A PROBLEM ON A RELIABILITY OF THE SYSTEM WITH REDUNDANCY (I)

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§ 1. Introduction.

In this paper there will be given the formulae for the reliability function, the mean time between failure (MTBF), the variance or the relative improvement in reliability of the system supported by many spare equipments such as machines, functional circuits, components ect., where the equipment can be replaced as soon as an equipment fails.

Suppose that the system consists of $m+n$ equipments independent to each other. We assume that the failure of the system means the failure of more than $n+1$ equipments. In other words when m group with n redundancy are given, m equipments are used indeed as functional unit (main operating) and n equipments are used as paralled or stand-by redundacy.

We shall treat in this paper two types of models. In model I, we assume that as soon as an equipment failed, failure is detected, where in model II the failure is detected only when the equipment is during the main operating, as the system is not equipped with the detecting functions on those which are in stand-by operating and therefore it will never be repaired if it fails during the stand-by operating. The model I will be treated in § 3 and the model II in § 4.

Throughout this paper we assume that reliability of the functions of detecting and switching is one. We also assume that the time until failure and that needed for repair are both subject to known distribution functions and all of those are statistically independent.

The motivation of this paper is to evaluate how system reliability can be improved by introducing redudant units of two different fashions. The auther believes that the formulae obtained in this paper are useful for the economical reliability designs of the system. The author is now preparing a tables of the reliabilities in order make usefully available at the disposal of those who are interested in the reliability design of electronic system. The auther wishes to discuss the methods of economical reliability design in a near for the coming paper.

§ 2. Structure of reliability probelm.

To characterize the model of reliability problem used in this paper, we may begin first discussing the distribution of times to failure for the equipment. We assume that the equipment is always in one of two possible states-good or failed.

The probability $R(t)$ that the equipment is good at time t is called the reliability function for the equipment. We assume that reliability $R(t)$ has the properties $R(0)=1$,

$R(\infty)=0$, $R(t)$ is twice differentiable, and $R'(t)<0$.

Suppose a complex equipment consisting of more than one stochastically failing components, and assume the equipment is good only if all components are good, then the reliability function $R(t)$ of the system may be given by

$$(2.1)\qquad R(t)=\prod_{j} r_j(t),$$

where $r_j(t)$ is the reliability function of the j-th part, as for the maintenance we assume that equipment is always in one of two possible states-repaired and not repaired.

The probability $M(t)$ that the equipment is repaired at time t after break down is called the maintainability function of the equipment. We assume that maintainability function $M(t)$ has the properties $M(0)=0$, $M(\infty)=1$, $M(t)$ is twice differentiable, and $M'(t)>0$.

We repeatedly use the easily verified relations

$$(2.2)\qquad R(t)=e^{-\int_0^t \lambda(x)dx},$$

$$(2.3)\qquad M(t)=1-e^{-\int_0^t \mu(x)dx},$$

where

$$(2.4)\qquad \lambda(x)=\frac{-R'(x)}{R(x)} \quad \text{for } R(x)>0,$$

$$(2.5)\qquad \mu(x)=\frac{M'(x)}{1-M(x)} \quad \text{for } M(x)<1,$$

and $\lambda(x)$ is called the failure rate. For the exponential failure and repair distribution, we have

$$(2.6)\qquad R(t)=e^{-\lambda t},$$

$$(2.7)\qquad M(t)=1-e^{-\mu t},$$

where
λ and μ are positive constants.

§ 3. Model I.

§ 3.1. Exponential failure and repair distribution.

Let E_i denotes the state that i equipments in the system of $m+n$ equipments are failure, and E_{n+1} means the failure of the system itself. The failure and repairment are transitions from one state to another. The probabilities of transition from E_i to E_{i+1} and E_{i-1} during time interval $(t, t+h)$ are assume to $\lambda_i h+0(h)$ and $\mu_i h+0(h)$, and those of the transition to other state are $0(h)$, where λ_i and μ_i are positive constants. This is nothing but a birth and death process as shown in Fig 1. Let $R_i(t)$ denotes the probability that the system starting from the state E_i remain within the states E_0, E_1, ...E_i until time t. Among $R_i(t)$ for $i=0,\cdots\cdots n$, we have the following relations

$$(3.1)\quad R_i(t)=e^{-(\lambda_i+\mu_i)t}+\int_0^t e^{-\lambda_i\tau}\mu_i e^{-\mu_i\tau}R_{i-1}(t-\tau)d\tau$$
$$-\iint_{\tau+\tau'\leq t} e^{-\lambda_i\tau}\mu_i e^{-\mu_i\tau}R'_{i-1}(\tau')R_i(t-\tau-\tau')d\tau d\tau', \qquad (i=1,2,\cdots n),$$
$$R_0(t)=e^{-\lambda_0 t}.$$

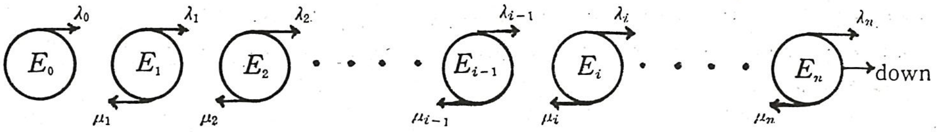

Fig. 1

The first term is the probability that the system remain in E_i until time t, the second term is the probability that the first transition of the system is to E_{i-1} and this occured between $(\tau, \tau+d\tau)$ and then the system remain $(E_0, E_1,\cdots E_{i-1})$ until t, the third term is that of the transition $E_i\to E_{i-1}\to E_i$ in the time interval $(\tau, \tau+d\tau)$ and $(\tau', \tau'+d\tau')$ and remaining within $(E_0, E_1, \cdots, E_i)$ afterward until t.

Let us put

$$(3.2)\quad \begin{cases} \varphi_i(s)=\int_0^\infty e^{-st}R_i(t)dt, & (i=0,1,\cdots,n),\\ \varphi_i^*(s)=\int_0^\infty e^{-st}dR_i(t), & (i=0,1,\cdots,n). \end{cases}$$

By applying the Laplace transformation to Eqs (3.1), we obtain

$$(3.3)\quad \begin{cases} \varphi_i(s)=\dfrac{1}{s+\lambda_i+\mu_i}+\dfrac{\mu_i}{s+\lambda_i+\mu_i}\varphi_{i-1}(s)-\dfrac{\mu_i}{s+\lambda_i+\mu_i}\varphi_{i-1}^*(s)\varphi_i(s), & (i=1,2,\cdots,n),\\ \varphi_0(s)=\dfrac{1}{s+\lambda_0}. \end{cases}$$

As we have the relation $\varphi_i^*(s)=-1+s\varphi_i(s)$, we get from (3.3)

$$(3.4)\quad \begin{cases} \varphi_i(s)=\dfrac{1+\mu_i\varphi_{i-1}(s)}{s+\lambda_i+\mu_i+\mu_i\varphi_{i-1}^*(s)}=\dfrac{1+\mu_i\varphi_{i-1}(s)}{s(1+\mu_i\varphi_{i-1}(s))+\lambda_i}\\ \qquad =\dfrac{Q_{i+1}(s)-Q_i(s)}{s}\cdot\dfrac{1}{Q_{i+1}(s)}, & (i=1,2,\cdots,n,)\\ \varphi_i^*(s)=\dfrac{-\lambda_i}{s+\lambda_i+\mu_i+\mu_i\varphi_{i-1}^*(s)}=\dfrac{-Q_i(s)}{Q_{i+1}(s)}, & (i=1,2,\cdots,n),\\ \varphi_0^*(s)=\dfrac{-\lambda_0}{s+\lambda_0}, \end{cases}$$

where

$$(3.5)\quad \begin{cases} Q_0(s)=1,\ \lambda_0Q_1(s)=(s+\lambda_0),\\ \lambda_iQ_{i+1}(s)=Q_i(s)(s+\lambda_i+\mu_i)-\mu_iQ_{i-1}(s), & (i=1,2,\cdots,n). \end{cases}$$

It is clear from (3.5) that the polynomial sequence $Q_{n+1}(s)$, $Q_n(s)$, $\cdots\cdots$ $Q_1(s)$, $Q_0(s)$ are modified Strum's function. Hence, from (3.5) and modified Strum's theorem, the polynomial $Q_i(s)$ has i distinct negative real roots. Moreover from (3.5) it is clear that $(Q_{i+1}(s)-Q_i(s))/s$ is a polynomial of degree i. Hence

$$\varphi_i(s)=\frac{Q_{i+1}(s)-Q_i(s)}{s}\cdot\frac{1}{Q_{i+1}(s)}=\frac{Q_{i+1}(s)-Q_i(s)}{s}\cdot\frac{\lambda_0\lambda_1\cdots\lambda_i}{\prod_{k=1}^{i+1}(s-s_k)}, \tag{3.6}$$

where $s_k(k=1, 2, \cdots i+1)$ are $i+1$ distinct negative real roots of $Q_{i+1}(s)$.

If we put $i=n$ in Eqs (3.1), we obtain the reliability function $R_n(t)$ of the system. The impotant special case is when

$$\begin{cases} \lambda_j=(m+n-j)\lambda, & (j=0,1,2,\cdots,n), \\ \mu_l=l\mu, & (l=1,2,\cdots,n), \end{cases} \tag{3.7}$$

which means that the system consists of identical equipments independent to each other and the reliability of stand-by operating and main operating are equal. In this case we have

$$\begin{cases} Q_0(s)=1, \\ (m+n)\lambda Q_1(s)=s+(m+n)\lambda, \\ (m+n-i)\lambda Q_{i+1}(s)=Q_i(s)[s+(m+n-i)\lambda+i\mu]-i\mu Q_{i-1}(s), & (i=1,2,\cdots,n). \end{cases} \tag{3.8}$$

By applying the inversion theorem to Eqs (3.6), we obtain the following theorem.

THEOREM 3.1. *The reliability function $R_n(t)$ of the model I is given by the following formula*

$$R_n(t)=\lambda_0\lambda_1\cdots\lambda_n\sum_{m=1}^{n+1}\frac{Q(s_m)}{P_m(s_m)}e^{s_mt}, \tag{3.9}$$

where

$$P(s)=\prod_{i=1}^{n+1}(s-s_i)=\lambda_0\lambda_1\cdots\lambda_nQ_{n+1}(s),$$

$$P_m(s)=\frac{\prod_{i=1}^{n+1}(s-s_i)}{(s-s_m)}=\frac{P(s)}{(s-s_m)},\quad Q(s)=\frac{Q_{n+1}(s)-Q_n(s)}{s},$$

and $Q_i(s)$ are given as (3.5) and given as (3.8) specially when (3.7) holds and $s_i(i=1,2,\cdots,n+1)$ are $n+1$ distinct negative real roots of $Q_{n+1}(s)$.

Next we calculate the MTBF T_{n+1} of the system.

Let us put

$$T_{i+1}=\int_0^\infty tf_i(t)dt,\quad (f_i(t)=F_i'(t)=(1-R_i(t))'),$$

then

(3.10)
$$T_{n+1}=\int_0^\infty tf_n(t)dt=-\int_0^\infty tR'_n(t)dt=\lim_{s\to 0}\varphi_n(s)$$
$$=\lim_{s\to 0}\left\{\frac{1}{\lambda_n}\left[\frac{\mu_n(Q_n(s)-Q_{n-1}(s))}{s}+Q_n(s)\right]\right\}\cdot\frac{1}{Q_{n+1}(s)}$$
$$=\frac{1}{\lambda_n}+\frac{\mu_n}{\lambda_n}T_n=\frac{1}{\lambda_n}+\sum_{l=1}^{n}\prod_{j=n-(l-1)}^{n}\mu_j/\prod_{j=n-l}^{n}\lambda_j,$$

where

(3.11)
$$\begin{cases} T_1=\lim\limits_{s\to 0}\dfrac{Q_1(s)-Q_0(s)}{s}=\dfrac{1}{\lambda_0}, \\ T_{i+1}=\lim\limits_{s\to 0}\dfrac{Q_{i+1}(s)-Q_i(s)}{s}=\lim\limits_{s\to 0}\dfrac{Q_{i+1}(s)-Q_i(s)}{sQ_{i+1}(s)}=\dfrac{1}{\lambda_i}+\dfrac{\mu_i}{\lambda_i}T_i, \qquad (i=1,2,\cdots,n). \end{cases}$$

From (3.5)

(3.12)
$$\lim_{s\to 0}Q_i(s)=1, \qquad (i=0,1,\cdots,n+1).$$

Specially when (3.7) holds then

(3.13)
$$T_{n+1}=\sum_{k=0}^{n}{}_{m+n}P_{n-k}\ {}_nP_k\lambda^{n-k}\mu^k/{}_{m+n}P_{n+1}\lambda^{n+1}$$
$$=(m-1)!\,n!\sum_{k=0}^{n}\frac{\mu^k}{(m+k)!(n-k)!\lambda^{k+1}}.$$

Our result may be sumarized in the following theorems.

THEOREM 3.2. *The MTBF T_{n+1} of the model I is given by*

(3.14)
$$T_{n+1}=\frac{1}{\lambda_n}+\frac{\mu_n}{\lambda_n}T_n=\frac{1}{\lambda_n}+\sum_{l=1}^{n}\prod_{j=n-(l-1)}^{n}\mu_j/\prod_{j=n-l}^{n}\lambda_j,$$

where

$$T_1=\frac{1}{\lambda_0},$$
$$T_{i+1}=\frac{1}{\lambda_i}+\frac{\mu_i}{\lambda_i}T_i, \qquad (i=1,2,\cdots,n),$$

and, specially when (3.7) is true, is given by

(3.15)
$$T_{n+1}=(m-1)!n!\sum_{k=0}^{n}\frac{\mu^k}{(m+k)!(n-k)!\lambda^{k+1}}.$$

THEOREM 3.3. *The relative improvement $\Delta T_{j+1,l+1}$ in MTBF of the system when number of spares increas from j to l is given by*

(3.16)
$$\Delta T_{j+1,l+1}=\frac{T_{l+1}}{T_{j+1}}=\frac{\lambda_j}{\lambda_l}\cdot\frac{1+\mu_l T_l}{1+\mu_j T_j}, \qquad (l=j+1,\cdots,n,j=0,\cdots,n-1),$$

and, specially when (3.7) *holds, is given by*

$$\Delta T_{j+1,l+1} = {}_lP_{l-j} \cdot \frac{\sum_{k=0}^{l} \frac{\mu^k}{(m+k)!(l-k)!\lambda^{k+1}}}{\sum_{k=0}^{j} \frac{\mu^k}{(m+k)!(j-k)!\lambda^{k+1}}}, \tag{3.17}$$

where T_{j+1} *are given as* (3.11).

In order to calculate the variance, let us put

$$E_{i+1}(T^2) = \int_0^\infty t^2 f_i(t)dt, \qquad (f_i(t) = F_i'(t) = (1-R_i(t))'),$$

then

$$E_{i+1}(T^2) = -\int_0^\infty t^2 R_i'(t)dt = -2\lim_{s\to 0}\varphi_i'(s) \tag{3.18}$$
$$= -2\lim_{s\to 0}\left\{\left[\frac{Q_{i+1}(s)-Q_i(s)}{s}\right]' \cdot \frac{1}{Q_{i+1}(s)} - \frac{Q_{i+1}(s)-Q_i(s)}{s} \cdot \frac{Q_{i+1}'(s)}{(Q_{i+1}(s))^2}\right\},$$
$$(i=0,1,\cdots,n).$$

Hence from $\lim_{s\to 0} Q_{i+1}(s)=1.$ and $\lim_{s\to 0} \frac{Q_{i+1}(s)-Q_i(s)}{s} = T_{i+1}$ for $i=0,1,\cdots,n$, we have

$$E_{i+1}(T^2) = -2[B_{i+1} - T_{i+1}A_{i+1}], \qquad (i=0,1,\cdots,n), \tag{3.19}$$

where

$$\begin{cases} A_0=0,\ A_1=\frac{1}{\lambda_0},\ A_{i+1}=\lim_{s\to 0} Q_{i+1}'(s) = \frac{[1+\mu_i(A_i-A_{i-1})+\lambda_i A_i]}{\lambda_i} & (i=1,2,\cdots,n), \\ B_1=\lim_{s\to 0}\left(\frac{Q_1(s)-Q_0(s)}{s}\right)'=0,\ B_{i+1}=\lim_{s\to 0}\left(\frac{Q_{i+1}(s)-Q_i(s)}{s}\right)' = \frac{\mu_i B_i + A_i}{\lambda_i}, & (i=1,2,\cdots,n). \end{cases} \tag{3.20}$$

Specially when (3.7) is true, we get

$$\begin{cases} A_0=0, \quad A_1=\frac{1}{(m+n)\lambda},\ A_{i+1}=\frac{[1+i\mu(A_i-A_{i-1})+(m+n-i)\lambda A_i]}{(m+n-i)\lambda}, & (i=1,2,\cdots,n), \\ B_1=0,\ B_{i+1}=\frac{i\mu B_i + A_i}{(m+n-i)\lambda}, & (i=1,2,\cdots,n). \end{cases} \tag{3.21}$$

Thus we obtain the following theorem.

THEOREM 3.4. *Variance* V_{n+1} *of the model I is given by the following formula*

$$V_{n+1} = T_{n+1}(2A_{n+1} - T_{n+1}) - 2B_{n+1} \tag{3.22}$$

where A_{i+1}, B_{i+1} *and* T_{i+1} *are given as* (3.20) *and* (3.14), *specially when* (3.7) *holds they are given as* (3.21).

EXAMPLE 1. *In case when* $m=n=1$, *since then* $\lambda_0=2\lambda$, $\lambda_1=\lambda$, *and* $\mu_1=\mu$,

$$Q_0(s)=1, \qquad Q_1(s)=\frac{(s+2\lambda)}{2\lambda},$$

$$Q_2(s)=\frac{1}{2\lambda^2}s^2+\frac{3\lambda+\mu}{2\lambda^2}\cdot s+1, \quad \frac{Q_2(s)-Q_1(s)}{s}=\frac{s+2\lambda+\mu}{2\lambda^2}.$$

The roots of $Q_2(s)$ *are*

$$s_1=\frac{-(3\lambda+\mu)+\sqrt{\lambda^2+6\lambda\mu+\mu^2}}{2}, \quad s_2=\frac{-(3\lambda+\mu)-\sqrt{\lambda^2+6\lambda\mu+\mu^2}}{2}.$$

Hence

$$R_1(t)=\frac{s_1+2\lambda+\mu}{s_1-s_2}e^{s_1t}+\frac{s_2+2\lambda+\mu}{s_2-s_1}e^{s_2t}$$

$$=e^{-\frac{(3\lambda+\mu)}{2}t}\left[\cosh\frac{\sqrt{\lambda^2+6\lambda\mu+\mu^2}}{2}t+\frac{\lambda+\mu}{\sqrt{\lambda^2+6\lambda\mu+\mu^2}}\sinh\frac{\sqrt{\lambda^2+6\lambda\mu+\mu^2}}{2}t\right].$$

On the other hand,

$$T_2=\lim_{s\to 0}\varphi_1(s)=\frac{2\lambda+\mu}{2\lambda^2},$$

$$\Delta T_{12}=1+\frac{\mu}{2\lambda},$$

$$V_2=T_2(2A_2-T_2)-2B_2=\frac{\mu^2}{4\lambda^4}+\frac{3\mu}{2\lambda^3}+\frac{1}{\lambda^2}.$$

EXAMPLE 2. *In case when* $m=2$ *and* $n=1$, *since then* $\lambda_0=3\lambda$, $\lambda_1=2\lambda$ *and* $\mu_1=\mu$,

$$Q_0(s)=1, \qquad Q_1(s)=\frac{s+3\lambda}{3\lambda},$$

$$Q_2(s)=\frac{1}{6\lambda^2}\cdot[s^2+(5\lambda+\mu)s+6\lambda^2],$$

$$\frac{Q_2(s)-Q_1(s)}{s}=\frac{s+3\lambda+\mu}{6\lambda^2}.$$

The roots of $Q_2(s)$ are

$$s_1=\frac{-(5\lambda+\mu)+\sqrt{\lambda^2+10\lambda\mu+\mu^2}}{2}, \quad s_2=\frac{-(5\lambda+\mu)-\sqrt{\lambda^2+10\lambda\mu+\mu^2}}{2}.$$

Hence

$$R_1(t)=\frac{s_1+3\lambda+\mu}{s_1-s_2}\cdot e^{s_1t}+\frac{s_2+3\lambda+\mu}{s_2-s_1}e^{s_2t}$$

$$=e^{-\frac{(5\lambda+\mu)}{2}t}\left\{\cosh\frac{\sqrt{\lambda^2+10\lambda\mu+\mu^2}}{2}t+\frac{\lambda+\mu}{\sqrt{\lambda^2+10\lambda\mu+\mu^2}}\cdot\sinh\frac{\sqrt{\lambda^2+10\lambda\mu+\mu^2}}{2}t\right\}.$$

And also we get

$$T_2=\lim_{s\to 0}\varphi(s)=\frac{3\lambda+\mu}{6\lambda^2},$$

$$\Delta T_{12}=\frac{3\lambda+\mu}{3\lambda},$$

$$V_2=T_2(2A_2-T_2)-2B_2=\frac{\mu^2}{36\lambda^4}+\frac{5\mu}{18\lambda^3}+\frac{1}{4\lambda^2}.$$

§ 3.2. General failure and repair distribution.

In the previous § 3.1 we considered the case when the distributions $F(t)$ $(=1-R(t))$, $M(t)$, are both exponential. Here we consider the case when

$$(3.23)\qquad \begin{cases} F(t)=1-R(t)=1-e^{-\int_0^t\lambda(x)dx}, \\ M(t)=1-e^{-\int_0^t\mu(x)dx}. \end{cases}$$

From the method as in § 3.1

$$(3.24)\qquad R_i(t)=e^{-\int_0^t[\lambda_i(x)+\mu_i(x)]dx}+\int_0^t\mu_i(\tau)e^{-\int_0^\tau[\lambda_i(x)+\mu_i(x)]dx}R_{i-1}(t-\tau)d\tau$$

$$-\iint_{\tau+\tau'\leq t}\mu_i(\tau)e^{-\int_0^\tau[\lambda_i(x)+\mu_i(x)]dx}R'_{i-1}(\tau')R_i(t-\tau-\tau')d\tau d\tau', \qquad (i=1,2,\cdots,n),$$

$$R_0(t)=e^{-\int_0^t\lambda_i(x)dx}.$$

Let us put

$$\varphi_i(s)=\int_0^\infty e^{-st}R_i(t)dt, \qquad (i=0,1,\cdots,n),$$

$$\varphi_i^*(s)=\int_0^\infty e^{-st}dR_i(t), \qquad (i=0,1,\cdots,n),$$

$${}^1\varphi(s,\lambda_i,\mu_i)=\int_0^\infty e^{-st-\int_0^t[\lambda_i(x)+\mu_i(x)]dx}dt, \qquad (i=0,1,\cdots,n),$$

$${}^2\varphi(s,\lambda_i,\mu_i)=\int_0^\infty \mu_i(t)e^{-st-\int_0^t[\lambda_i(x)+\mu_i(x)]dx}dt, \qquad (i=0,1,\cdots,n).$$

where $\mu_0=0$

By applying the Laplace transformation to Eqs (3.24)

$$(3.25)\qquad \varphi_i(s)={}^1\varphi(s,\lambda_i,\mu_i)+{}^2\varphi(s,\lambda_i,\mu_i)\varphi_{i-1}(s)-{}^2\varphi(s,\lambda_i,\mu_i)\varphi_{i-1}^*(s)\varphi_i(s), \qquad (i=1,2,\cdots,n),$$

$$\varphi_0(s)=\int_0^\infty e^{-st-\int_0^t\lambda_0(x)dx}dt.$$

Hence from (3.25)

$$(3.26)\qquad \varphi_i(s)=\frac{{}^1\varphi(s,\lambda_i,\mu_i)+{}^2\varphi(s,\lambda_i,\mu_i)\varphi_{i-1}(s)}{1+{}^2\varphi(s,\lambda_i,\mu_i)\varphi^*_{i-1}(s)}$$

$$=\frac{{}^1\varphi(s,\lambda_i,\mu_i)+{}^2\varphi(s,\lambda_i,\mu_i)\varphi_{i-1}(s)}{s^2\varphi(s,\lambda_i,\mu_i)\varphi_{i-1}(s)+1-{}^2\varphi(s,\lambda_i,\mu_i)},\qquad (i=1,2,\cdots,n),$$

$$(3.27)\qquad \begin{cases} \varphi^*_i(s)=\dfrac{s^1\varphi(s,\lambda_i,\mu_i)+{}^2\varphi(s,\lambda_i,\mu_i)-1}{1+{}^2\varphi(s,\lambda_i,\mu_i)\varphi^*_{i-1}(s)}, & (i=1,2,\cdots,n),\\ \varphi^*_0(s)=-\displaystyle\int_0^\infty \lambda_0(t)e^{-st-\int_0^t\lambda_0(x)dx}dt. & \end{cases}$$

It is difficult to obtain the exact formulae for ${}^1\varphi(s,\lambda_i,\mu_i)$ and ${}^2\varphi(s,\lambda_i,\mu_i)$, but we shall obtain the approximate formulae of these by inserting the appropriate functions (e.g. trigonometric series) into $e^{-\int_0^t\lambda_i(x)dx}$ and $e^{-\int_0^t\mu_i(x)dx}$.
We omit the details. To obtain the MTBF, the variance and the relative improvement in reliability of the system will be left to the reader.

§ 4. Model II.

§ 4.1. Exponential failure and repair distribution.

In this section we shall treat Model II. In this model there are two kinds of failures, as explained in the Introduction, repairable failure and non repairable one. Let us consider a system with $m+n$ equipments and the system is good only if at least m equipments are good.

Let us E_i^{m+n-j} denotes the state that i equipments in the system of $m+n$ equipments are in the repairable failure and $m+n-j$ equipments are operating, and when the system reaches one of the states $E_h^{m-1}(h=1,2,\cdots,n+1)$ it fails. The transitions from one state to another are caused by either the occurance of failure or the accomplishment of repair. The probabilities of transitions from E_i^{m+n-j} to $E_i^{m+n-(j+1)}$, $E_{i+1}^{m+n-(j+1)}$ and $E_{i-1}^{m+n-(j-1)}$ during time interval $(t,\ t+h)$ are assumed to given by $(\lambda_j-\lambda_n)h+0(h)$, $\lambda_n h+0(h)$ and $\mu_i h+0(h)$ respectively, and these of the transitions to other state are $0(h)$ (Fig 2), where λ_j, λ_n and μ_i are the positive constants. Let $R_i^{m+n-j}(t)$ denotes the probability that the system starting from E_i^{m+n-j} remain with the states E_k^{m+n-l} $(l=k+j-i,\ k+j-i+1\cdots j;\ k=0,1,\cdots i)$ at time t. Among $R_i^{m+n-j}(t)$ $(j=i,\ i+1,\cdots n;\ i=1,2,\cdots n)$ we have the following relations

$$(4.1)\qquad \begin{cases} R_i^{m+n-j}(t)=e^{-(\lambda_j+\mu_i)t}+\displaystyle\int_0^t \mu_i e^{-(\lambda_j+\mu_i)\tau}R_{i-1}^{m+n-(j-1)}(t-\tau)d\tau \\ \qquad +\displaystyle\iint\limits_{\tau+\tau'\le t}\mu_i e^{-(\lambda_j+\mu_i)\tau} \\ \qquad \cdot\left[\displaystyle\sum_{k=1}^{i} g_{i-1k-1}^{m+n-(j-1)}(\tau')\left(p_{k-1k}^{j-1j}R_k^{m+n-j}(t-\tau-\tau')+p_{k-1k-1}^{j-1j}R_{k-1}^{m+n-j}(t-\tau-\tau)\right)\right]d\tau d\tau', \\ \qquad\qquad (j=i,i+1,\cdots,n;i=1,2,\cdots n), \\ R_0^{m+n-j}(t)=e^{-\lambda_j t}, \qquad (j=0,1,\cdots,n), \end{cases}$$

where p_{ik}^{jl} is the conditional transition probability from E_i^{m+n-j} to E_k^{m+n-l} $(k=i,\ i+1;$

$l=j+1$; $j=i, i+1. \cdots n-1$; $i=0,1,\cdots n-1$) when the failure occured at the state E_i^{m+n-j} and $g_{ih}^{m+n-j}(t)dt$ is the transition probability that the system starting from E_i^{m+n-j} remain with the state E_k^{m+n-l} ($l=k+j-i,\ k+j-i+1\cdots,\ j$; $k=0, 1, \cdots i$) and are transition from E_h^{m+n-j} to $E_h^{m+n-(j+1)}$ or $E_{h+1}^{m+n-(j+1)}$ ($h=0,1,\cdots, i$) during time interval $(t, t+dt)$.

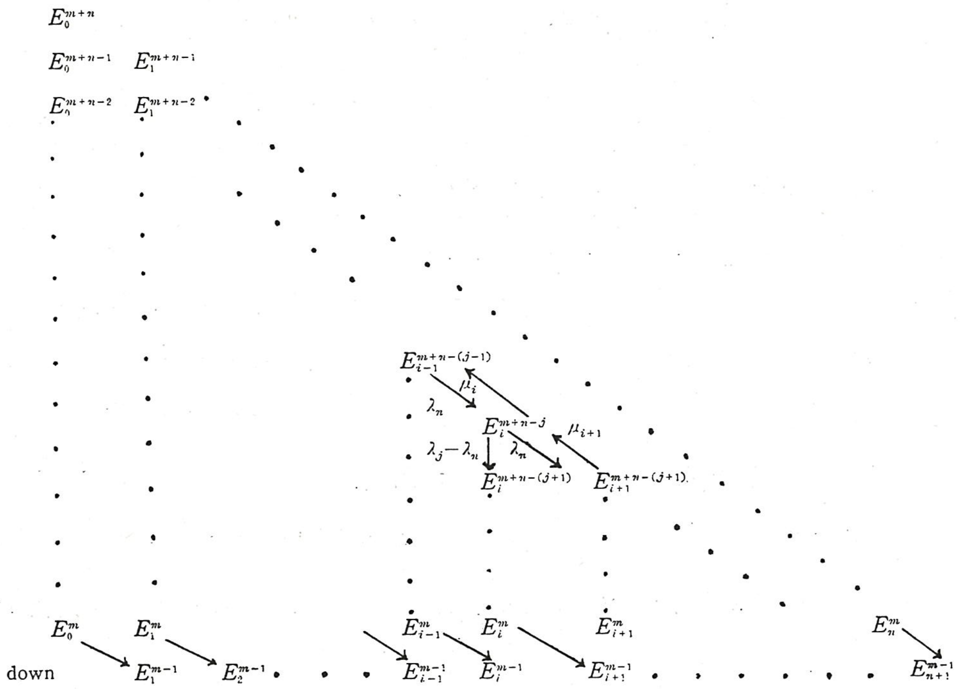

Fig. 2

Among $g_{ih}^{m+n-j}(t)$ ($j=i, i+1, \cdots n$; $i=1,2,\cdots n$; $h=0,1,2,\cdots i$) we have the following relations.

$$
(4.2)\quad
\begin{cases}
g_{ih}^{m+n-j}(t)=\displaystyle\iint_{\tau+\tau'\leq t} \mu_i e^{-(\lambda_j+\mu_i)\tau}\Big[g_{i-1i-1}^{m+n-(j-1)}(\tau')p_{i-1i}^{j-1j}g_{ih}^{m+n-j}(t-\tau-\tau') \\
\quad +\displaystyle\sum_{l=h}^{i-1}\left(p_{ll}^{j-1j}g_{i-1l}^{m+n-(j-1)}(\tau')+p_{l-1l}^{j-1j}g_{i-1l-1}^{m+n-(j-1)}(\tau')\right)g_{lh}^{m+n-j}(t-\tau-\tau')\Big]d\tau d\tau', \\
\qquad\qquad (j=i,\ i+1,\cdots,n;\ i=1,2,\cdots n; h=0,1,\cdots i-1), \\
g_{ii}^{m+n-j}(t)=\lambda_j e^{-(\lambda_j+\mu_i)t}+\displaystyle\iint_{\tau+\tau'\leq t} \mu_i e^{-(\lambda_j+\mu_i)\tau}g_{i-1i-1}^{m+n-(j-1)}(\tau')p_{i-1i}^{j-1j}g_{ii}^{m+n-j}(t-\tau-\tau')d\tau d\tau', \\
\qquad\qquad (j=i,\ i+1,\cdots n;\ i=1,2,\cdots n), \\
g_{00}^{m+n-j}(t)=\lambda_j e^{-\lambda_j t}, \qquad (j=0,1,\cdots n).
\end{cases}
$$

Let us define

$$
(4.3)\quad
\begin{cases}
\varphi_i^{m+n-j}(s)=\displaystyle\int_0^\infty e^{-st}R_i^{m+n-j}(t)dt, & (j=i,\ i+1,\cdots n;\ i=0,1,\cdots n) \\
\varphi_i^{*m+n-j}(s)=\displaystyle\int_0^\infty e^{-st}dR_i^{m+n-j}(t), & (j=i,\ i+1,\cdots n;\ i=0,1,\cdots n) \\
\psi_{ih}^{m+n-j}(s)=\displaystyle\int_0^\infty e^{-st}g_{ih}^{m+n-j}(t)dt, & (j=i,\ i+1,\cdots n;\ i=0,1,\cdots n;\ h=0,1,\cdots i)
\end{cases}
$$

$\varphi_i^{m+n-j}(s)$ and $\psi_{ih}^{m+n-j}(s)$ are the Laplace transform of (4.1) and (4.2), and are given by

(4.4)
$$\begin{cases} \varphi_i^{m+n-j}(s)=\dfrac{1}{s+\lambda_j+\mu_i}+\dfrac{\mu_i}{s+\lambda_j+\mu_i}\varphi_{i-1}^{m+n-(j-1)}(s) \\ \qquad +\dfrac{\mu_i}{s+\lambda_j+\mu_i}\sum_{k=1}^{i}\psi_{i-1k-1}^{m+n-(j-1)}(s)(p_{k-1k}^{j-1j}\varphi_k^{m+n-j}(s)+p_{k-1k-1}^{j-1j}\varphi_{k-1}^{m+n-j}(s)), \\ \qquad\qquad (j=i,\ i+1,\ \cdots n;\ i=1,2,\cdots n), \\ \varphi_0^{m+n-j}(s)=\dfrac{1}{s+\lambda_j}, \qquad (j=0,1,\ \cdots,n), \\ \psi_{ih}^{m+n-j}(s)=\dfrac{\mu_i}{s+\lambda_j+\mu_i}[\psi_{i-1i-1}^{m+n-(j-1)}(s)p_{i-1i}^{j-1j}\psi_{ih}^{m+n-j}(s) \\ \qquad +\sum_{l=h}^{i-1}(p_{ll}^{j-1j}\psi_{i-1l}^{m+n-(j-1)}(s)+p_{l-1l}^{j-1j}\psi_{i-1l-1}^{m+n-(j-1)}(s))\psi_{lh}^{m+n-j}(s)], \\ \qquad\qquad (j=i,\ i+1,\cdots n;\ i=1.2,\cdots n;\ h=0,1,\cdots i-1), \\ \psi_{ii}^{m+n-j}(s)=\dfrac{\lambda_j}{s+\lambda_j+\mu_i}+\dfrac{\mu_i}{s+\lambda_j+\mu_i}\psi_{i-1i-1}^{m+n-(j-1)}(s)p_{i-1i}^{j-1j}\psi_{ii}^{m+n-j}(s), \\ \qquad\qquad (j=i,\ i+1,\cdots n;\ i=1,2,\cdots n), \\ \psi_{00}^{m+n-j}(s)=\dfrac{\lambda_j}{s+\lambda_j}, \qquad (j=0,1,\cdots n). \end{cases}$$

As we have relation $\varphi_i^{*m+n-j}(s)=-1+s\varphi_i^{m+n-j}(s)$, we get from (4.4)

(4.5)
$$\begin{cases} \varphi_i^{m+n-j}(s)=\dfrac{1+\mu_i\varphi_{i-1}^{m+n-(j-1)}(s)+\mu_i\sum_{k=0}^{i-1}(\psi_{i-1k}^{m+n-(j-1)}(s)p_{kk}^{j-1j}+\psi_{i-1k-1}^{m+n-(j-1)}(s)p_{k-1k}^{j-1j})\varphi_k^{m+n-j}(s)}{s+\lambda_j+\mu_i-\mu_i p_{i-1i}^{j-1j}\psi_{i-1i-1}^{m+n-(j-1)}} \\ \qquad =\dfrac{Q_{i+1}^{j+1}(s)-P_{i+1}^{j+1}(s)}{s}\cdot\dfrac{1}{Q_{i+1}^{j+1}(s)}, \qquad (j=i,\ i+1,\cdots n;\ i=1,2,\cdots n), \\ \varphi_0^{m+n-j}(s)=\dfrac{1}{\lambda_j Q_1^{j+1}(s)}, \qquad (j=0,1,\cdots n), \\ \varphi_i^{*m+n-j}(s)=\{-\lambda_j+\mu_i[\varphi_{i-1}^{*m+n-(j-1)}(s)+\sum_{k=0}^{i-1}(\psi_{i-1k}^{m+n-(j-1)}(s)p_{kk}^{j-1j}+\psi_{i-1k-1}^{m+n-(j-1)}(s)p_{k-1k}^{j-1j})\varphi_k^{*m+n-j}(s) \\ \qquad +\sum_{k=0}^{i}(\psi_{i-1k}^{m+n-(j-1)}(s)p_{kk}^{j-1j}+\psi_{i-1k-1}^{m+n-(j-1)}(s)p_{k-1k}^{j-1j})]\}/s+\lambda_j+\mu_i-\mu_i p_{i-1i}^{j-1j}\psi_{i-1i-1}^{m+n-(j-1)}(s) \\ \qquad =\dfrac{-P_{i+1}^{j+1}(s)}{Q_{i+1}^{j+1}(s)}, \qquad (j=i,\ i+1,\cdots n;\ i=1,2,\cdots n), \\ \varphi_0^{*m+n-j}(s)=-\dfrac{1}{Q_1^{j+1}(s)}, \qquad (j=0,1,\cdots n), \\ \psi_{ih}^{m+n-j}(s)=\dfrac{\mu_i\sum_{l=h}^{i-1}\psi_{lh}^{m+n-j}(s)(p_{ll}^{j-1j}\psi_{i-1l}^{m+n-(j-1)}(s)+p_{l-1l}^{j-1j}\psi_{i-1l-1}^{m+n-(j-1)}(s)}{s+\lambda_j+\mu_i-\mu_i p_{i-1i}^{j-1j}\psi_{i-1i-1}^{m+n-(j-1)}(s)}, \\ \qquad\qquad (j=i,\ i+1,\cdots n;\ i=1,2,\cdots n;\ h=0,1,\cdots i-1), \\ \psi_{ii}^{m+n-j}(s)=\dfrac{\lambda_j}{s+\lambda_j+\mu_i-\mu_i p_{i-1i}^{j-1j}\psi_{i-1i-1}^{m+n-(j-1)}(s)}, \qquad (j=i,\ i+1,\cdots n;\ i=1,2,\cdots n), \\ \psi_{00}^{m+n-j}(s)=\dfrac{1}{Q_1^{j+1}(s)}, \qquad (j=0,1,\cdots n). \end{cases}$$

where

$$
(4.6)\quad \begin{cases}
p_{-10}^{j-1j}=0,\ Q_0^{j+1}(s)=P_1^{j+1}(s)=1,\ \lambda_j Q_1^{j+1}(s)=(s+\lambda_j), & (j=0,1,\cdots n),\\
\psi_{i-1i}^{m+n-j}(s)=0,\ \lim_{s\to 0}\psi_{ik}^{m+n-j}(s)=L_{ik}^{j}, & (j=i,\ i+1\cdots n;\ i=1,2,\cdots n;\ k=0,1,\cdots i,),\\
\lambda_j Q_{i+1}^{j+1}(s)=L_{ii}^{j}(s+\lambda_j+\mu_i-\mu_i p_{i-1i}^{j-1j}\psi_{i-1i-1}^{m+n-(j-1)}(s))Q_i^j(s)\prod_{k=0}^{i}Q_k^{j+1}(s), & \\
 & (j=i,\ i+1,\cdots n;\ i=1,2,\cdots n),\\
\lambda_j P_{i+1}^{j+1}(s)=L_{ii}^{j}\{[\lambda_j-\mu_i\sum_{k=0}^{i}(\psi_{i-1k}^{m+n-(j-1)}(s)p_{kk}^{j-1j}+\psi_{i-1k-1}^{m+n-(j-1)}(s)p_{k-1k}^{j-1j}]Q_i^j(s)\prod_{k=0}^{i}Q_k^{i+1}(s) & \\
\quad +\mu_i[P_i^j(s)\prod_{k=0}^{i}Q_k^{j+1}(s)+Q_i^j(s)\sum_{l=0}^{i-1}P_{l+1}^{j+1}(s)\prod_{k\neq l+1}^{i}Q_k^{j+1}(s)(\psi_{i-1l}^{m+n-(j-1)}(s)p_{ll}^{j-1j}+\psi_{i-1l-1}^{m+n-(j-1)}(s)p_{l-1l}^{j-1j})]\}, & \\
 & (j=i,\ i+1,\cdots n;\ i=1,2,\cdots n).
\end{cases}
$$

The reliability function can be obtained by putting $i=j=n$ in (4.1). The case when the following hold is impotant.

$$
(4.7)\quad \begin{cases}
\lambda_j=(m+n-j)\lambda,\ \mu_l=l\mu, & (j=0,1,\cdots n;\ l=1,2,\cdots,n),\\
p_{k-1k-1}^{j-1j}=\dfrac{(n-j+1)\lambda}{(m+n-j+1)\lambda}=\dfrac{n-j+1}{m+n-j+1}, & (j=k,\ k+1,\cdots,n;\ k=1,2,\cdots n),\\
p_{k-1k}^{j-1j}=\dfrac{m\lambda}{(m+n-j+1)\lambda}=\dfrac{m}{m+n-j+1}, & (j=k,\ k+1,\cdots,n;\ k=1,2,\cdots n).
\end{cases}
$$

This is the case that the system consists of equipment identical and independent to each other and the reliabilities in stand-by operating and main operating are equal. In this case we have

$$
(4.8)\quad \begin{cases}
p_{-10}^{j-1j}=0,\ Q_0^{j+1}(s)=P_1^{j+1}(s)=1,\ (m+n-j)\lambda Q_1^{j+1}(s)=[s+(m+n-j)\lambda], & \\
 & (j=0,1,\cdots n),\\
\psi_{i-1i}^{m+n-j}(s)=0,\ \lim_{s\to 0}\psi_{ik}^{m+n-j}(s)=L_{ik}^{j}, & (j=i,\ i+1,\cdots n;\ i=1,2,\cdots n;\ k=0\ 1,\cdots,i),\\
[(m+n-j)(m+n-j+1)\lambda]Q_{i+1}^{j+1}(s) & \\
\quad =L_{ii}^{j}\{(m+n-j+1)[s+(m+n-j)\lambda]+i\mu(m+n-j+1-mL_{i-1i-1}^{j-1})\}Q_i^j(s)\prod_{k=0}^{i}Q_k^{j+1}(s), & \\
 & (j=i,\ i+1,\cdots n;\ i=1,2,\cdots n),\\
[(m+n-j)(m+n-j+1)\lambda]P_{i+1}^{j+1}(s) & \\
\quad =L_{ii}^{j}\{[(m+n-j)(m+n-j+1)\lambda-i\mu\sum_{k=0}^{i}((n-j+1)\psi_{i-1k}^{m+n-(j+1)}(s) & \\
\quad +m\psi_{i-1k-1}^{m+n-(j-1)}(s))]Q_i^j(s)\prod_{k=0}^{i}Q_k^{j+1}+[i\mu(m+n-j+1)P_i^j(s)\prod_{k=0}^{i}Q_k^{j+1}(s) & \\
\quad +Q_i^j(s)\sum_{l=0}^{i-1}P_{l+1}^{j+1}(s)\prod_{k\neq l+1}^{i}Q_k^{j+1}(s) & \\
\qquad \cdot((n-j+1)\psi_{i-1l}^{m+n-(j-1)}+m\psi_{i-1k-1}^{m+n-(j-1)}(s))]\}, & \\
 & (j=i,\ i+1,\cdots,n;i=1,2,\cdots n).
\end{cases}
$$

Consequently we have proved the following theorem.

THEOREM 4.1 *The Laplace transformation $\varphi_n^m(s)$ of the reliability function in the model II is given by the following formula*

$$\varphi_n^m(s)=\frac{Q_{n+1}^{n+1}(s)-P_{n+1}^{n+1}(s)}{s}\cdot\frac{1}{Q_{n+1}^{n+1}(s)}$$

where $Q_{n+1}^{n+1}(s)$ and $P_{n+1}^{n+1}(s)$ are given as (4.6) and specially when (4.7) holds true they are given as (4.8). The reliability function can be obtained by the inverse Laplace transformation of the above functions.

Next we calculate the MTBF T_{n+1}^{n+1} of the system. After some simple calculation we may derive the following identities

$$(4.9)\quad \begin{cases} \lim\limits_{s\to 0} Q_{i+1}^{j+1}(s)=1, & (j=i,\ i+1,\cdots n;\ i=1,2,\cdots n),\\ \lim\limits_{s\to 0} P_{i+1}^{j+1}(s)=1, & (j=i,\ i+1,\cdots n;\ i=1,2,\cdots n). \end{cases}$$

Let us here put

$$T_{i+1}^{j+1}=\lim_{s\to 0}\varphi_i^{m+n-j}(s),$$

then

$$(4.10)\quad \begin{cases} T_{i+1}^{j+1}=\lim\limits_{s\to 0}\left[\dfrac{Q_{i+1}^{j+1}(s)-P_{i+1}^{j+1}(s)}{s}\cdot\dfrac{1}{Q_{i+1}^{j+1}(s)}\right]=\lim\limits_{s\to 0}\dfrac{Q_{i+1}^{j+1}(s)-P_{i+1}^{j+1}(s)}{s}\\ =\lim\limits_{s\to 0}\dfrac{L_{ii}^j}{\lambda_j s}\{[s+\mu_i-\mu_i p_{i-1i}^{j-1j}\psi_{i-1i-1}^{m+n-(j-1)}(s)+\mu_i\sum\limits_{l=0}^{i}(\psi_{i-1l}^{m+n-(j-1)}(s)p_{ll}^{j-1j}\\ +\psi_{i-1l-1}^{m+n-(j-1)}(s)p_{l-1l}^{j-1j})]Q_i^j(s)\prod\limits_{k=0}^{i}Q_k^{j+1}(s)-\mu_i[P_i^j(s)\prod\limits_{k=0}^{i}Q_k^{j+1}(s)+Q_i^j(s)\sum\limits_{l=0}^{i-1}P_{l+1}^{j+1}(s)\prod\limits_{k\neq l+1}^{i}Q_k^{j+1}(s)\\ \cdot(\psi_{i-1l}^{m+n-(j-1)}(s)p_{ll}^{j-1j}+\psi_{i-1l-1}^{m+n-(j-1)}p_{l-1l}^{j-1j})]\}\\ =\lim\limits_{s\to 0}\dfrac{L_{ii}^j}{\lambda_j}\left[Q_i^j\prod\limits_{k=0}^{i}Q_k^{j+1}(s)+\mu_i\prod\limits_{k=0}^{i}Q_k^{j+1}(s)\left(\dfrac{Q_i^j(s)-P_i^j(s)}{s}\right)\right.\\ \left.+\mu_iQ_i^j(s)\sum\limits_{l=0}^{i-1}\left(\dfrac{Q_{l+1}^{j+1}(s)-P_{l+1}^{j+1}(s)}{s}\right)\prod\limits_{k\neq l+1}^{i}Q_k^{j+1}(s)(\psi_{i-1l}^{m+n-(j-1)}(s)p_{ll}^{j-1j}+\psi_{i-1l-1}^{m+n-(j-1)}(s)p_{l-1l}^{j-1j})\right]\\ =\dfrac{L_{ii}^j[1+\mu_iT_i^j+\mu_i\sum\limits_{l=0}^{i-1}T_{l+1}^{j+1}(p_{ll}^{j-1j}L_{i-1l}^{j-1}+p_{l-1l}^{j-1j}L_{i-1l-1}^{j-1})]}{\lambda_j} \end{cases}$$

where

$$T_1^{j+1}=\frac{1}{\lambda_j},\ L_{00}^j=1, \qquad (j=0,1,\cdots n),$$

$$T_i^{j+1}(s)=\lim_{s\to 0}\frac{Q_i^{j+1}(s)-P_i^{j+1}(s)}{s},\ L_{ii}^j=\lim_{s\to 0}\psi_{ii}^{m+n-j}(s)=\frac{\lambda_j}{\lambda_j+\mu_i-\mu_i p_{i-1i}^{j-1j}L_{i-1i-1}^{j-1}},\ L_{i-1-1}^{j-1}=0,$$

$$L_{ik}^j=\lim_{s\to 0}\psi_{ik}^{m+n-j}(s), \qquad (j=i,\ i+1,\cdots,n;\ i=1,2,\cdots n;\ k=0,1,\cdots,i-1).$$

Specially when (4.7) holds, we have

$$(4.11)\quad \begin{cases} T_1^{j+1}=\dfrac{1}{(m+n-j)\lambda}, & (j=0,1,\cdots n), \\ T_{i+1}^{j+1}=\dfrac{L_{ii}^j\{(m+n-j+1)(1+i\mu T_i^j)+i\mu\sum\limits_{l=0}^{i-1}T_{l+1}^{j+1}[(n-j+1)L_{i-1l}^{j-1}+mL_{i-1l-1}^{j-1}]\}}{(m+n-j)(m+n-j+1)\lambda}, \\ & (j=i,\ i+1,\cdots n;\ i=2,3,\cdots n). \end{cases}$$

Hence we obtain the following theorems.

THEOREM 4.2. *The MTBF T_{n+1}^{n+1} of the model II is expressed by*

$$(4.12)\qquad T_{n+1}^{n+1}=\frac{L_{nn}^n[1+\mu_nT_n^n+\mu_n\sum\limits_{l=0}^{n-1}T_{l+1}^{n+1}(p_{ll}^{n-1n}L_{n-1l}^{n-1}+p_{l-1l}^{n-1n}L_{n-1l-1}^{n-1})]}{\lambda_n},$$

where T_1^{j+1}, L_{ii}^j and T_{i+1}^{j+1} are given as (4.10), and given by

$$(4.13)\qquad T_{n+1}^{n+1}=\frac{L_{nn}^j[(m+1)(1+n\mu T_n^n)+n\mu\sum\limits_{l=0}^{n-1}T_{l+1}^{n+1}(L_{n-1l}^{n-1}+mL_{n-1l-1}^{n-1})]}{m(m+1)\lambda},$$

where T_1^{j+1}, T_2^{j+1} and T_{i+1}^{j+1} are given as (4.11), when (4.7) is true.

THEOREM 4.3. *The relative improvement ΔT_{j+1l+1} in MTBF of the system when the spare increase from j to l is equal to*

$$(4.14)\qquad \Delta T_{j+1l+1}=\frac{T_{l+1}^{l+1}}{T_{j+1}^{j+1}}=\frac{\lambda_jL_{ll}^l[1+\mu_lT_l^l+\mu_l\sum\limits_{k=0}^{l-1}T_{k+1}^{l+1}(p_{kk}^{l-1l}L_{l-1k}^{l-1}+p_{k-1k}^{l-1l}L_{l-1k-1}^{l-1})]}{\lambda_lL_{jj}^j[1+\mu_jT_j^j+\mu_j\sum\limits_{l=0}^{j-1}T_{l+1}^{j+1}(p_{ll}^{j-1j}L_{j-1l}^{j-1}+p_{l-1l}^{j-1j}L_{j-1l-1}^{j-1})]},$$

where T_{i+1}^{j+1} is given (4.10), and equal to

$$(4.15)\qquad \Delta T_{j+1l+1}=\frac{T_{l+1}^{l+1}}{T_{j+1}^{j+1}}=\frac{L_{ll}^l[(m+1)(1+l\mu T_l^l)+l\mu\sum\limits_{k=0}^{l-1}T_{k+1}^{l+1}(L_{l-1k}^{l-1}+mL_{l-1k-1}^{l-1})]}{L_{jj}^j[(m+1)(1+j\mu T_j^j)+j\mu\sum\limits_{l=0}^{j-1}T_{l+1}^{j+1}(L_{j-1l}^{j-1}+mL_{j-1l-1}^{j-1})]},$$

where T_1^{j+1}, T_2^{j+1} and T_{i+1}^{j+1} are given as (4.10), when (4.7) holds true for $n=l$ and $n=j$.

Now we calculate the variance.
Let us put

$$E_{i+1}^{j+1}(T^2)=\int_0^\infty t^2f_i^{m+n-j}(t)dt,\quad (f_i^{m+n-j}(t)=F_i'^{m+n-j}(t)=(1-R_i^{m+n-j}(t))'),$$

then it follows that

$$
(4.16)\quad \begin{cases} E_{i+1}^{j+1}(T^2)=2\int_0^\infty tR_i^{m+n-j}(t)dt=-2\lim\limits_{s\to 0}\varphi'^{m+n-j}(s) \\ \quad =-2\lim\limits_{s\to 0}\left[\left(\dfrac{Q_{i+1}^{j+1}(s)-P_{i+1}^{j+1}(s)}{s}\right)'\dfrac{1}{Q_{i+1}^{j+1}(s)}-\dfrac{(Q_{i+1}^{j+1}(s)-P_{i+1}^{j+1}(s))}{s}\cdot\dfrac{Q'^{j+1}_{i+1}(s)}{(Q_{i+1}^{j+1}(s))^2}\right] \\ \quad =-2[C_{i+1}^{j+1}-T_{i+1}^{j+1}A_{i+1}^{j+1}], \end{cases}
$$

where

$$
(4.17)\quad \begin{cases}
A_1^{j+1}=\lim\limits_{s\to 0}Q'^{j+1}_1(s)=\dfrac{1}{\lambda_j},\ B_1^{j+1}=\lim\limits_{s\to 0}P_1^{j+1}(s)=0,\ T_1^{j+1}=\lim\limits_{s\to 0}\left(\dfrac{Q_1^{i+1}(s)-P_1^{i+1}(s)}{s}\right)=\dfrac{1}{\lambda_j} \\
\hfill (j=0,1,\cdots n), \\
C_1^{j+1}=\lim\limits_{s\to 0}\left(\dfrac{Q_1^{j+1}(s)-P_1^{j+1}(s)}{s}\right)'=0,\ L_{00}^j=\lim\limits_{s\to 0}\psi_{00}^{m+n-j}(s)=1, \qquad (j=0,1,\cdots n), \\
D_{00}^j=\lim\limits_{s\to 0}\psi'^{m+n-j}_{00}(s)=-\dfrac{1}{\lambda_j}, \qquad (j=0,1,\cdots n), \\
L_{ii}^j=\dfrac{\lambda_j}{\lambda_j+\mu_i-\mu_i p_{i-1i}^{j-1j}L_{i-1i-1}^{j-1}},\ L_{ik}^j=\lim\limits_{s\to 0}\psi_{ik}^{m+n-j}(s),\ D_{ik}^j=\lim\limits_{s\to 0}\psi'^{m+n-j}_{ik}(s), \\
\hfill (j=i,\ i+1,\cdots n;\ i=1,2,\cdots n;\ k=0,1,\cdots i-1) \\
D_{ii}^j=\lim\limits_{s\to 0}\psi'^{m+n-j}_{ii}(s)=-\dfrac{(1-\mu_i p_{i-1i}^{j-1j}D_{i-1i-1}^{j-1j})(L_{ii}^j)^2}{\lambda_j}, \qquad (j=i,\ i+1,\cdots n;\ i=1,2,\cdots n), \\
A_{i+1}^{j+1}=\lim\limits_{s\to 0}Q'^{j+1}_{i+1}(s)=(A_i^j+\sum\limits_{k=0}^{i-1}A_{k+1}^{j+1})+\dfrac{L_{ii}^j(1-\mu_i p_{i-1i}^{j-1j}D_{i-1i-1}^{j-1})}{\lambda_j}, \\
\hfill (j=i,\ i+1,\cdots n;\ i=1,2,\cdots n), \\
B_{i+1}^{j+1}=\lim\limits_{s\to 0}P'^{j+1}_{i+1}(s)=L_{ii}^j\{[(\lambda_j-\mu_i p_{i-1i}^{j-1j}L_{i-1i-1}^{j-1})A_i^j+\mu_i(B_i^j-p_{i-1i}^{j-1j}D_{i-1i-1}^{j-1}) \\
\quad +[\lambda_j+\mu_i(1-p_{i-1i}^{j-1j}L_{i-1i-1}^{j-1})]\sum\limits_{k=0}^{i}A_k^{j+1}+\mu_i\sum\limits_{l=0}^{i-1}(p_{ll}^{j-1j}L_{i-1l}^{j-1}+p_{l-1l}^{j-1j}L_{i-1l-1}^{j-1})(B_{l+1}^{j+1}-A_{l+1}^{j+1})]\}/\lambda_j, \\
\hfill (j=i,\ i+1,\cdots n;\ i=1,2,\cdots n), \\
T_{i+1}^{j+1}=\lim\limits_{s\to 0}\left(\dfrac{Q_{i+1}^{j+1}(s)-P_{i+1}^{j+1}(s)}{s}\right)=\dfrac{L_{ii}^j[1+\mu_iT_i^j+\mu_i\sum\limits_{l=0}^{i-1}T_{l+1}^{j+1}(p_{ll}^{j-1j}L_{i-1l}^{j-1}+p_{l-1l}^{j-1j}L_{i-1l-1}^{j-1})]}{\lambda_j}, \\
\hfill (j=i,\ i+1,\cdots n;\ i=1,2,\cdots n), \\
C_{i+1}^{j+1}=\lim\limits_{s\to 0}\left(\dfrac{Q_{i+1}^{j+1}(s)-P_{i+1}^{j+1}(s)}{s}\right)'=L_{ii}^j\{A_i^j+\mu_iC_i^j+(1+\mu_iT_i^j)\sum\limits_{k=0}^{i}A_k^{j+1} \\
\quad +\mu_i\sum\limits_{l=0}^{i-1}(L_{i-1l}^{j-1}p_{ll}^{j-1j}+L_{i-1l-1}^{j-1}p_{l-1l}^{j-1j}) \\
\qquad \cdot[C_{l+1}^{j+1}+T_{l+1}^{j+1}(\sum\limits_{k=l+1}^{i}A_k^{j+1}+A_i^j)]+\mu_i\sum\limits_{l=0}^{i-1}T_{l+1}^{j+1}(D_{i-1l}^{j-1}p_{ll}^{j-1j}+D_{i-1l-1}^{j-1}p_{l-1l}^{j-1j})\}/\lambda_j, \\
\hfill (j=i,\ i+1,\cdots n;\ i=1,2,\cdots,n).
\end{cases}
$$

Specially when (4.7) is true we have

$$
(4.18)\quad
\left\{
\begin{aligned}
&A_1^{j+1}=\frac{1}{(m+n-j)\lambda},\ B_1^{j+1}=0,\ T_1^{j+1}=\frac{1}{(m+n-j)\lambda}, && (j=0,1,\cdots n),\\
&C_1^{j+1}=0,\ L_{00}^{j}=1,\ D_{00}^{j}=-\frac{1}{(m+n-j)\lambda}, && (j=0,1,\cdots n),\\
&L_{ii}^{j}=\frac{(m+n-j)(m+n-j+1)\lambda}{(m+n-j)(m+n-j+1)\lambda+i\mu(m+n-j+1-mL_{i-1i-1}^{j-1})},\\
& && (j=i+1,\cdots n;\ i=1,2,\cdots n),\\
&D_{ii}^{j}=\frac{-(m+n-j+1-im\mu D_{i-1i-1}^{j-1})(L_{ii}^{j})^2}{(m+n-j)(m+n-j+1)\lambda}, && (j=i,i+1,\cdots n;\ i=1,2,\cdots n),\\
&A_{i+1}^{j+1}=(A_i^j+\sum_{k=0}^{i-1}A_{k+1}^{j+1})+\frac{L_{ii}^{j}(m+n-j+1-im\mu D_{i-1i-1}^{j-1})}{(m+n-j)(m+n-j+1)\lambda},\\
& && (j=i,i+1,\cdots n;\ i=1,2,\cdots n),\\
&B_{i+1}^{j+1}=L_{ii}^{j}\{[(m+n-j)(m+n-j+1)\lambda-im\mu L_{i-1i-1}^{j-1}]A_i^j+i\mu[(m+n-j+1)B_i^j\\
&\qquad -mD_{i-1i-1}^{j-1}]+[(m+n-j)(m+n-j+1)\lambda+i\mu(m+n-j+1-mL_{i-1i-1}^{j-1})]\sum_{k=0}^{i}A_k^{j+1}\\
&\quad +i\mu\sum_{l=0}^{i-1}[(n-j+1)L_{i-1l}^{j-1}+mL_{i-1l-1}^{j-1}[(B_{l+1}^{j+1}-A_{l+1}^{j+1})\}/(m+n-j)(m+n-j+1)\lambda,\\
& && (j=i,i+1,\cdots n;\ i=1,2,\cdots n),\\
&T_{i+1}^{j+1}=\frac{L_{ii}^{j}\{(m+n-j+1)(1+i\mu T_i^j)+i\mu\sum_{l=0}^{i-1}T_{l+1}^{j+1}[(n-j+1)L_{i-1l}^{j-1}+mL_{i-1l-1}^{j-1}]\}}{(m+n-j)(m+n-j+1)\lambda},\\
& && (j=i,\ i+1,\cdots n;\ i=2,3,\cdots n),\\
&C_{i+1}^{j+1}=L_{ii}^{j}\{(m+n-j+1)(A_i^j+i\mu C_i^j)+(m+n-j+1)(1+\mu_i T_i^j)\sum_{k=0}^{i}A_k^{j+1}\\
&\qquad +i\mu\sum_{l=0}^{i-1}[(n-j+1)L_{i-1l}^{j-1}+mL_{i-1l-1}^{j-1}]\\
&\qquad\quad \cdot[C_{l+1}^{j+1}+T_{l+1}^{j+1}(\sum_{k,l+1}^{i}A_k^{j+1}+A_i^j)]+i\mu\sum_{l=0}^{i-1}T_{l+1}^{j+1}[(n-j+1)D_{i-1l}^{j-1}\\
&\qquad\qquad +mD_{i-1l-1}^{j-1}]\}/(m+n-j)(m+n-j+1)\lambda,\\
& && (j=i,\ i+1,\cdots n;\ i=1,2,\cdots n).
\end{aligned}
\right.
$$

We have obtained the following theorem.

THEOREM 4.4. *The variance V_{n+1}^{n+1} of the model II is given by following formula*

$$(4.19)\qquad V_{n+1}^{n+1}=T_{n+1}^{n+1}(2A_{n+1}^{n+1}-T_{n+1}^{n+1})-2C_{n+1}^{n+1},$$

where T_{i+1}^{j+1}, A_{i+1}^{j+1} and C_{i+1}^{j+1} are given as (4.17), *and given by* (4.18) *when* (4.7) *holds true.*

EXAMPLE 3.

For the case $m=n=1$, since then $\lambda_0=2\lambda$, $\lambda_1=\lambda$ and $p_{00}^{01}=p_{01}^{01}=\frac{1}{2}$, we have

$$Q_1^1(s)=\frac{s+\lambda_0}{\lambda_0}=\frac{s+2\lambda}{2\lambda},\ Q_1^2(s)=\frac{s+\lambda_1}{\lambda_1}=\frac{s+\lambda}{\lambda},$$

$$Q_2^2(s)=\frac{(s+\lambda)^2(s+2\lambda+\mu)}{\lambda^2(2\lambda+\mu)},\ \frac{Q_2^2(s)-P_2^2(s)}{s}=\frac{(s+2\lambda)(s+\lambda+\mu)}{\lambda^2(2\lambda+\mu)},$$

$$\varphi_1^1(s)=\frac{Q_2^2(s)-P_2^2(s)}{s}\cdot\frac{1}{Q_2^2(s)}=\frac{(s+2\lambda)(s+\lambda+\mu)}{(s+\lambda)^2(s+2\lambda+\mu)},$$

which leads to that

$$R_1^1(t)=e^{-\lambda t}+\frac{\lambda\mu}{\lambda+\mu}te^{-\lambda t}+\frac{\lambda\mu}{(\lambda+\mu)^2}e^{-\lambda t}[e^{-(\lambda+\mu)t}-1],$$

$$T_2^2=\frac{2(\lambda+\mu)}{\lambda(2\lambda+\mu)},\ \Delta T_{12}=\frac{2(\lambda+\mu)}{(2\lambda+\mu)},$$

$$V_2^2=T_2^2(2A_2^2-T_2^2)-2C_2^2=\frac{16}{(2\lambda+\mu)^2}+\frac{2(11-4\lambda)\mu}{\lambda(2\lambda+\mu)^2}+\frac{2(1-2\lambda)\mu^2}{\lambda^2(2\lambda+\mu)^2}.$$

§ 4.2. General failure and repair distribution.

In the previous § 4.1 we considered the case when the distributions $F(t)$ $(=1-R(t))$, $M(t)$ are both exponential. Here we consider the case when (3.23) holds. By the same method as in § 4.1 we get

$$(4.20)\quad \left\{\begin{aligned}
&R_i^{m+n-j}(t)=e^{-\int_0^t[\lambda_j(x)+\mu_i(x)]dx}+\int_0^t\mu_i(\tau)e^{-\int_0^\tau[\lambda_j(x)+\mu_i(x)dx}R_{i-1}^{m+n-(j-1)}(t-\tau)d\tau\\
&\qquad+\iint\limits_{\tau+\tau'\leq t}\mu_i(\tau)e^{-\int_0^\tau[\lambda_j(x)+\mu_i(x)]dx}\\
&\qquad\cdot[\sum_{k=1}^{i}g_{i-1k-1}^{m+n-(j-1)}(\tau')[p_{k-1k}^{j-1j}R_k^{m+n-j}(t-\tau-\tau')+p_{k-1k-1}^{j-1j}R_{k-1}^{m+n-j}(t-\tau-\tau')]d\tau d\tau',\\
&\qquad\qquad(j=i,\ i+1,\cdots n;\ i=1,2,\cdots n),\\
&R_0^{m+n-j}(t)=e^{-\int_0^t\lambda_j(x)dx},\qquad (j=0,1,\cdots n),\\
&g_{ih}^{m+n-j}(t)=\iint\limits_{\tau+\tau'\leq t}\mu_i(\tau)e^{-\int_0^\tau[\lambda_j(x)+\mu_i(x)]dx}[g_{i-1i-1}^{m+n-(j-1)}(\tau')p_{i-1i}^{j-1j}g_{ih}^{m+n-j}(t-\tau-\tau')\\
&\qquad+\sum_{l=h}^{i-1}(p_{ll}^{j-1j}g_{i-1l}^{m+n-(j-1)}(\tau')+p_{l-1l}^{j-1j}g_{i-1l-1}^{m+n-(j-1)}(\tau'))g_{lh}^{m+n-j}(t-\tau-\tau')]d\tau d\tau',\\
&\qquad\qquad(j=i,\ i+1,\cdots n;\ i=1,2,\cdots n;\ h=0,1,\cdots i-1),\\
&g_{ii}^{m+n-j}(t)=\lambda_j(t)e^{-\int_0^t[\lambda_j(x)+\mu_i(x)]dx}+\iint\limits_{\tau+\tau'\leq t}\mu_i(\tau)e^{-\int_0^\tau[\lambda_j(x)+\mu_i(x)]dx}\\
&\qquad\cdot g_{i-1i-1}^{m+n-(j-1)}(\tau')p_{i-1i}^{j-1j}g_{ii}^{m+n-j}(t-\tau-\tau')d\tau d\tau',\\
&\qquad\qquad(j=i,\ i+1,\cdots n;\ i=1,2,\cdots n),\\
&g_{00}^{m+n-j}(t)=\lambda_j(t)e^{-\int_0^t\lambda_j(x)dx}\qquad (j=0,1,\cdots n).
\end{aligned}\right.$$

Let us put

$$
(4.21)\quad
\begin{cases}
\varphi_i^{m+n-j}(s)=\int_0^\infty e^{-st}R_i^{m+n-j}(t)dt, \qquad \varphi_i^{*m+n-j}(s)=\int_0^\infty e^{-st}dR_i^{m+n-j}(t), & (j=i,\ i+1,\cdots n;\ i=0,1,\cdots n),\\
\psi_{ik}^{m+n-j}(s)=\int_0^\infty e^{-st}g_{ik}^{m+n-j}(t)dt, & (j=i,\ i+1,\cdots n;\ i=0,1,\cdots n),\\
{}^1\varphi(s,\lambda_j,\mu_i)=\int_0^\infty e^{-st-\int_0^t[\lambda_j(x)+\mu_i(x)]dx}dt, & (j=i,\ i+1,\cdots n;\ i=0,1,\cdots n),\\
{}^2\varphi(s,\lambda_j,\mu_i)=\int_0^\infty \mu_i(t)e^{-st-\int_0^t[\lambda_j(x)+\mu_i(x)]dx}dt, & (j=i,\ i+1,\cdots n;\ i=0,1,\cdots n)\\
{}^3\varphi(s,\lambda_j,\mu_i)=\int_0^\infty \lambda_j(t)e^{-st-\int_0^t[\lambda_j(x)+\mu_i(x)]dx}dt, & (j=i,\ i+1,\cdots n;\ i=0,1,\cdots n)
\end{cases}
$$

where $\mu_0(x)=0$

By applying the laplace transformation to Eqs (4.20) we obtain

(4.22)

$$
\varphi_i^{m+n-j}(s)=\frac{{}^1\varphi(s,\lambda_j,\mu_i)+{}^2\varphi(s,\lambda_j,\mu_i)\varphi_{i-1}^{m+n-(j-1)}(s)+{}^2\varphi(s,\lambda_j,\mu_i)\sum_{k=0}^{i-1}(\psi_{i-1k}^{m+n-(j-1)}(s)p_{kk}^{j-1j}+\psi_{i-1k-1}^{m+n-(j-1)}p_{k-1k}^{j-1j})\varphi_k^{m+n-j}(s)}{1-p_{i-1i}^{j-1j}{}^2\varphi(s,\lambda_j,\mu_i)\psi_{i-1i-1}^{m+n-(j-1)}(s)}
$$

$$(j=i,\ i+1,\cdots,n;\ i=1,2,\cdots n),$$

$$
\begin{aligned}
\varphi_i^{*m+n-j}(s)=\{&-1+{}^1\varphi(s,\lambda_j,\mu_i)s+{}^2\varphi(s,\lambda_j,\mu_i)[1+\varphi_{i-1}^{*m+n-(j-1)}(s)\\
&+\sum_{k=0}^{i}(\psi_{i-1k}^{m+n-(j-1)}(s)p_{kk}^{j-1j}+\psi_{i-1k-1}^{m+n-(j-1)}(s)p_{k-1k}^{j-1j})\\
&+\sum_{k=0}^{i-1}(\psi_{i-1k}^{m+n-(j-1)}(s)p_{kk}^{j-1j}+\psi_{i-1k-1}^{m+n-(j-1)}(s)p_{k-1k}^{j-1j})\varphi_k^{*m+n-j}(s)\}/1-p_{i-1i}^{j-1j}{}^2\varphi(s,\lambda_j,\mu_i)\psi_{i-1i-1}^{m+n-(j-1)},
\end{aligned}
$$

$$(j=i,\ i+1,\cdots,n;\ i=1,2,\cdots n),$$

$$\varphi_0^{m+n-j}(s)={}^1\varphi(s,\lambda_j,0), \qquad (j=0,1,\cdots,n),$$

$$\varphi_0^{*m+n-j}(s)=-{}^3\varphi(s,\lambda_j,0), \qquad (j=0,1,\cdots,n),$$

$$
\psi_{ih}^{m+n-j}(s)=\frac{{}^2\varphi(s,\lambda_j,\mu_i)\sum_{l=h}^{i-1}(p_{ll}^{j-1j}\psi_{i-1l}^{m+n-(j-1)}(s)+p_{l-1l}^{j-1j}\psi_{i-1l-1}^{m+n-(j-1)}(s))\psi_{lh}^{m+n-j}(s)}{1-p_{i-1i}^{j-1j}{}^2\varphi(s,\lambda_j,\mu_i)\psi_{i-1i-1}^{m+n-(j-1)}(s)}
$$

$$(j=i,\ i+1,\cdots,n;\ i=1,2,\cdots n)$$

$$
\psi_{ii}^{m+n-j}(s)=\frac{{}^3\varphi(s,\lambda_j,\mu_i)}{1-p_{i-1i}^{j-1j}\varphi(s,\lambda_j,\mu_i)\psi_{i-1i-1}^{m+n-(j-1)}(s)}, \qquad (j=i,\ i+1,\cdots,n;\ i=1,2,\cdots,n)
$$

$$\psi_{00}^{m+n-j}(s)={}^3\varphi(s,\lambda_j,0), \qquad (j=0,1,\cdots,n).$$

And we obtain the $\varphi_i^{m+n-j}(s)$ by applying the same method as in the § 3.2.

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