z, w)]f=\iota([xyz], w)f+\iota(z, [xyw])f \qquad \text{for } f\in C^{2p}(M),\ p=0, 1, 2,\cdots.$$

Next, we obtain for $f\in C^{2p}(M)$, $p=0, 1, 2,\cdots$,

$$(4.6)\quad [\kappa(x, y),\ \kappa(z, w)]f=\kappa([xyz], w)f+\kappa(z, [xyw])f.$$

For, in case $p=0$, (4.6) is trivial and if $p=1$ then a direct computation implies (4.6).

6) $\{a, b\}$ denotes the Jordan product $ab+ba$.

Hence, we assume that this relation has been proved for all $f \in C^{2p}(M)$ and let $f \in C^{2p+2}(M)$, $p \geqq 1$. Then for arbitrary $u, v \in M$, using (4.5) we obtain

$$\begin{aligned}&\iota(u, v)[\kappa(x, y),\ \kappa(z, w)]f\\ &=[\kappa(x, y),\ \kappa(z, w)]\iota(u, v)f+\iota([zw[xyu]], v)f-\iota([xy[zwu]], v)f\\ &\quad+\iota(u, [zw[xyv]])f-\iota(u, [xy[zwv]])f.\end{aligned}$$

On the other hand, we have by (4.5)

$$\begin{aligned}&\iota(u, v)(\kappa([xyz], w)+\kappa(z, [xyw]))f\\ &=(\kappa([xyz], w)+\kappa(z, [xyw]))\iota(u, v)f-\iota([[xyz]wu], v)f\\ &\quad-\iota([z[xyw]u], v)f-\iota(u, [[xyz]wv])f-\iota(u, [z[xyw]v])f.\end{aligned}$$

Hence by (1.7) $\iota(u, v)([\kappa(x, y),\ \kappa(z, w)]-\kappa([xyz], w)-\kappa(z, [xyw]))f=0$. Since u, v are arbitrary (4.6) holds for all $f \in C^{2p+2}(M)$.

For $f \in C^{2p}(M)$, $p=0, 1, 2, \cdots$,

$$\kappa(x, y)\delta f=\delta\kappa(x, y)f. \tag{4.7}$$

PROOF. If $p=0$ (4.7) holds trivially and if $p=1$ (4.7) is proved by a direct computation. Hence, we assume that this relation has been proved for all $f \in C^{2p}(M)$ and let $f \in C^{2p+2}(M)$, $p \geqq 1$. Then for arbitrary $z, w \in M$, by (4.4), (4.5), (4.6) $\iota(z, w)[\kappa(x, y), \delta]f =[\delta, \kappa(x, y)]\iota(z, w)f+[\kappa(x, y),\ \kappa(z, w)]f-\{\iota([xyz], w), \delta\}f-\{\iota(z, [xyw]), \delta\}f=0$. Since z, w are arbitrary, (4.7) holds for $f \in C^{2p+2}(M)$.

Using these facts we have the following relation:

$$\delta\delta f=0 \qquad \text{for } f \in C^{2p}(M),\ p=0, 1, 2, \cdots. \tag{4.8}$$

For (4.8) holds trivially in case $p=0$ and is easily proved in case $p=1$. Therefore, we assume that (4.8) has been proved for all $f \in C^{2p}(M)$ and let $f \in C^{2p+2}(M)$, $p \geqq 1$. Then by (4.4), (4.7) for arbitrary $x, y \in M$ $\iota(x, y)\delta\delta f=(-\delta\iota(x, y)+\kappa(x, y))\delta f=\delta\delta\iota(x, y)f-\delta\kappa(x, y)f +\kappa(x, y)\delta f=0$. Since x, y are arbitrary, (4.8) holds for $f \in C^{2p+2}(M)$.

A cochain $f \in C^{2p}(M)$ is called a cocycle if $\delta f=0$ and a coboundary if f is of the form δg for some $g \in C^{2p-2}(M)$. Denote $Z^{2p}(M)$ and $B^{2p}(M)$ the subspaces of $C^{2p}(M)$ spanned by the $2p$-dimensional cocycles and coboundaries respectively. From (4.8), $B^{2p}(M)$ is a subspace of $Z^{2p}(M)$. The factor space $H^{2p}(M)=Z^{2p}(M)/B^{2p}(M)$, $p=0, 1, 2, \cdots$, is called the *2pth cohomology group of the Malcev algebra M*, where we identify $B^0(M)$ with 0.

Let two cochains $f \in C^{2p}(M)$ and $g \in C^{2q}(M)$ be given. We define a cochain $f \cup g \in C^{2(p+q)}(M)$ by setting

$$\begin{aligned}&(f\cup g)(x_1, \cdots, x_{2p})=f(x_1, \cdots, x_{2p})\cdot g &&\text{for } q=0,\\ &(f\cup g)(x_1, \cdots, x_{2q})=f\cdot g(x_1, \cdots, x_{2q}) &&\text{for } p=0,\\ &(f\cup g)(x_1, \cdots, x_{2(p+q)})\\ &=\sum_{\substack{i_1<\cdots<i_p\\ i_{p+1}<\cdots<i_{p+q}}} \operatorname{sgn}\binom{1 \cdots\cdots p+q}{i_1 \cdots i_{p+q}} f(x_{2i_1-1}, x_{2i_1}, \cdots, x_{2i_p-1}, x_{2i_p})g(x_{2i_{p+1}-1}, \cdots, x_{2i_{p+q}}),\end{aligned} \tag{4.9}$$

where

$$\operatorname{sgn}\binom{1 \cdots\cdots p+q}{i_1 \cdots i_{p+q}}=\begin{cases}\ \ 1 \text{ if } (i_1 \cdots i_p) \text{ is an even permutation of } (1\,2 \cdots n),\\ -1 \text{ if } (i_1 \cdots i_p) \text{ is an odd permutation of } (1\,2 \cdots n).\end{cases}$$

This $\cup$-product is distributive and associative:

$$(f\cup g)\cup h=f\cup(g\cup h) \tag{4.10}$$

for $f\in C^{2p}(M)$, $g\in C^{2q}(M)$, $h\in C^{2r}(M)$.

PROOF. We assume $p>0$, $q>0$, $r>0$ since (4.10) is easily proved in the other cases.

$$((f\cup g)\cup h)(x_1,\cdots,x_{2(p+q+r)})$$

$$=\sum_{\substack{i_1<\cdots<i_{p+q}\\ i_{p+q+1}<\cdots<i_{p+q+r}}}\operatorname{sgn}\binom{1\cdots\cdots p+q+r}{i_1\cdots i_{p+q+r}}(f\cup g)(x_{2i_1-1},\cdots,x_{2i_{p+q}})h(x_{2i_{p+q+1}-1},\cdots,x_{2i_{p+q+r}})$$

$$=\sum_{\substack{i_1<\cdots<i_{p+q}\\ i_{p+q+1}<\cdots<i_{p+q+r}}}\sum_{\substack{j_1<\cdots<j_p\\ j_{p+1}<\cdots<j_{p+q}}}\operatorname{sgn}\binom{1\cdots\cdots p+q+r}{i_1\cdots i_{p+q+r}}\operatorname{sgn}\binom{i_1\cdots i_{p+q}}{j_1\cdots j_{p+q}}$$

$$\cdot f(x_{2j_1-1},\cdots,x_{2j_p})g(x_{2j_{p+1}-1},\cdots,x_{2j_{p+q}})h(x_{2i_{p+q+1}-1},\cdots,x_{2i_{p+q+r}}).$$

$$(f\cup(g\cup h))(x_1,\cdots,x_{2(p+q+r)})$$

$$=\sum_{\substack{i_1<\cdots<i_p\\ i_{p+1}<\cdots<i_{p+q+r}}}\sum_{\substack{j_{p+1}<\cdots<j_{p+q}\\ j_{p+q+1}<\cdots<j_{p+q+r}}}\operatorname{sgn}\binom{1\cdots\cdots p+q+r}{i_1\cdots i_{p+q+r}}\operatorname{sgn}\binom{i_{p+1}\cdots i_{p+q+r}}{j_{p+1}\cdots j_{p+q+r}}$$

$$\cdot f(x_{2i_1-1},\cdots,x_{2i_p})g(x_{2j_{p+1}-1},\cdots,x_{2j_{p+q}})h(x_{2j_{p+q+1}-1},\cdots,x_{2j_{p+q+r}}).$$

These two expressions equal to

$$\sum_{\substack{j_1<\cdots<j_p\\ j_{p+1}<\cdots<j_{p+q}\\ j_{p+q+1}<\cdots<j_{p+q+r}}}\operatorname{sgn}\binom{1\cdots\cdots p+q+r}{j_1\cdots j_{p+q+r}}f(x_{2j_1-1},\cdots,x_{2j_p})g(x_{2j_{p+1}-1},\cdots,x_{2j_{p+q}})h(x_{2j_{p+q+1}-1},\cdots,x_{2j_{p+q+r}}),$$

therefore, (4.10) follows.

We obtain easily the relation:

$$f\cup g=(-1)^{pq}g\cup f \qquad \text{for } f\in C^{2p}(M),\ g\in C^{2q}(M). \tag{4.11}$$

Also, we have

$$\kappa(x,y)(f\cup g)=\kappa(x,y)f\cup g+f\cup\kappa(x,y)g \qquad \text{for } f\in C^{2p}(M),\ g\in C^{2q}(M). \tag{4.12}$$

PROOF. (4.12) holds trivially in case $p=0$ or $q=0$. Hence we assume $p>0$ and $q>0$.

$$(\kappa(x,y)(f\cup g))(x_1,\cdots,x_{2(p+q)})$$

$$=-\sum_{j=1}^{2(p+q)}(f\cup g)(x_1,\cdots,[xyx_j],\cdots,x_{2(p+q)})$$

$$=-\sum_{j=1}^{p}\sum_{\substack{i_1<\cdots<i_p\\ i_{p+1}<\cdots<i_{p+q}}}\operatorname{sgn}\binom{1\cdots\cdots p+q}{i_1\cdots i_{p+q}}f_{ij}\cdot g(x_{2i_{p+1}-1},\cdots,x_{2i_{p+q}})$$

$$-\sum_{j=p+1}^{p+q}\sum_{\substack{i_1<\cdots<i_p\\ i_{p+1}<\cdots<i_{p+q}}}\operatorname{sgn}\binom{1\cdots\cdots p+q}{i_1\cdots i_{p+q}}f(x_{2i_1-1},\cdots,x_{2i_p})\cdot g_{ij},$$

where

$$f_{ij}=f(x_{2i_1-1},\cdots,[x\,y\,x_{2i_j-1}],\cdots,x_{2i_p})+f(x_{2i_1-1},\cdots,[x\,y\,x_{2i_j}],\cdots,x_{2i_p}),$$
$$g_{ij}=g(x_{2i_{p+1}-1},\cdots,[x\,y\,x_{2i_j-1}],\cdots,x_{2i_{p+q}})+g(x_{2i_{p+1}-1},\cdots,[x\,y\,x_{2i_j}],\cdots,x_{2i_{p+q}}).$$

Hence $(\kappa(x, y)(f\cup g))(x_1,\cdots, x_{2(p+q)})=(\kappa(x, y)f\cup g+f\cup\kappa(x, y)g)(x_1,\cdots, x_{2(p+q)})$ and (4.12) follows.

For $f\in C^{2p}(M)$, $g\in C^{2q}(M)$

$$(4.13)\qquad \iota(x, y)\,(f\cup g)=\iota(x, y)f\cup g+(-1)^p f\cup\iota(x, y)g.$$

PROOF. This is trivial if $p=0$ or $q=0$. Assume $p>0$ and $q>0$.

$$\begin{aligned}
&\iota(x_1, x_2)(f\cup g)\,(x_3,\cdots, x_{2(p+q)})\\
&=(f\cup g)\,(x_1,\cdots, x_{2(p+q)})\\
&=\sum_{\substack{i_2<\cdots<i_p\\ i_{p+1}<\cdots<i_{p+q}}} \operatorname{sgn}\binom{2\cdots\cdots p+q}{i_2\cdots i_{p+q}} f(x_1, x_2, x_{2i_2-1},\cdots, x_{2i_p})g(x_{2i_{p+1}-1},\cdots, x_{2i_{p+q}})\\
&+\sum_{\substack{i_2<\cdots<i_{p+1}\\ i_{p+2}<\cdots<i_{p+q}}} \operatorname{sgn}\binom{1\cdots\cdots\cdots\cdots\cdots\cdots p+q}{i_2\cdots i_{p+1}\,1\,i_{p+2}\cdots i_{p+q}} f(x_{2i_2-1},\cdots, x_{2i_{p+1}})g(x_1, x_2, x_{2i_{p+2}-1},\cdots, x_{2i_{p+q}})\\
&=(\iota(x_1, x_2)f\cup g)\,(x_3,\cdots, x_{2(p+q)})+(-1)^p(f\cup\iota(x_1, x_2)g)(x_3,\cdots, x_{2(p+q)}).
\end{aligned}$$

Hence (4.13) is proved.

For $f\in C^{2p}(M)$, $g\in C^{2q}(M)$

$$(4.14)\qquad \delta(f\cup g)=\delta f\cup g+(-1)^p f\cup\delta g.$$

PROOF. (4.14) is trivial if $p=0$ or $q=0$ and can be proved by direct calculation if $p=q=1$. Hence assume that (4.14) has been proved for all $f\in C^2(M)$ and $g\in C^{2q-2}(M)$. For $g\in C^{2q}(M)$, $q>1$, by (4.4), (4.12), (4.13)

$$\begin{aligned}
&\iota(x, y)\,(\delta(f\cup g)-\delta f\cup g+f\cup\delta g)\\
&=-\delta\iota(x, y)\,(f\cup g)+\kappa(x, y)\,(f\cup g)-\iota(x, y)\,(\delta f\cup g)+\iota(x, y)\,(f\cup\delta g)\\
&=-\delta(\iota(x, y)f\cup g)+\delta(f\cup\iota(x, y)g)+\kappa(x, y)(f\cup g)-\iota(x, y)(\delta f\cup g)+\iota(x, y)\,(f\cup\delta g)\\
&=(\kappa(x, y)-\delta\iota(x, y)-\iota(x, y)\delta)f\cup g+f\cup(\kappa(x, y)-\delta\iota(x, y)-\iota(x, y)\delta)g=0.
\end{aligned}$$

Since this holds for arbitrary $x, y\in M$, (4.14) holds for every $q\geqq 0$ in case $p=1$. Hence, by the induction on p, assume that (4.14) holds for any $q\geqq 0$ in case p-1 and let $f\in C^{2p}(M)$. Since by (4.11) if $q=1$ the formula holds for any p, we assume the formula holds for all $g\in C^{2q-2}(M)$ and let $g\in C^{2q}(M)$. Then the same manner stated above implies $\iota(x, y)(\delta(f\cup g)-\delta f\cup g-(-1)^p f\cup\delta g)=0$ for arbitrary $x, y\in M$, hence (4.14) holds for $f\in C^{2p}(M)$, $g\in C^{2q}(M)$ and (4.14) is proved.

From (4.14)

$$\begin{aligned}
&f\in Z^{2p}(M),\ g\in Z^{2q}(M) \text{ implies } f\cup g\in Z^{2(p+q)}(M),\\
&f\in Z^{2p}(M),\ g\in B^{2q}(M) \text{ implies } f\cup g\in B^{2(p+q)}(M),\\
&f\in B^{2p}(M),\ g\in Z^{2q}(M) \text{ implies } f\cup g\in B^{2(p+q)}(M),
\end{aligned}$$

hence, it can be defined the $\cup$-product between two chomology classes of arbitrary dimensions and by this product the direct sum $\sum_{p=0}^{\infty} H^{2p}(M)$ becomes a ring, which is called the *cohomology ring* of Malcev algebra M.

5. Cohomology groups associated with a generalized representation. Let (P, V) be a generalized representation of a Malcev algebra M and let f be a $2p$-linear mapping of $M\times\cdots\times M$ ($2p$ times) into V satisfying

$$f(x_1, x_2,\cdots, x_{2k-1}, x_{2k},\cdots, x_{2p})=0 \qquad \text{for } x_{2k-1}=x_{2k},\ k=1,2,\cdots,p,$$

f is called a $2p$-V-cochain. Denote $C^{2p}(M, V)$, $p=0,1,2,\cdots$, a vector space spanned by $2p$-V-cochains, where we identify $C^0(M, V)$ with V. We define a linear mapping δ of $C^{2p}(M, V)$ into $C^{2p+2}(M, V)$ as follows:

(5.1)
$$(\delta f)(x_1, x_2)=P(x_1, x_2)f \qquad \text{for } f\in C^0(M, V),$$
$$\begin{aligned}(\delta f)&(x_1, x_2,\cdots, x_{2p+2})\\ &=\sum_{k=1}^{p+1}(-1)^{k+1}P(x_{2k-1}, x_{2k})f(x_1, x_2,\cdots, \hat{x}_{2k-1}, \hat{x}_{2k},\cdots, x_{2p+2})\\ &+\sum_{k=1}^{p}\sum_{j=2k+1}^{2p+2}(-1)^k f(x_1,\cdots, \hat{x}_{2k-1}, \hat{x}_{2k},\cdots, [x_{2k-1}\ x_{2k}\ x_j],\cdots, x_{2p+2})\end{aligned}$$
$$\text{for } f\in C^{2p}(M, V),\ p=1,2,\cdots,$$

where the sign $\wedge$ over a letter indicates that this letter is to be omitted. We shall prove $\delta\delta f=0$ for any f. In case $f\in C^0(M, V)$, this follows from (3.6)'. To prove the general case, we consider two linear mappings.

For a pair x, y in M, a linear mapping $\kappa(x, y)$ of $C^{2p}(M, V)$ into itself is defined by the formulas:

(5.2)
$$\kappa(x, y)f=P(x, y)f \qquad \text{for } f\in C^0(M, V),$$
$$(\kappa(x, y)f)(x_1, x_2,\cdots, x_{2p})=P(x, y)f(x_1, x_2,\cdots, x_{2p})-\sum_{j=1}^{2p} f(x_1, x_2,\cdots, [x\,y\,x_j],\cdots, x_{2p}),$$
$$\text{for } f\in C^{2p}(M, V),\ p=1,2,3,\cdots,$$

and a linear mapping $\iota(x, y)$ of $C^{2p}(M, V)$ into $C^{2p-2}(M, V)$ is defined by the formulas:

(5.3)
$$\iota(x, y)f=0 \qquad \text{for } f\in C^0(M, V),$$
$$(\iota(x, y)f)(x_1, x_2,\cdots, x_{2p-2})=f(x, y, x_1, x_2,\cdots, x_{2p-2})$$
$$\text{for } f\in C^{2p}(M, V),\ p=1,2,3,\cdots.$$

Then by a direct calculation we have:

(5.4)
$$\{\iota(x, y), \delta\}f=\kappa(x, y)f$$

and

(5.5)
$$[\kappa(x, y),\ \iota(z, w)]f=\iota([xyz], w)f+\iota(z, [xyw])f.$$

We can next verify the following two formulas by induction on n and (1.7), (3.6)', (5.5):

$$[\kappa(x, y), \kappa(z, w)] f = \kappa([xyz], w) f + \kappa(z, [xyw]) f \quad (5.6)$$

for $f \in C^{2p}(M, V)$, $p = 0, 1, 2, \cdots$,

and

$$\kappa(x, y)\, \delta f = \delta \kappa(x, y) f \quad (5.7)$$

for $f \in C^{2p}(M, V)$, $p = 0, 1, 2, \cdots$.

Assume that $\delta\delta f = 0$ for every $f \in C^{2p}(M, V)$ and let $f \in C^{2p+2}(M, V)$. Then $\iota(x, y)\delta\delta f = 0$ for all x, y in M and $\delta\delta f = 0$. Therefore, we proved that for the coboundary operator δ defined by (5.1) we have

$$\delta\delta f = 0. \quad (5.8)$$

Denote $Z^{2p}(M, V)$ a subspace spanned by elements f of $C^{2p}(M, V)$ such that $\delta f = 0$, which is called a $2p$-P-cocycle and $B^{2p}(M, V)$ a subspace spanned by elements of $C^{2p}(M, V)$ of the form δf, $f \in C^{2p-2}(M, V)$, which is called a $2p$-P-coboundary, where $B^0(M, V) = 0$ by definition. Then $B^{2p}(M, V)$ is a subspace of $Z^{2p}(M, V)$ by (5.8). The quotient space $H^{2p}(M, V) = Z^{2p}(M, V)/B^{2p}(M, V)$ is called the *$2p$th cohomology group of M relative to the generalized representation P.*

Let (P, V) be a generalized representation of M. A Killing form φ of M is a bilinear symmetric form defined by $\varphi(x, y) = \mathrm{Tr}\,(L_x L_y)$, L_x being the left multiplication by x. From $[[L_x, L_y] + L_{xy}, L_z L_w] = L_{[xyz]} L_w + L_z L_{[xyw]}$ we get

$$\varphi([xyz], w) + \varphi(z, [xyw]) = 0^{7)}. \quad (5.9)$$

Let $X_1, X_2, \cdots, X_n$ be a base of M. If the matrix $(\varphi(X_i, X_j))$ is regular, then φ is called to be non-degenerate. Assume φ is non-degenerate and let (π_{ij}) be an inverse matrix of $(\varphi(X_i, X_j))$, then (π_{ij}) is symmetric and $\bar{X}_i = \sum_{j=1}^{n} \pi_{ij} X_j$, $i = 1, 2, \cdots, n$, is also a base of M. In this case a linear operator Γ of V is defined by

$$\Gamma = \sum_{i,j=1}^{n} P(X_i, \bar{X}_j) P(\bar{X}_i, X_j).$$

Γ may also be expressed as $\sum_{i,j=1}^{n} P(X_i, X_j) P(\bar{X}_i, \bar{X}_j)$, since $\pi_{ij} = \pi_{ji}$.

LEMMA 5.1. *Let (P, V) be a generalized representation of a Malcev algevra M with non-degenerate Killing form φ. Then Γ commutes with $P(x, y)$ for all x, y in M.*

PROOF. For x, y in M, put $[x y X_i] = \sum_j \alpha_{ij} X_j$, $[x y \bar{X}_i] = \sum_j \beta_{ij} \bar{X}_j$, then by using (5.9) we have $\alpha_{ij} + \beta_{ji} = 0$. Therefore, $[P(x, y), \Gamma] = \sum_{i,j} [P(x, y), P(X_i, \bar{X}_j)] P(\bar{X}_i, X_j) + \sum_{i,j} P(X_i, \bar{X}_j)[P(x, y), P(\bar{X}_i, X_j)] = \sum_{i,j} (P([xyX_i], \bar{X}_j) + P(X_i, [xy\bar{X}_j])) P(\bar{X}_i, X_j) + \sum_{i,j} P(X_i, \bar{X}_j)(P([xy \bar{X}_i], X_j) + P(\bar{X}_i, [xyX_j])) = \sum_{i,j,k} (\alpha_{ik} + \beta_{ki}) P(X_k, \bar{X}_j) P(\bar{X}_i, X_j) + \sum_{i,j,k} (\beta_{jk} + \alpha_{kj}) P(X_i, \bar{X}_k) P(\bar{X}_i, X_j) = 0$, and the lemma is proved.

THEOREM 5.1. *Let (P, V) be a generalized representation of a Malcev algebra M with*

7) (5.9) follows also from [6, Theorem 7.16].

non-degenerate Killing form. Assume Γ is not a zero-operator and V is irreducible for P. Then $H^2(M, V)=(0)$.

PROOF. We use the same notations as in Lemma 5.1. Let f be a 2-P-cocycle, then we have $P(x, y)f(z, w)-P(z, w)f(x, y)-f([xyz], w)-f(z, [xyw])=0$. Put $e=\sum_{i,j=1}^{n} P(X_i, \bar{X}_j) f(X_i, X_j)$, then

$$
\begin{aligned}
P(x, y)e &= \sum_{i,j} [P(x, y), P(X_i, \bar{X}_j)]f(\bar{X}_i, X_j)+\sum_{i,j} P(X_i, \bar{X}_j)P(x, y)f(\bar{X}_i, X_j) \\
&= \sum_{i,j,k} \alpha_{ik}P(X_k, \bar{X}_j)f(\bar{X}_i, X_j)+\sum_{i,j,k} \beta_{jk}P(X_i, \bar{X}_k)f(\bar{X}_i, X_j) \\
&\quad + \sum_{i,j} P(X_i, \bar{X}_j)P(x, y)f(\bar{X}_i, X_j) \\
&= -\sum_{i,j} P(X_i, \bar{X}_j) (f([xy\bar{X}_i], X_j)+f(\bar{X}_i, [xyX_j])) \\
&\quad + \sum_{i,j} P(X_i, \bar{X}_j)P(x, y)f(\bar{X}_i, X_j) \\
&= \sum_{i,j} P(X_i, \bar{X}_j)P(\bar{X}_i, X_j)f(x, y) \\
&= \Gamma f(x, y).
\end{aligned}
$$

Therefore we have $P(x, y)e=\Gamma f(x, y)$. From Lemma 5.1 ΓV is an invariant subspace of V for P. Since Γ is not a zero-operator, $\Gamma V=V$ and there exists an inverse operator Γ^{-1}. Then $f(x, y)=\Gamma^{-1}P(x, y)e=P(x, y)\Gamma^{-1}e$ and f is a 2-P-coboundary. Thus, this theorem is proved.

6. Cohomology groups associated with a weak representation. Let (ρ, V) be a weak representation of a Malcev algebra M and let f be a $2p$-1-linear mapping of $M\times\cdots\times M$ ($2p$-1 times) into V such that if $x_{2k-1}=x_{2k}$, $k=1, 2,\cdots, p-1$,

$$f(x_1, x_2,\cdots, x_{2k-1}, x_{2k},\cdots, x_{2p-1})=0.$$

We denote $C^{2p-1}(M, V)$ a vector space spanned by such mappings and identify $C^0(M, V)$ with V. The coboundary operator δ is a linear mapping of $C^{2p-1}(M, V)$ into $C^{2p+1}(M, V)$ defined by the formulas:

$$(\delta f)(x)=\rho(x)f \qquad \text{for } f\in C^0(M, V),$$

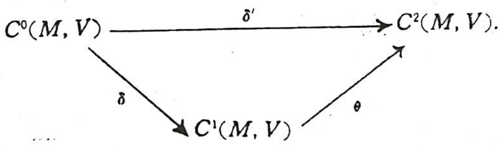
$$
\begin{aligned}
&(\delta f)(x_1, x_2,\cdots, x_{2p+1}) \\
&=(-1)^p\rho(x_{2p+1})[\rho(x_{2p-1})f(x_1,\cdots, x_{2p-2}, x_{2p})-\rho(x_{2p})f(x_1,\cdots, x_{2p-1}) \\
&\quad +f(x_1, \cdots, x_{2p-2}, x_{2p-1}x_{2p})] \\
&\quad +\sum_{k=1}^{p} (-1)^{k+1}D(x_{2k-1}, x_{2k})f(x_1, x_2,\cdots, \hat{x}_{2k-1}, \hat{x}_{2k},\cdots, x_{2p+1}) \\
&\quad +\sum_{k=1}^{p} \sum_{j=2k+1}^{2p+1} (-1)^k f(x_1, x_2,\cdots, \hat{x}_{2k-1}, \hat{x}_{2k},\cdots, [x_{2k-1}x_{2k}x_j],\cdots, x_{2p+1})
\end{aligned}
\tag{6.1}
$$

for $f\in C^{2p-1}(M, V)$, $p=1, 2, 3,\cdots$,

where the sign $\wedge$ over a letter indicates that this letter is to be omitted. Then by using the method in § 5, we obtain $\delta\delta f=0$ for every q-V-cochain f and we can define the $2p$-1th cohomology group $H^{2p-1}(M, V)$ relative to the weak representation ρ by the quotient space $Z^{2p-1}(M, V)/B^{2p-1}(M, V)$, where $Z^{2p-1}(M, V)$ is a vector space spanned by f such that $\delta f=0$ and $B^{2p-1}(M, V)$ is a vector space spanned by f of the form $f=\delta g$, $g\in C^{2p-3}(M, V)$. By definition $H^0(M, V)=Z^0(M, V)$, a subspace of $C^0(M, V)$ spanned by f such that $\rho(x)f=0$ for all $x\in M$.

Let a weak representation ρ of M into V be given, then ρ induces a generalized representation of M by putting $P(x, y)=D(x, y)$ hence the statements in § 5 is applied to this case and the even-dimensional cohomology groups of M relative to P can be considered. We denote by δ' the coboundary operator for $2p$-V-cochains. Then from (3.2) by direct computation we have

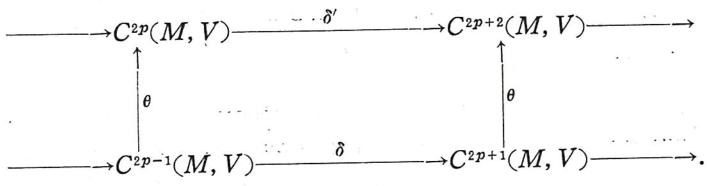
THEOREM 6.1. *Let (ρ, V) be a weak representation of a Malcev algebra M. Define a linear mapping θ of $C^1(M, V)$ into $C^2(M, V)$ by $(\theta f)(x, y)=\rho(x)f(y)-\rho(y)f(x)+f(xy)$. Then the following diagram is commutative:*



Define a linear mapping θ of $C^{2p-1}(M, V)$ into $C^{2p}(M, V)$ by

$$(\theta f)(x_1,\cdots, x_{2p})=\rho(x_{2p-1})f(x_1,\cdots, x_{2p-2}, x_{2p})-\rho(x_{2p})f(x_1,\cdots, x_{2p-1})+f(x_1,\cdots, x_{2p-2}, x_{2p-1}x_{2p}).$$

Then the following diagram is commutative:



COROLLARY. *θ induces a homomorphism θ^*: $H^{2p-1}(M, V)\to H^{2p}(M, V)$, $p=1, 2, 3, \cdots$.*

Let (ρ, V) be a weak representation of M with base $X_1, X_2, \cdots, X_n$. Assume the Killing form φ of M is non-degenerate and put $\bar{X}_i=\sum_{j=1}^{n}\pi_{ij}X_j$, (π_{ij}) being an inverse matrix of $(\varphi(X_i, X_j))$. A Casimir operator C of ρ is defined as $C=\sum_{i=1}^{n}\rho(X_i)\rho(\bar{X}_i)$.

THEOREM 6.2. *Let (ρ, V) be a weak representation of a Malcev algebra M with non-degenerate Killing form φ. Then the Casimir operator C of ρ commutes with $D(x, y)$ for all x, y in M.*

PROOF. From (3.2) we have $[D(x, y), \rho(z)\rho(w)]=\rho([xyz])\rho(w)+\rho(z)\rho([xyw])$, hence $[D(x, y), C]=\sum_i \rho([x\,y\,X_i])\rho(\bar{X}_i)+\sum_i \rho(X_i)\rho([x\,y\,X_i])$. Put $[x\,y\,X_i]=\sum_j \alpha_{ij}X_j$,

$[xy\bar{X}_i]=\sum_j \beta_{ij}\bar{X}_j$, then by (5.9) $\alpha_{ij}+\beta_{ji}=0$. Therefore $[D(x,y),C]=\sum_{i,j}\alpha_{ij}\rho(X_j)\rho(\bar{X}_i)+\sum_{i,j}\beta_{ij}\rho(X_i)\rho(\bar{X}_j)=0$.

COROLLARY (Casimir). *Let ρ be a representation of a semi-simple Lie algebra $\mathfrak{L}$. Then the Casimir operator C of ρ commutes with all $\rho(x)$, x in $\mathfrak{L}$.*

PROOF. It is well known that the Killing form of $\mathfrak{L}$ is non-degenerate if and only if $\mathfrak{L}$ is semi-simple and in this case any element x of $\mathfrak{L}$ is of the form $\sum_i [y_i, z_i]$. Hence $\rho(x)C=\sum_i \rho([y_i,z_i])C=\frac{1}{2}\sum_i D(y_i,z_i)C=\frac{1}{2}\sum_i CD(y_i,z_i)=\sum_i C\rho([y_i,z_i])=C\rho(x)$.

Let f and g be linear mappings of a vector space V into a vector space W and W into itself respectively, then a linear mapping gf of V into W is defined by $(gf)(v)=g(f(v))$ for $v\in V$. Under the same notations as Theorems 6.1 and 6.2 we have

THEOREM 6.3. *Let (ρ, V) be a weak representation of a Malcev algebra M with non-degenerate Killing form, then $(C\theta)^*(H^1(M,V))=(0)$. Assume V is irreducible for D and C is not a zero-operator, then $\theta^*(H^1(M,V))=(0)$.*

PROOF. For arbitrary $x, y\in M$, if we put $[xyX_i]=\sum_j \alpha_{ij}X_j$ then $[xy\bar{X}_i]=-\sum_j \alpha_{ji}\bar{X}_j$ by (5.9). Let f be 1-ρ-cocycle and put $e=\sum_{i=1}^{n}\rho(X_i)f(\bar{X}_i)$. Then $(\delta'e)(x,y)=\sum_i [D(x,y), \rho(X_i)]f(\bar{X}_i)+\sum_i \rho(X_i)D(x,y)f(\bar{X}_i)=\sum_i \rho([xyX_i])f(\bar{X}_i)+\sum_i \rho(X_i)D(x,y)f(\bar{X}_i)=-\sum_i \rho(X_i)f([xy\bar{X}_i])+\sum_i \rho(X_i)D(x,y)f(\bar{X}_i)$. From $f([xyz])=D(x,y)f(z)-\rho(z)\rho(x)f(y)+\rho(z)\rho(y)f(x)-\rho(z)f(xy)$ we have $(\delta'e)(x,y)=(C\theta f)(x,y)$. This proves $(C\theta)^*(H^1(M,V))=(0)$. Next suppose that V is irreducible for D, then $CV\neq(0)$ is invariant under D from Theorem 6.2, so $CV=V$ and there exists an inverse operator C^{-1} of C. Then $(\theta f)(x,y)=C^{-1}\delta'e(x,y)=C^{-1}D(x,y)e=D(x,y)C^{-1}e=\delta'C^{-1}e(x,y)$ hence $\theta^*(H^1(M,V))=(0)$.

7. Representations. In this section, we shall study a representation which is more restrictive than that defined in § 3. We first prove the following

THEOREM 7.1. *Let ρ be a linear mapping of a Malcev algebra M into the algebra of linear transformations of a vector space V. Then the following three conditions are equivalent each other.*

(7.1) $$\rho(xy\cdot z)-\rho(x\cdot yz)=[\rho(xz),\rho(y)]+[\rho(x)\rho(z),\rho(y)]+\rho(x)\rho(yz)-\rho(xy)\rho(z).$$

(7.2) $$\rho(xy\cdot z)=\rho(x)\rho(yz)+\rho(xz)\rho(y)+[\rho(z),\rho(y)\rho(x)].$$

(7.3 a) $$\rho(x\cdot yz)-\rho(z)\{\rho(x),\rho(y)\}+\rho(y)\{\rho(x),\rho(z)\}=\{\rho(x),\rho(yz)\}-\rho(z)\rho(xy)+\rho(y)\rho(xz),$$

(7.3 b) $$[D(x,y),\rho(z)]=\rho([xyz]),$$

where $D(x,y)=[\rho(x),\rho(y)]+\rho(xy)$.

PROOF. (7.1)⇒(7.2): $\mathfrak{S}\,\rho(xy\cdot z-x\cdot yz)=\mathfrak{S}([\rho(xz),\rho(y)]+[\rho(x)\rho(z),\rho(y)]+\rho(x)\rho(yz)-\rho(xy)\rho(z))$, where $\mathfrak{S}$ denotes the summation obtained by cyclic permutations of x, y, z, implies $\rho(J(x, y, z))+[\rho(xy), \rho(z)]+[\rho(yz), \rho(x)]+[\rho(zx), \rho(y)]=0$. Using this relation and (7.1) we have (7.2).

(7.2)⇒(7.3 a): In (7.2) interchange x with y and x with z and add to (7.2), then we obtain (7.3 a).

(7.2)⇒(7.3 b): In (7.2) interchange y with z and x with z and add to (3.2), then we obtain (7.3 b).

(7.3)⇒(7.1): (7.3 a) is rewritten as (α): $\Delta(x, yz)-2\Delta(y, z)\rho(x)-\rho(y)\Delta(x, z)+\rho(z)\Delta(x, y)=0$. In ($\alpha$) interchanging x and y we have (β): $\Delta(y, xz)-2\Delta(x, z)\rho(y)-\rho(x)\Delta(y, z)-\rho(z)\Delta(x, y)=0$. (7.3 b) is rewritten as (γ): $[\Delta(x, y), \rho(z)]+2\Delta(xy, z)+\rho(J(x, y, z))=0$, hence by using ($\alpha$) we obtain ($\delta$): $[\Delta(x, y), \rho(z)]-4\Delta(x, y)\rho(z)+2\rho(y)\Delta(z, x)-2\rho(x)\Delta(z, y)+\rho(J(x, y, z))=0$. In ($\gamma$) interchanging y and z we have (ε): $2[\Delta(x, z), \rho(y)]+4\Delta(xz, y)-2\rho(J(x, y, z))=0$. Then ($\gamma$)$-$($\delta$)$+$($\varepsilon$) implies $\rho(J(x, y, z))+\Delta(y, xz)+\rho(x)\Delta(y,z)-\Delta(x, y)\rho(z)=0$ and this gives (7.1).

A linear mapping $x\to\rho(x)$ of a Malcev algebra M into the algebra of linear transformations of a vector space V is called a *representation* of M if $\rho(x)$'s satisfy one of the conditions in Theorem 7.1.

COROLLARY. *A representation of a Malcev algebra M is a weak representation of M. A weak representation* ρ *of M reduces to a representation of M if and only if* ρ *satisfies* (7.3 a).

If ρ is the linear mapping $x\to L_x$ in M, (7.2) is an expression in Proposition 2.21 in [6] hence (7.2) and the anti-commutative law characterize a Malcev algebra. Therefore we obtain

COROLLARY.[8] *In an anti-commutative algebra, the Malcev condition* (1.9) *is equivalent to any one of the following conditions.*

(7.1)′ $$(xy\cdot z)w-(x\cdot yz)w=(xz)(yw)-(xy)(zw)+x(yz\cdot w)-y(xz\cdot w)+x(z\cdot yw)-y(x\cdot zw).$$

(7.2)′ $$(xz)(yw)=(xy\cdot z)w+(yz\cdot w)x+(zw\cdot x)y+(wx\cdot y)z.$$

(7.3)′ $$[x, y, zw]=[xyz]w+z[xyw],$$

where $[xyz]=x(yz)-y(xz)+(xy)z$.

The regular representation $x\to L_x$ of M is a representation of M into M. Let A be a subalgebra of M and let B be an ideal of M, then the regular representation of M induces a representation of A into B. Since (7.3 a) may be expressed as $\Delta(x, yz)=2\Delta(y, z)\rho(x)+\rho(y)\Delta(x, z)-\rho(z)\Delta(x, y)$ a special representation of M is also a representation. In the following, we base our argument on the relation (7.3). Let (ρ, V) be a representation of M. If we put $\rho(x)v=xv=-vx$ for v in V, we have

8) (7.1)′, (7.2)′, and (7.3)′ are (2.11), Proposition 2.21, and Proposition 8.3 in [6] respectively. That in an anti-commutative algebra (1.9) is equivalent to (7.3)′ has stated in [12, Theorem 1.1].

(7.4) $$xv=-vx,$$

(7.5) $$(x\cdot yz)v-z(x\cdot yv)-z(y\cdot xv)+y(x\cdot zv)+y(z\cdot xv) = x(yz\cdot v)+(yz)(xv)-z(xy\cdot v)+y(xz\cdot v),$$

(7.6) $$x(y\cdot zv)-y(x\cdot zv)-z(x\cdot yv)+z(y\cdot xv)+(xy)(zv)-z(xy\cdot v) = (x\cdot yz)v-(y\cdot xz)v+(xy\cdot z)v$$

for all x, y, z in M, v in V. Following Eilenberg a vector space V with bilinear compositions xv, vx, x in M, v in V, such that xv, vx in V, is called a *Malcev module* if these compositions satisfy (7.4), (7.5), and (7.6). Conversely, let V be a Malcev module for M. If we define a linear mapping $\rho(x)$ of V by $\rho(x)v=xv$, then ρ is a representation of M with representation space V. Therefore the concept of representation of M is equivalent to that of Malcev module for M.

Let (L, R) be a representation of an alternative algebra A into V and let $\breve{A}$ be a vector space direct sum $A\oplus V$. If we define a product in $\breve{A}$ by

$$(x_1+v_1)(x_2+v_2)=x_1x_2+L_{x_1}(v_2)+R_{x_2}(v_1),$$

where $x_i\in A$, $v_i\in V$, $i=1,2$, then $\breve{A}$ is an alternative algebra and V is an ideal of $\breve{A}$. Hence V becomes an ideal of an associated Malcev algebra $\breve{A}^{(-)}$ of $\breve{A}$ and $x\to L_x-R_x$ is a regular representation of $\breve{A}^{(-)}$ hence a representation of $A^{(-)}$ into V. Thus we have

PROPOSITION 7.1.[9] *Let (L, R) be a representation of an alternative algebra A. Then $x\to L_x-R_x$ is a representation of a Malcev algebra associated with A.*

EXAMPLE 7.1. Let M be a Malcev algebra over Φ with base X_1, X_2, X_3, X_4, in which a multiplication is defined by $X_1X_2=-X_2$, $X_1X_3=-X_3$, $X_1X_4=X_4$, $X_2X_3=2X_4$, $X_iX_j=-X_jX_i$, $i\neq j$, and the others equal zero.[10] $\Phi X_1+\Phi X_2$ forms a subalgebra V of M and $\Phi X_3+\Phi X_4$ forms an ideal W of M. Let ρ be the regular representation of V and σ a representation of V into the space W, which is induced from the regular representation of M. For $x\in V$, if we define a linear transformation $\tau(x)$ of $\mathfrak{L}(V, W)$ by

$$(\tau(x)f)(v)=\sigma(x)f(v)-f(\rho(x)v), \qquad f\in\mathfrak{L}(V, W),\ v\in V,$$

then τ is a weak representation of V, but not a representation. For example, define f by $f(\lambda_1X_1+\lambda_2X_2)=(\lambda_1-\lambda_2)X_3$, $\lambda_i\in\Phi$, and let $x=X_2$, $y=X_1$, $z=X_1+X_2$, then $(\tau(xy\cdot z)-\tau(x)\tau(yz)-\tau(xz)\tau(y)-\tau(z)\tau(y)\tau(x)+\tau(y)\tau(x)\tau(z))f(X_1+X_2)=12X_4\neq 0$.

PROPOSITION 7.2. *Let ρ be a representation of a Malcev algebra M. If we put*

(7.7) $$\theta(x, y)=\{\rho(x), \rho(y)\}-\rho(xy)$$

for all x, y in M, then the bilinear mapping θ satisfies the following relations:

9) The proof of this proposition follows the method in [8] and this is a refinement of Proposition 3.1.

10) [6, Example 3.1].

(7. 8) $$D(x, y)+\theta(x, y)-\theta(y, x)-\Delta(x, y)=0,$$

(7. 9) $$\theta(x, yz)-\rho(y)\theta(x, z)+\rho(z)\theta(x, y)=0,$$

(7.10) $$\theta(xy, z)-\theta(x, z)\rho(y)+\theta(y, z)\rho(x)=0,$$

(7.11) $$\theta(z, w)\theta(x, y)-\theta(y, w)\theta(x, z)-\theta(x, [yzw])+D(y, z)\theta(x, w)=0,$$

(7.12) $$[D(x, y),\ \theta(z, w)]=\theta([xyz], w)+\theta(z, [xyw]).$$

PROOF. (7.8) is clear and (7.9) follows from (7.3a).

(7.10): From (7.9) $\theta(y, zx)-\rho(z)\theta(y, x)+\rho(x)\theta(y, z)=0$. Adding this relation to (7.9) and subtracting (7.4) we have (7.10).

(7.11): We apply twice (7.9) to $\theta(x, [yzw])$ and we obtain (7.11).

(7.12): Using a relation $[D(x, y),\ \rho(z)\rho(w)]=\rho([xyz])\rho(w)+\rho(z)\rho([xyw])$ we have (7.12).

Let ρ be a linear mapping of a general L.t.s. T into the algebra $\mathfrak{E}(V)$ of linear transformations of a vector space V over Φ and let θ and D be bilinear mappings of $T\times T$ into $\mathfrak{E}(V)$. (ρ, θ, D) is called a representation of T into V provided these mappings satisfy (7.3b), (7.8), (7.9), $\cdots$, (7.12). Hence a representation of a Malcev algebra M induces a representation of a general L.t.s. associated with M. For a representation of a general L.t.s. we can prove that $D(xy, z)+D(yz, x)+D(zx, y)=0$ and $[D(x, y), D(z, w)]=D([xyz], w)+D(z, [xyw])$ hence the vector space spanned by $\sum_i D(x_i, y_i)$ forms a subalgebra of $\mathfrak{gl}(V)$.

Under the notations and definitions in § 6 we have

THEOREM 7.2. *Let ρ be a representation of a Malcev algebra M with non-degenerate Killing form φ. Then the Casimir operator C of ρ commutes with all $\rho(x)$, x in M.*

PROOF. From (7.2) in Theorem 7.1 we have $\rho(\bar{X}_iX_i\cdot x)=\rho(\bar{X}_i)\rho(X_ix)+\rho(\bar{X}_ix)\rho(X_i)+\rho(x)\rho(X_i)\rho(\bar{X}_i)-\rho(X_i)\rho(\bar{X}_i)\rho(x)$. Hence

$$[C, \rho(x)]=\rho((\sum_{i=1}^{n} X_i\bar{X}_i)x)-\sum_{i=1}^{n} (\rho(x\bar{X}_i)\rho(X_i)+\rho(\bar{X}_i)\rho(xX_i)).$$

Since $\pi_{ij}=\pi_{ji}$, $\sum_{i=1}^{n} X_i\bar{X}_i=\sum_{i,j=1}^{n}\pi_{ij}X_iX_j=0$, therefore the first term on the right hand vanishes. We show the second term is also zero. If we put $xX_i=\sum_j \alpha_{ij}X_j$ and $x\bar{X}_i=\sum_j \beta_{ij}\bar{X}_j$, $i=1, 2,\cdots, n$, then $\alpha_{ij}+\beta_{ji}=0$ since $\varphi(xX_i, \bar{X}_k)+\varphi(X_i, x\bar{X}_k)=0$ [6, Theorem 7.16] and $\varphi(X_i, \bar{X}_k)=\delta_{ik}$. Hence $\sum_i (\rho(x\bar{X}_i)\rho(X_i)+\rho(\bar{X}_i)\rho(xX_i))=\sum_{i,j} (\beta_{ij}\rho(\bar{X}_j)\rho(X_i)+\alpha_{ij}\rho(\bar{X}_i)\rho(X_j))=0$, and we obtain $[C, \rho(x)]=0$. This completes the proof.

Let (ρ, V) be a representation of a Malcev algebra M and let f be a multilinear mapping of $M\times\cdots\times M$ ($2p$ times) into V such that if $x_{2k-1}=x_{2k}$, $k=1, 2,\cdots, p$,

$$f(x_1, x_2,\cdots, x_{2k-1}, x_{2k},\cdots, x_{2p})=0.$$

We denote by $C^{2p}(M, V)$, $n=0, 1, 2,\cdots$, the vector space spanned by such $2p$-linear mappings, where we identify $C^0(M, V)$ with V and denote $C^1(M, V)$ the vector space of linear mappings of M into V. We call f in $C^q(M, V)$ a q-V-cochain.

We define a linear mapping δ of $C^{2p}(M, V)$ into $C^{2p+2}(M, V)$, $n=0, 1, 2, \cdots$, and of $C^1(M, V)$ into $C^2(M, V)$ as follows:

$$(\delta f)(x_1, x_2)=\varDelta(x_1, x_2)f \qquad \text{for } f\in C^0(M, V),$$

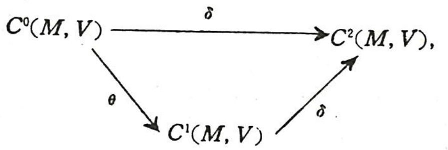
$$(\delta f)(x_1, x_2)=\rho(x_1)f(x_2)-\rho(x_2)f(x_1)-f(x_1x_2) \qquad \text{for } f\in C^1(M, V),$$

$$\begin{aligned}
&(\delta f)(x_1, x_2, \cdots, x_{2p+2})\\
&=(-1)^p[\rho(x_{2p+1})\rho(x_{2p-1})f(x_1, \cdots, x_{2p-2}, x_{2p}, x_{2p+2})-\rho(x_{2p+1})\rho(x_{2p})f(x_1, \cdots, x_{2p-1}, x_{2p+2})\\
&\quad -\rho(x_{2p+2})\rho(x_{2p-1})f(x_1, \cdots, x_{2p-2}, x_{2p}, x_{2p+1})+\rho(x_{2p+2})\rho(x_{2p})f(x_1, \cdots, x_{2p-1}, x_{2p+1})\\
&\quad -\varDelta(x_{2p+1}, x_{2p+2})f(x_1, \cdots, x_{2p})\\
&\quad -\rho(x_{2p-1})f(x_1, \cdots, x_{2p-2}, x_{2p}, x_{2p+1}x_{2p+2})+\rho(x_{2p})f(x_1, \cdots, x_{2p-1}, x_{2p+1}x_{2p+2})\\
&\quad +\rho(x_{2p+1})f(x_1, \cdots, x_{2p-1}, x_{2p}x_{2p+2})-\rho(x_{2p+1})f(x_1, \cdots, x_{2p-2}, x_{2p}, x_{2p-1}x_{2p+2})\\
&\quad +\rho(x_{2p+1})f(x_1, \cdots, x_{2p-2}, x_{2p-1}x_{2p}, x_{2p+2})\\
&\quad -\rho(x_{2p+2})f(x_1, \cdots, x_{2p-1}, x_{2p}x_{2p+1})+\rho(x_{2p+2})f(x_1, \cdots, x_{2p-2}, x_{2p}, x_{2p-1}x_{2p+1})\\
&\quad -\rho(x_{2p+2})f(x_1, \cdots, x_{2p-2}, x_{2p-1}x_{2p}, x_{2p+1})\\
&\quad -f(x_1, \cdots, x_{2p-1}, x_{2p}(x_{2p+1}x_{2p+2}))+f(x_1, \cdots, x_{2p-2}, x_{2p}, x_{2p-1}(x_{2p+1}x_{2p+2}))\\
&\quad -f(x_1, \cdots, x_{2p-2}, x_{2p-1}x_{2p}, x_{2p+1}x_{2p+2})]\\
&+\sum_{k=1}^{p}(-1)^{k+1}D(x_{2k-1}, x_{2k})f(x_1, x_2, \cdots, \hat{x}_{2k-1}, \hat{x}_{2k}, \cdots, x_{2p+2})\\
&+\sum_{k=1}^{p}\sum_{j=2k+1}^{2p+2}(-1)^k f(x_1, \cdots, \hat{x}_{2k-1}, \hat{x}_{2k}, \cdots, [x_{2k-1}x_{2k}x_j], \cdots, x_{2p+2})
\end{aligned}$$

$$\text{for } f\in C^{2p}(M, V),\ p=1, 2, 3, \cdots,$$

where the sign $\wedge$ over a letter indicates that this letter is to be omitted.

Then by the method stated in § 5 we have $\delta\delta f=0$ for every cochain f. A q-V-cochain f is called a q-ρ-cocycle if $\delta f=0$. Denote $Z^q(M, V)$ a subspace of $C^q(M, V)$ spanned by q-ρ-cocycles. Put $B^2(M, V)=\delta C^1(M, V)$ and $B^{2p}(M, V)=\delta C^{2p-2}(M, V)$, $n=2, 3, 4, \cdots$. $B^0(M, V)=B^1(M, V)=(0)$ by definition. If we define a linear mapping θ of $C^0(M, V)$ into $C^1(M, V)$ by $(\theta f)(x)=\rho(x)f$, then the following diagram is commutative:

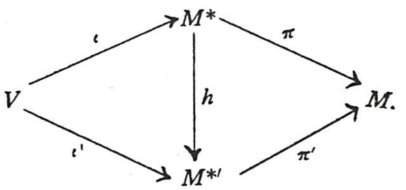


from which $\delta C^0(M, V)\subseteqq B^2(M, V)$. $B^q(M, V)$ is a subspace of $Z^q(M, V)$ and the quotient space $H^q(M, V)=Z^q(M, V)/B^q(M, V)$ is called the qth cohomology group of M relative to the representation ρ.

$H^0(M, V)$ is the subspace of V spanned by the invariant elements for Δ. In particular, if ρ is a regular representation then $H^0(M, M)$ is the J-nucleus of M. A linear mapping f of M into a representation space V is called a derivation of M into V if $f(xy)=\rho(x)f(y)-\rho(y)f(x)$. If ρ is regular f is a derivation of M. Then,

$H^1(M, V)$ is the vector space spanned by derivations of M into V.

Let M, M^*, V be the Malcev algebras over a same field. An *extension of M by V* is an exact sequence $0\rightarrow V\overset{\iota}{\rightarrow}M^*\overset{\pi}{\rightarrow}M\rightarrow 0$ of Malcev algebras, or M^* is called an extension of M by V simply. Two extensions $0\rightarrow V\overset{\iota}{\rightarrow}M^*\overset{\pi}{\rightarrow}M\rightarrow 0$ and $0\rightarrow V\overset{\iota'}{\rightarrow}M^{*\prime}\overset{\pi'}{\rightarrow}M\rightarrow 0$ are said to be equivalent if there is a homomorphism h of M^* into $M^{*\prime}$ such that the following diagram is commutative:



We consider an extension M^* of a Malcev algebra M with an abelian kernel V, i.e. $VV=(0)$. Let l be a linear mapping of M into M^* such that $\pi l(x)=x$ for all $x\in M$. For $x\in M$ if we put $\rho(x)v=l(x)v$, $v\in V$, then $\rho(x)$ is a linear transformation of V and this definition is independent of the choice of l. ρ is a representation of M into V because $\rho(xy\cdot z)v-\rho(x)\rho(yz)v-\rho(xz)\rho(y)v-[\rho(z), \rho(y)\rho(x)]v=l(xy\cdot z)v-l(x)\cdot l(yz)v-l(xz)\cdot l(y)v-l(z)\ (l(y)\cdot l(x)v)+l(y)\ (l(x)\cdot l(z)v)=(l(xy\cdot z)-l(x)l(y)\cdot l(z))v-l(x)\ ((l(yz)-l(y)l(z))v)-(l(xz)-l(x)l(z))\ (l(y)v)=0$, $x, y, z\in M$, $v\in V$, by (7.2)$'$, hence ρ satisfies (7.2). Since π is a homomorphism,

$$f(x, y)=l(x)l(y)-l(xy)$$

is an element of V and f is a 2-V-cochain. For $x, y, z, w\in M$ we have $[l(x), l(y), l(z)\cdot l(w)]=[l(x)l(y)l(z)]l(w)+l(z)[l(x)l(y)l(w)]$, from which it follows:

$$D(x, y)f(z, w)+\Delta(z, w)f(x, y)-\rho(z)\rho(x)f(y, w)+\rho(z)\rho(y)f(x, w)+\rho(w)\rho(x)f(y, z)-\rho(w)\rho(y)f(x, z)+\rho(x)f(y, zw)-\rho(y)f(x, zw)-\rho(z)\ (f(x, yw)-f(y, xw)+f(xy, w))+\rho(w)\ (f(x, yz)-f(y, xz)+f(xy, z))+f(x, y\cdot zw)-f(y, x\cdot zw)+f(xy, zw)-f([xyz], w)-f(z, [xyw])=0,$$

i.e. $(\delta f)\ (x, y, z, w)=0$, therefore f is a 2-ρ-cocycle. Let l' be another linear mapping of M into M^* such that $\pi l'(x)=x$ for all $x\in M$, then $g(x)=l'(x)-l(x)$ is in V and g is a 1-V-cochain. If we put $f'(x, y)=l'(x)l'(y)-l'(xy)$, then $f'(x, y)=f(x, y)+(\delta g)(x, y)$, hence $f'=f+\delta g$ and f and f' belong to the same cohomology class.

Let ρ be a representation of a Malcev algebra M into a vector space V and f be a 2-ρ-cocycle. If we put $M^*=V\oplus M$ (vector space direct sum) and define a multiplication in M^* by

$$(v_1, x_1)(v_2, x_2)=(\rho(x_1)v_2-\rho(x_2)v_1+f(x_1, x_2),\ x_1x_2),$$

then M^* is a Malcev algebra and we have an extension $0 \to V \xrightarrow{\iota} M^* \xrightarrow{\pi} M \to 0$ with $\iota(v) = (v, 0)$ and $\pi(v, x) = x$. Put $l(x) = (0, x)$, then $l(x_1)l(x_2) = (f(x_1, x_2), 0) + l(x_1x_2)$ and f is one of cocycles defined by this extension. Therefore we have

THEOREM 7.3. *To each equivalent class of extensions of a Malcev algebra M by abelian Malcev algebra V corresponds an element of $H^2(M, V)$. Let ρ be a representation of a Malcev algebra M into a vector space V. If f is a cocycle belonging to the element of $H^2(M, V)$, then there is an extension of M by V such that f is one of cocycles defined by this extension.*

Now, we assume the base field Φ is algebraically closed. Let (P, V) be a generalized representation of a nilpotent Malcev algebra M and $P(M)$ the Lie algebra generated by all $\sum_i P(x_i, y_i)$. A linear form λ on $P(M)$ is called a weight of the generalized representation P of M if there exists a non-zero element v of V such that $\mathsf{P}(v) = \lambda(\mathsf{P})v$ for all $\mathsf{P} \in P(M)$. Let (ρ, V) be a weak representation of M, then a generalized representation of M is induced from ρ by putting $P(x, y) = D(x, y)$ and a weight of the weak representation ρ of M is defined similarly. Theorem 3.4 and Lie's theorem imply for a nilpotent Malcev algebra there is at least a weight of P (or ρ). Let A be a linear transformation of a vector space V. For A and $\lambda \in \Phi$, put $V(A, \lambda) = \{v \in V\colon\ (A - \lambda I)^n v = 0$ for some integer $n > 0\}$. For a generalized representation (P, V) let λ be a linear form on $P(M)$ and let V_λ a subspace of V spanned by v such that there is an integer $n > 0$ satisfying $(\mathsf{P} - \lambda(\mathsf{P})I)^n v = 0$ for all $\mathsf{P} \in P(M)$. Then $V_\lambda = \bigcap_{\mathsf{P} \in P(M)} V(\mathsf{P}, \lambda(\mathsf{P}))$. By applying the well known result in Lie algebras to $P(M)$, (e.g. [9, Exposé n^0 9, Théorème 1]), we have the following

THEOREM 7.4. *Let P be a generalized representation of a nilpotent Malcev algebra M into a vector space V. Then,*

(i) *V_λ is P-invariant subspace of V,*

(ii) *if $V_\lambda \neq (0)$, then λ is a weight of P, and vice versa,*

(iii) *$V = \sum_\lambda V_\lambda$ (direct sum).*

THEOREM 7.5. *Let (ρ, V) be a weak representation of a nilpotent Malcev algebra M. If P is a generalized representation induced from ρ by putting $P(x, y) = D(x, y)$, then each V_λ is invariant under ρ.*

PROOF. It is sufficient to prove that $V(\sum_i D(x_i, y_i), \mu)$, $\mu \in \Phi$, is ρ-invariant for every $\sum_i D(x_i, y_i)$. From (3.2) we have

$$(\sum_i D(x_i, y_i) - \mu I)\rho(z) = \rho(z)(\sum_i D(x_i, y_i) - \mu I) + \rho(\sum_i [x_i y_i z]).$$

By the induction on n, we obtain

$$(\sum_i D(x_i, y_i) - \mu I)^n \rho(z) = \sum_{k=0}^{n} \binom{n}{k} \rho((\sum_i \bar{D}(x_i, y_i)^k \cdot z)(\sum_i D(x_i, y_i) - \mu I)^{n-k},$$

where $\bar{D}(x, y)z=[xyz]$. Since M is nilpotent, there is an integer s such that $(\sum_i \bar{D}(x_i, y_i))^s=0$ for all $x_i, y_i \in M$ and $(\sum_i D(x_i, y_i)-\mu I)^{\dim V(\sum D(x_i, y_i), \mu)} \cdot v=0$ for all $v \in V(\sum_i D(x_i, y_i), \mu)$, hence for some integer t it holds $(\sum_i D(x_i, y_i)-\mu I)^t \rho(z)v=0$ for all $z \in M$, $v \in V(\sum_i D(x_i, y_i), \mu)$, so that $V(\sum_i D(x_i, y_i), \mu)$ is invariant under ρ.

Let N be a nilpotent subalgebra of a Malcev algebra M and ρ a representation of N into M induced from the regular representation of M. A weight of ρ is called a root of N in M. Since N is nilpotent, there exists a non-zero element $z \in N$ such that $\sum_i [x_i y_i z]=0$ for all $x_i, y_i \in N$, hence a zero form on $\mathfrak{D}(N)$ is a root. We denote M_α instead of V_α. Then, we have the following

THEOREM 7.6. *If N is a nilpotent subalgebra of a Malcev algebra M, then*

(i) *$M=M_0 \oplus M_\alpha \oplus \cdots \oplus M_\gamma$, where $0, \alpha, \cdots, \gamma$ are roots of N in M,*

(ii) *$M_\alpha M_\beta \subseteqq M_{\alpha+\beta}$ if $\alpha+\beta$ is a root, and $M_\alpha M_\beta=(0)$ otherwise,*

(iii) *M_0 is a subalgebra of M and $N \subseteqq M_0$.*

8. **Simple Malcev algebra C^* derived from Cayley-Dickson algebra.** In [6, §§ 3, 8] Sagle showed that a 7-dimensional simple Malcev algebra C^*, which is not a Lie algebra, can be derived from the Cayley-Dickson algebra and discussed the left multiplications and derivations of C^*. In this section we shall determine the Lie algebra $\mathfrak{L}=C^* \oplus \mathfrak{D}(C^*)$ (standard construction) since it seems that C^* is useful as an example of a simple Malcev algebra which is not a Lie algebra. For the sake of completeness, we first recall the known results of C^*. Let $e_1, e_2, \cdots, e_7$ be a base of vector space C^*, then a Malcev algebra C^* is defined by the following multiplication table:

	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	0	$2e_2$	$2e_3$	$2e_4$	$-2e_5$	$-2e_6$	$-2e_7$
e_2	$-2e_2$	0	$2e_7$	$-2e_6$	e_1	0	0
e_3	$-2e_3$	$-2e_7$	0	$2e_5$	0	e_1	0
e_4	$-2e_4$	$2e_6$	$-2e_5$	0	0	0	e_1
e_5	$2e_5$	$-e_1$	0	0	0	$-2e_4$	$2e_3$
e_6	$2e_6$	0	$-e_1$	0	$2e_4$	0	$-2e_2$
e_7	$2e_7$	0	0	$-e_1$	$-2e_3$	$2e_2$	0

Let $x=\sum_{i=1}^{7} x_i e_i$ and $y=\sum_{i=1}^{7} y_i e_i$ be arbitrary elements of C^*, then the derivation $D(x, y)=[L_x, L_y]+L_{xy}$ of C^* is expressed by

$$\begin{bmatrix}
0 & \begin{matrix}4x_{[1}y_{5]}\\+2x_{[3}y_{4]}\end{matrix} & \begin{matrix}4x_{[1}y_{6]}\\-2x_{[2}y_{4]}\end{matrix} & \begin{matrix}4x_{[1}y_{7]}\\+2x_{[2}y_{3]}\end{matrix} & \begin{matrix}4x_{[1}y_{2]}\\+2x_{[6}y_{7]}\end{matrix} & \begin{matrix}4x_{[1}y_{3]}\\-2x_{[5}y_{7]}\end{matrix} & \begin{matrix}4x_{[1}y_{4]}\\+2x_{[5}y_{6]}\end{matrix}\\
\begin{matrix}-8x_{[1}y_{2]}\\-4x_{[6}y_{7]}\end{matrix} & \begin{matrix}4x_{[2}y_{5]}\\-2x_{[3}y_{6]}\\-2x_{[4}y_{7]}\end{matrix} & 6x_{[2}y_{6]} & 6x_{[2}y_{7]} & 0 & \begin{matrix}4x_{[1}y_{7]}\\+2x_{[2}y_{3]}\end{matrix} & \begin{matrix}-4x_{[1}y_{6]}\\+2x_{[2}y_{4]}\end{matrix}\\
\begin{matrix}-8x_{[1}y_{3]}\\+4x_{[5}y_{7]}\end{matrix} & 6x_{[3}y_{5]} & \begin{matrix}-2x_{[2}y_{5]}\\+4x_{[3}y_{6]}\\-2x_{[4}y_{7]}\end{matrix} & 6x_{[3}y_{7]} & \begin{matrix}-4x_{[1}y_{7]}\\-2x_{[2}y_{3]}\end{matrix} & 0 & \begin{matrix}4x_{[1}y_{5]}\\+2x_{[3}y_{4]}\end{matrix}\\
\begin{matrix}-8x_{[1}y_{4]}\\-4x_{[5}y_{6]}\end{matrix} & 6x_{[4}y_{5]} & 6x_{[4}y_{6]} & \begin{matrix}-2x_{[2}y_{5]}\\-2x_{[3}y_{6]}\\+4x_{[4}y_{7]}\end{matrix} & \begin{matrix}4x_{[1}y_{6]}\\-2x_{[2}y_{4]}\end{matrix} & \begin{matrix}-4x_{[1}y_{5]}\\-2x_{[3}y_{4]}\end{matrix} & 0\\
\begin{matrix}-8x_{[1}y_{5]}\\-4x_{[3}y_{4]}\end{matrix} & 0 & \begin{matrix}4x_{[1}y_{4]}\\+2x_{[5}y_{6]}\end{matrix} & \begin{matrix}-4x_{[1}y_{3]}\\+2x_{[5}y_{7]}\end{matrix} & \begin{matrix}-4x_{[2}y_{5]}\\+2x_{[3}y_{6]}\\+2x_{[4}y_{7]}\end{matrix} & -6x_{[3}y_{5]} & -6x_{[4}y_{5]}\\
\begin{matrix}-8x_{[1}y_{6]}\\+4x_{[2}y_{4]}\end{matrix} & \begin{matrix}-4x_{[1}y_{4]}\\-2x_{[5}y_{6]}\end{matrix} & 0 & \begin{matrix}4x_{[1}y_{2]}\\+2x_{[6}y_{7]}\end{matrix} & -6x_{[2}y_{6]} & \begin{matrix}2x_{[2}y_{5]}\\-4x_{[3}y_{6]}\\+2x_{[4}y_{7]}\end{matrix} & -6x_{[4}y_{6]}\\
\begin{matrix}-8x_{[1}y_{7]}\\-4x_{[2}y_{3]}\end{matrix} & \begin{matrix}4x_{[1}y_{3]}\\-2x_{[5}y_{7]}\end{matrix} & \begin{matrix}-4x_{[1}y_{2]}\\-2x_{[6}y_{7]}\end{matrix} & 0 & -6x_{[2}y_{7]} & -6x_{[3}y_{7]} & \begin{matrix}2x_{[2}y_{5]}\\+2x_{[3}y_{6]}\\-4x_{[4}y_{7]}\end{matrix}
\end{bmatrix},$$

where $x_{[i}y_{j]}$ denotes $x_iy_j-x_jy_i$. We see that

$$D(e_1, e_2)=2D(e_6, e_7), \qquad D(e_1, e_3)=-2D(e_5, e_7),$$
$$D(e_1, e_4)=2D(e_5, e_6), \qquad D(e_1, e_5)=2D(e_3, e_4),$$
$$D(e_1, e_6)=-2D(e_2, e_4), \qquad D(e_1, e_7)=2D(e_2, e_3),$$
$$D(e_2, e_5)=-D(e_3, e_6)-D(e_4, e_7).$$

If we put

$$Y_1=D(e_2, e_3),\ Y_2=D(e_2, e_4),\ Y_3=D(e_2, e_6),\ Y_4=D(e_2, e_7),$$
$$Y_5=D(e_3, e_4),\ Y_6=D(e_3, e_5),\ Y_7=D(e_3, e_6),\ Y_8=D(e_3, e_7),$$
$$Y_9=D(e_4, e_5),\ Y_{10}=D(e_4, e_6),\ Y_{11}=D(e_4, e_7),\ Y_{12}=D(e_5, e_6),$$
$$Y_{13}=D(e_5, e_7),\ Y_{14}=D(e_6, e_7),$$

then the Lie algebra $\mathfrak{D}(C^*)$ generated by $\sum_i D(x_i, y_i)$ is a (14-dimensional) simple Lie algebra of type G_2 with base $Y_1,\cdots, Y_{14}$, therefore $\mathfrak{L}=C^*\oplus\mathfrak{D}(C^*)$ is a 21-dimensional Lie algebra. The multiplication in $\mathfrak{L}$ is given by Table 1.

Let us consider the following base transformation:

$$X_1=e_1,\ X_2=e_2-Y_{14},\ X_3=e_3+Y_{13},\ X_4=e_4-Y_{12},$$
$$X_5=e_5+Y_5,\ X_6=e_6-Y_2,\ X_7=e_7+Y_1,\ X_8=2e_7-Y_1,$$
$$X_9=2e_6+Y_2,\ X_{10}=Y_3,\ X_{11}=Y_4,\ X_{12}=2e_5-Y_5,$$
$$X_{13}=Y_6,\ X_{14}=Y_7,\ X_{15}=Y_8,\ X_{16}=Y_9,\ X_{17}=Y_{10},$$
$$X_{18}=Y_{11},\ X_{19}=2e_4+Y_{12},\ X_{20}=2e_3-Y_{13},\ X_{21}=2e_2+Y_{14}.$$

Then, a multiplication of this base $X_1, \cdots, X_{21}$ is given by Table 2. In the following we assume the base field Φ is algebraically closed. From Theorem 2.2 $\mathfrak{L}$ is a semi-simple Lie algebra. X_1, X_{14}, X_{18} form a Cartan subalgebra $\mathfrak{H}$ of $\mathfrak{L}$. For an arbitrary element $\alpha X_1+\beta X_{14}+\gamma X_{18}$ of $\mathfrak{H}$, if we put $\omega_1=-2\alpha-2\beta-2\gamma$, $\omega_2=-2\alpha+4\beta-2\gamma$, $\omega_3=-2\alpha-2\beta+4\gamma$, then the roots of $\mathfrak{H}$ in $\mathfrak{L}$ and the elements belonging to these roots are:

$$\begin{array}{cccccc} \omega_1, & -\omega_1, & \omega_2, & -\omega_2, & \omega_3, & -\omega_3, \\ X_2, & X_5, & X_3, & X_6, & X_4, & X_7, \end{array}$$

$$\begin{array}{cccccc} \omega_1+\omega_2, & -\omega_1-\omega_2, & \omega_1+\omega_3, & -\omega_1-\omega_3, & \omega_2+\omega_3, & -\omega_2-\omega_3, \\ X_8, & X_{19}, & X_9, & X_{20}, & X_{12}, & X_{21}, \end{array}$$

$$\begin{array}{cccccc} \omega_1-\omega_2, & -\omega_1+\omega_2, & \omega_1-\omega_3, & -\omega_1+\omega_3, & \omega_2-\omega_3, & -\omega_2+\omega_3, \\ X_{10}, & X_{13}, & X_{11}, & X_{16}, & X_{15}, & X_{17}. \end{array}$$

Hence, the roots $\omega_1-\omega_2$, $\omega_2-\omega_3$, ω_3 form a simple system of roots with Dynkin diagram B_3 and $\mathfrak{L}$ is a simple Lie algebra of type B_3. Since the derivation algebra of C^* reduces to $\mathfrak{D}(C^*)$ we have the following

THEOREM 8.1. *Let $\mathfrak{D}$ be a derivation algebra of C^* over an algebraically closed field, then the Lie algebra $C^*\oplus\mathfrak{D}$, in which a multiplication is defined by* (1.12), *is a simple Lie algebra of type B_3.*

Conversely, let $\mathfrak{L}$ be a simple Lie algebra of type B_3 with base $X_1, X_2, \cdots, X_{21}$, then we may assume the multiplication in $\mathfrak{L}$ is given by Table 2. If we define a new base $e_1, \cdots, e_7, Y_1, \cdots, Y_{14}$ by (8.1), then we obtain Table 1 as a multiplication of this base. Let M and $\mathfrak{D}$ be subspaces of $\mathfrak{L}$ spanned by $e_1, \cdots, e_7$ and $Y_1, \cdots, Y_{14}$ respectively, then $\mathfrak{L}$ is a vector space direct sum of M and $\mathfrak{D}$. For $x, y \in M$, define a multiplication xy in M by the M-component of $[x, y]$, then this multiplication in M coincides with that in C^*, hence an algebra M is isomorphic to the 7-dimensional simple non-Lie Malcev algebra C^*. Next, $\mathfrak{D}$ forms a 14-dimensional simple Lie subalgebra of $\mathfrak{L}$ and Y_7, Y_{11} form a Cartan subalgebra $\mathfrak{H}$ of $\mathfrak{D}$. For an arbitrary element $\alpha Y_7+\beta Y_{11}$ of $\mathfrak{H}$, if we put $\omega_1=2\alpha-4\beta$, $\omega_2=-4\alpha+2\beta$ then the roots of $\mathfrak{H}$ in $\mathfrak{D}$ are: $\pm\omega_1$, $\pm\omega_2$, $\pm(\omega_1-\omega_2)$, $\pm(\omega_1+\omega_2)$, $\pm(2\omega_1+\omega_2)$, $\pm(\omega_1+2\omega_2)$ and the roots $\omega_1-\omega_2$, ω_2 form a simple system with Dynkin diagram G_2. Hence, $\mathfrak{D}$ is an exceptional simple Lie algebra of type G_2 relative to the multiplication $[x, y]$. Therefore we have the following

THEOREM 8.2. *Let $\mathfrak{L}$ be a simple Lie algebra of type B_3 over an algebraically closed field and assume a multiplication $[x, y]$ in $\mathfrak{L}$ is given by Table 2. Define a new base $e_1, \cdots, e_7, Y_1, \cdots, Y_{14}$ by* (8.1). *Then, the subspace of $\mathfrak{L}$ spanned by $e_1, \cdots, e_7$ forms a simple Malcev algebra isomorphic to C^* relative to a multiplication $[x, y]_M$, an M-component of $[x, y]$. The subspace of $\mathfrak{L}$ spanned by $Y_1, \cdots, Y_{14}$ forms an exceptional simple Lie algebra of type G_2 relative to $[x, y]$.*

TABLE 1

	e_1	e_2	e_3	e_4	e_5	e_6	e_7	Y_1	Y_2	Y_3
e_1	0	$2e_2+2Y_{14}$	$2e_3-2Y_{13}$	$2e_4+2Y_{12}$	$-2e_5+2Y_5$	$-2e_6-2Y_2$	$-2e_7+2Y_1$	$4e_7$	$-4e_6$	0
e_2	$-2e_2-2Y_{14}$	0	$2e_7+Y_1$	$-2e_6+Y_2$	$e_1-Y_7-Y_{11}$	Y_3	Y_4	0	0	0
e_3	$-2e_3+2Y_{13}$	$-2e_7-Y_1$	0	$2e_5+Y_5$	Y_6	e_1+Y_7	Y_8	0	$2e_1$	$-6e_2$
e_4	$-2e_4-2Y_{12}$	$2e_6-Y_2$	$-2e_5-Y_5$	0	Y_9	Y_{10}	e_1+Y_{11}	$-2e_1$	0	0
e_5	$2e_5-2Y_5$	$-e_1+Y_7+Y_{11}$	$-Y_6$	$-Y_9$	0	$-2e_4+Y_{12}$	$2e_3+Y_{13}$	$2e_3$	$2e_4$	$6e_6$
e_6	$2e_6+2Y_2$	$-Y_3$	$-e_1-Y_7$	$-Y_{10}$	$2e_4-Y_{12}$	0	$-2e_2+Y_{14}$	$-2e_2$	0	0
e_7	$2e_7-2Y_1$	$-Y_4$	$-Y_8$	$-e_1-Y_{11}$	$-2e_3-Y_{13}$	$2e_2-Y_{14}$	0	0	$-2e_2$	0
Y_1	$-4e_7$	0	0	$2e_1$	$-2e_3$	$2e_2$	0	0	$-4Y_{14}$	0
Y_2	$4e_6$	0	$-2e_1$	0	$-2e_4$	0	$2e_2$	$4Y_{14}$	0	0
Y_3	0	0	$6e_2$	0	$-6e_6$	0	0	0	0	0
Y_4	0	0	0	$6e_2$	$-6e_7$	0	0	0	0	0
Y_5	$-4e_5$	$2e_1$	0	0	0	$-2e_4$	$2e_3$	$-4Y_{13}$	$4Y_{12}$	$-6Y_2$
Y_6	0	$6e_3$	0	0	0	$-6e_5$	0	0	$6Y_5$	$12Y_7+6Y_{11}$
Y_7	0	$-2e_2$	$4e_3$	$-2e_4$	$2e_5$	$-4e_6$	$2e_7$	$2Y_1$	$-4Y_2$	$-6Y_3$
Y_8	0	0	0	$6e_3$	0	$-6e_7$	0	0	$6Y_1$	$-6Y_4$
Y_9	0	$6e_4$	0	0	0	0	$-6e_5$	$-6Y_5$	0	$6Y_{10}$
Y_{10}	0	0	$6e_4$	0	0	0	$-6e_6$	$6Y_2$	0	0
Y_{11}	0	$-2e_2$	$-2e_3$	$4e_4$	$2e_5$	$2e_6$	$-4e_7$	$-4Y_1$	$2Y_2$	0
Y_{12}	$-4e_4$	$-2e_6$	$2e_5$	0	0	0	$2e_1$	$-2Y_{11}$	$2Y_{10}$	0
Y_{13}	$4e_3$	$-2e_7$	0	$2e_5$	0	$-2e_1$	0	$2Y_8$	$-2Y_7$	$6Y_{14}$
Y_{14}	$-4e_2$	0	$-2e_7$	$2e_6$	$2e_1$	0	0	$-2Y_4$	$2Y_3$	0

TABLE 1 (Continued)

	Y_4	Y_5	Y_6	Y_7	Y_8	Y_9	Y_{10}	Y_{11}	Y_{12}	Y_{13}	Y_{14}
e_1	0	$4e_5$	0	0	0	0	0	0	$4e_4$	$-4e_3$	$4e_2$
e_2	0	$-2e_1$	$-6e_3$	$2e_2$	0	$-6e_4$	0	$2e_2$	$2e_6$	$2e_7$	0
e_3	0	0	0	$-4e_3$	0	0	$-6e_4$	$2e_3$	$-2e_5$	0	$2e_7$
e_4	$-6e_2$	0	0	$2e_4$	$-6e_3$	0	0	$-4e_4$	0	$-2e_5$	$-2e_6$
e_5	$6e_7$	0	0	$-2e_5$	0	0	0	$-2e_5$	0	0	$-2e_1$
e_6	0	$2e_4$	$6e_5$	$4e_6$	$6e_7$	0	0	$-2e_6$	0	$2e_1$	0
e_7	0	$-2e_3$	0	$-2e_7$	0	$6e_5$	$6e_6$	$4e_7$	$-2e_1$	0	0
Y_1	0	$4Y_{13}$	0	$-2Y_1$	0	$6Y_5$	$-6Y_2$	$4Y_1$	$2Y_{11}$	$-2Y_8$	$2Y_4$
Y_2	0	$-4Y_{12}$	$-6Y_5$	$4Y_2$	$-6Y_1$	0	0	$-2Y_2$	$-2Y_{10}$	$2Y_7$	$-2Y_3$
Y_3	0	$6Y_2$	$-12Y_7-6Y_{11}$	$6Y_3$	$6Y_4$	$-6Y_{10}$	0	0	0	$-6Y_{14}$	0
Y_4	0	$-6Y_1$	$-6Y_8$	0	0	$-6Y_7-12Y_{11}$	$6Y_3$	$6Y_4$	$6Y_{14}$	0	0
Y_5	$6Y_1$	0	0	$-2Y_5$	0	0	0	$-2Y_5$	$2Y_9$	$-2Y_6$	$-2Y_7-2Y_{11}$
Y_6	$6Y_8$	0	0	$-6Y_6$	0	0	$-6Y_9$	0	0	0	$-6Y_{13}$
Y_7	0	$2Y_5$	$6Y_6$	0	$6Y_3$	0	$-6Y_{10}$	0	$-2Y_{12}$	$4Y_{13}$	$-2Y_{14}$
Y_8	0	0	0	$-6Y_8$	0	$6Y_6$	$6Y_7-6Y_{11}$	$6Y_8$	$-6Y_{13}$	0	0
Y_9	$6Y_7+12Y_{11}$	0	0	0	$-6Y_6$	0	0	$-6Y_9$	0	0	$6Y_{12}$
Y_{10}	$-6Y_3$	0	$6Y_9$	$6Y_{10}$	$-6Y_7+6Y_{11}$	0	0	$-6Y_{10}$	0	$-6Y_{12}$	0
Y_{11}	$-6Y_4$	$2Y_5$	0	0	$-6Y_8$	$6Y_9$	$6Y_{10}$	0	$4Y_{12}$	$-2Y_{13}$	$-2Y_{14}$
Y_{12}	$-6Y_{14}$	$-2Y_9$	0	$2Y_{12}$	$6Y_{13}$	0	0	$-4Y_{12}$	0	$-4Y_5$	$4Y_2$
Y_{13}	0	$2Y_6$	0	$-4Y_{13}$	0	0	$6Y_{12}$	$2Y_{13}$	$4Y_5$	0	$-4Y_1$
Y_{14}	0	$2Y_7+2Y_{11}$	$6Y_{13}$	$2Y_{14}$	0	$-6Y_{12}$	0	$2Y_{14}$	$-4Y_2$	$4Y_1$	0

TABLE 2

	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8	X_9	X_{10}	X_{11}
X_1	0	$-2X_2$	$-2X_3$	$-2X_4$	$2X_5$	$2X_6$	$2X_7$	$-4X_8$	$-4X_9$	0	0
X_2	$2X_2$	0	$3X_8$	$-3X_9$	$-3(X_1+X_{14}+X_{18})$	$3X_{10}$	$3X_{11}$	0	0	0	0
X_3	$2X_3$	$-3X_8$	0	$3X_{12}$	$3X_{13}$	$-3X_1+3X_{14}$	$3X_{15}$	0	0	$-6X_2$	0
X_4	$2X_4$	$3X_9$	$-3X_{12}$	0	$3X_{16}$	$3X_{17}$	$-3X_1+3X_{18}$	0	0	0	$-6X_2$
X_5	$-2X_5$	$3(X_1+X_{14}+X_{18})$	$-3X_{13}$	$-3X_{16}$	0	$-3X_{19}$	$3X_{20}$	$6X_3$	$-6X_4$	$6X_6$	$6X_7$
X_6	$-2X_6$	$-3X_{10}$	$3X_1-3X_{14}$	$-3X_{17}$	$3X_{19}$	0	$-3X_{21}$	$-6X_2$	0	0	0
X_7	$-2X_7$	$-3X_{11}$	$-3X_{15}$	$3X_1-3X_{18}$	$-3X_{20}$	$3X_{21}$	0	0	$6X_2$	0	0
X_8	$4X_8$	0	0	0	$-6X_3$	$6X_2$	0	0	0	0	0
X_9	$4X_9$	0	0	0	$6X_4$	0	$-6X_2$	0	0	0	0
X_{10}	0	0	$6X_2$	0	$-6X_6$	0	0	0	0	0	0
X_{11}	0	0	0	$6X_2$	$-6X_7$	0	0	0	0	0	0
X_{12}	$4X_{12}$	0	0	0	0	$-6X_4$	$6X_3$	0	0	$6X_9$	$6X_8$
X_{13}	0	$6X_3$	0	0	0	$-6X_5$	0	0	$-6X_{12}$	$12X_{14}+6X_{18}$	$6X_{15}$
X_{14}	0	$-2X_2$	$4X_3$	$-2X_4$	$2X_5$	$-4X_6$	$2X_7$	$2X_8$	$-4X_9$	$-6X_{10}$	0
X_{15}	0	0	0	$6X_3$	0	$-6X_7$	0	0	$-6X_8$	$-6X_{11}$	0
X_{16}	0	$6X_4$	0	0	0	0	$-6X_5$	$-6X_{12}$	0	$6X_{17}$	$6X_{14}+12X_{18}$
X_{17}	0	0	$6X_4$	0	0	0	$-6X_6$	$-2X_9$	0	0	$-6X_{10}$
X_{18}	0	$-2X_2$	$-2X_3$	$4X_4$	$2X_5$	$2X_6$	$-4X_7$	$-4X_8$	$2X_9$	0	$-6X_{11}$
X_{19}	$-4X_{19}$	$6X_6$	$-6X_5$	0	0	0	0	$12X_1+6X_{18}$	$6X_{17}$	0	$-6X_{21}$
X_{20}	$-4X_{20}$	$-6X_7$	0	$6X_5$	0	0	0	$6X_{15}$	$12X_1+6X_{14}$	$-6X_{21}$	0
X_{21}	$-4X_{21}$	0	$6X_7$	$-6X_6$	0	0	0	$6X_{11}$	$6X_{10}$	0	0

TABLE 2 (Continued)

	X_{12}	X_{13}	X_{14}	X_{15}	X_{16}	X_{17}	X_{18}	X_{19}	X_{20}	X_{21}
X_1	$-4X_{12}$	0	0	0	0	0	0	$4X_{19}$	$4X_{20}$	$4X_{21}$
X_2	0	$-6X_3$	$2X_2$	0	$-6X_4$	0	$2X_2$	$-6X_6$	$6X_7$	0
X_3	0	0	$-4X_3$	0	0	$-6X_4$	$2X_3$	$6X_5$	0	$-6X_7$
X_4	0	0	$2X_4$	$-6X_3$	0	0	$-4X_4$	0	$-6X_5$	$6X_6$
X_5	0	0	$-2X_5$	0	0	0	$-2X_5$	0	0	0
X_6	$6X_4$	$6X_5$	$4X_6$	$6X_7$	0	0	$-2X_6$	0	0	0
X_7	$-6X_3$	0	$-2X_7$	0	$6X_5$	$6X_6$	$4X_7$	0	0	0
X_8	0	0	$-2X_8$	0	$6X_{12}$	$2X_9$	$4X_8$	$-12X_1-6X_{18}$	$-6X_{15}$	$-6X_{11}$
X_9	0	$6X_{12}$	$4X_9$	$6X_8$	0	0	$-2X_9$	$-6X_{17}$	$-12X_1-6X_{14}$	$-6X_{10}$
X_{10}	$-6X_9$	$-12X_{14}-6X_{18}$	$6X_{10}$	$6X_{11}$	$-6X_{17}$	0	0	0	$6X_{21}$	0
X_{11}	$-6X_8$	$-6X_{15}$	0	0	$-6X_{14}-12X_{18}$	$6X_{10}$	$6X_{11}$	$6X_{21}$	0	0
X_{12}	0	0	$-2X_{12}$	0	0	0	$-2X_{12}$	$-6X_{16}$	$-6X_{13}$	$-12X_1+6X_{14}+6X_{18}$
X_{13}	0	0	$-6X_{13}$	0	0	$-6X_{16}$	0	0	0	$6X_{20}$
X_{14}	$2X_{12}$	$6X_{13}$	0	$6X_{15}$	0	$-6X_{17}$	0	$-2X_{19}$	$4X_{20}$	$-2X_{21}$
X_{15}	0	0	$-6X_{15}$	0	$6X_{13}$	$6X_{14}-6X_{18}$	$6X_{15}$	$6X_{20}$	0	0
X_{16}	0	0	0	$-6X_{13}$	0	0	$-6X_{16}$	0	0	$6X_{19}$
X_{17}	0	$6X_{16}$	$6X_{17}$	$-6X_{14}+6X_{18}$	0	0	$-6X_{17}$	0	$6X_{19}$	0
X_{18}	$2X_{12}$	0	0	$-6X_{15}$	$6X_{16}$	$6X_{17}$	0	$4X_{19}$	$-2X_{20}$	$-2X_{21}$
X_{19}	$6X_{16}$	0	$2X_{19}$	$-6X_{20}$	0	0	$-4X_{19}$	0	0	0
X_{20}	$6X_{13}$	0	$-4X_{20}$	0	0	$-6X_{19}$	$2X_{20}$	0	0	0
X_{21}	$12X_1-6X_{14}-6X_{18}$	$-6X_{20}$	$2X_{21}$	0	$-6X_{19}$	0	$2X_{21}$	0	0	0

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